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Estimation of the Instantaneous Signal Parameters Using a Modified Prony's Method

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Abstract—A method for approximation of an arbitrary continuous function in the neighborhood of a given point by complex exponential functions (exponentials) is described. The method is similar to the Taylor series to the effect that approximation is performed by the derivatives at the given point. Determination of parameters of the exponentials is made on the basis of a modified Prony's method. It is shown that the method makes it possible to simulate the local behavior of periodic and quasiperiodic processes more effectively than a Taylor series. An algorithm for estimating the instantaneous parameters of the single-component quasiperiodic continuous signals is obtained. The relation to a known energy separation algorithm (ESA) is shown. A separate algorithm for discrete signals by using finite differences is obtained. The applicability of the algorithm for practical analysis of discrete signals with consideration of the influence of additive noise on the accuracy of the parameter estimates is investigated. Experimental comparison of the algorithm with the known methods of estimating instantaneous parameters on the basis of the Prony's method and ESA is performed.

Keywords: time–frequency analysis, determination of parameters of decaying exponentials, instantaneous frequency estimation, algorithm of energy separation, Prony's method

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1. INTRODUCTION

Using the Taylor series makes it possible to approximate a given real-valued function $f(x)$ in the neighborhood of point x_0 by means of a power series $f(x) \approx \sum_{k=0}^{n} a_k (x - x_0)^k$ with real-valued coefficients a_k that are calculated from the derivatives of the function at the point x_0. In this work, the problem of approximating a given complex function $f(x)$ in the neighborhood of point x_0 is solved using the exponential power series $f(x) \approx \sum_{k=1}^{n} h_k z_k^{x-x_0}$ having the complex parameters h_k and z_k that also are calculated from the derivatives of the function at the point x_0. To solve the problem, the Prony's method is used, which makes it possible to simulate the selected data as a linear combination of exponential functions [1], while all the parameters of these functions (amplitude, frequency, damping factor, and initial phase) are provided directly from the data themselves. As the Prony's method resolves the function into exponential components and not into polynomials, when modeling quasiperiodic processes, it has the following advantages:

(1) significantly longer approximation interval when using the same number of derivatives;

(2) the possibility of direct interpretation of the model parameters in terms of time and instantaneous frequency [2].

The Prony's method consists of three main phases. At first, the estimation of the linear prediction coefficients is implemented and they form the characteristic polynomial. Then the roots of a polynomial are calculated, by which the damping factor and frequency of each exponential are determined. Then the system of linear equations is solved, resulting in an estimation of amplitudes and initial phases. The initial Prony's method and all of its modifications known to the authors imply that the analyzed data are a sequence of measurements selected at regular intervals, and thus they apply only to digital signals. However, as is shown in this article, the Prony's method can be modified in such a way as to obtain a local approximation of continuous functions using its derivatives.

The method of approximation of continuous functions obtained in the article can be used to estimate the instantaneous parameters of discrete signals. For this purpose, an approximate calculation of derivatives in the form of finite differences is performed. As is shown in the work, the error in the calculation of derivatives affects the evaluation of instantaneous frequency and can be corrected by using the appropriate expressions. Application of the method is parametric modeling of quasiperiodic processes (including the modeling of voiced speech signals using instantaneous sinusoidal parameters [3, 4]). A feature of the method is the ability to process not only real-valued but also complex signals.

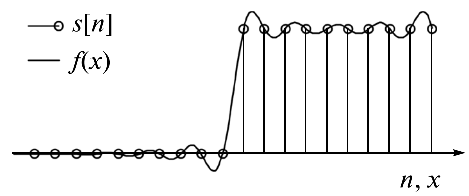

Fig. 1. Recovery of the continuous function (signal) $f(x)$ from digital sequence $s[n]$.

Approximation of the signal using the derivatives of can be applied only to continuous and smooth functions on the observation interval. For discontinuous or noisy signals, the approximation method on the basis of derivatives is not applicable. Therefore, in the work, it is assumed that the analyzed digital signal has a restricted spectrum (if it is not specified particularly, it is assumed that the spectrum is restricted by the Nyquist frequency) and, consequently, according to the Nyquist–Shannon–Kotelnikov theorem, the continuous signal can be recovered from this spectrum and be described by a continuous and smooth function. Figure 1 shows an example of the recovery of a continuous signal from a discrete sequence corresponding to a rectangular pulse. The assumption of restricted spectrum makes it possible to calculate the derivatives of any signals, including noisy ones.

Unlike analytical functions, for which it is possible to predict the behavior of the approximated signal for an arbitrary point in time (far away from the time of calculation of all derivatives), in the case of a continuous signal, the interval of effective approximation is determined by the width of the signal spectrum (the narrower spectrum, the longer the available interval of approximation).

Because the behavior of the continuous signal is completely determined by the values of its discrete samples, then the calculation of derivatives at a given point in time and the approximation in the given interval do not result in the receipt of new information. However, the value of the evolved method is in the possibility to describe the signal using an exponential (sinusoidal) model with instantaneous (i.e., changing at each moment of time) parameters. Sinusoidal modeling of the signal provides a basis for solution of many practical tasks (for example, processing, coding, and speech synthesis). Interpolation methods based on derivatives also may be useful in the practical tasks of multirate processing of the signal values at any points of time that are not multiple to the sampling interval.

The work is organized as follows. Section 2 outlines the initial Prony's method. The modified Prony's method for approximation of continuous functions by its derivatives is proposed in Section 3. In Section 4, the modified Prony's method is compared with Taylor series by a few examples. Several continuous functions are used for which the region of approximation by the Taylor series is appreciably limited. It is shown that the proposed method provides a wider region of approximation for the selected functions. Section 5 discusses the special case where the function is a decaying sinusoid. This case is chosen as the simplest and most appropriate example for comparison with alternative methods, as well as for research of robustness to errors and additive noise. The corresponding estimation algorithm is deduced and its relation to the known algorithm ESA is shown [5]. The discrete version of the algorithm is performed using the finite difference method and the expressions of the error correction in calculation of the derivative are deduced. It is shown that, in the case of a three-point differentiator, the derived discrete version of algorithm is more tolerant to additive white noise compared to discrete versions of the ESA. In Section 6, a practical comparison of the obtained algorithm with similar methods using pure and noisy synthetic signals is performed.

2. THE INITIAL PRONY'SMETHOD

In accordance with the Prony's method, the discrete complex signal $s[n]$ can be represented as a sum of damped complex exponentials:

$$s[n] = \sum_{k=1}^{p} h_k z_k^n,$$

where p is the number of complex exponentials, $h_k = A_k e^{j\theta_k}$ is the initial amplitude, and $z_k^n = e^{n(\alpha_k + j\omega_k)}$ is the complex exponential with the damping factor α_k and normalized angular frequency ω_k. In order to evaluate the exact parameters of the model, $2p$ complex samples of the signal are needed. The desired solution can be obtained using the following set of equations:

$$\begin{pmatrix} z_1^0 & z_2^0 & \cdots & z_p^0 \\ z_1^1 & z_2^1 & \cdots & z_p^1 \\ \vdots & \vdots & & \vdots \\ z_1^{p-1} & z_2^{p-1} & \cdots & z_p^{p-1} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_p \end{pmatrix} = \begin{pmatrix} s[1] \\ s[2] \\ \vdots \\ s[p] \end{pmatrix}. \tag{1}$$

Complex exponents $z_1, z_2, \ldots, z_p$ are the roots of the characteristic polynomial

$$\psi(z) = \sum_{m=0}^{p} a[m] z^{p-m} \tag{2}$$

with complex coefficients $a[m]$, which in turn are the solution of the set of linear equations

$$\begin{pmatrix} s[p] & s[p-1] & \cdots & s[1] \\ s[p+1] & s[p] & \cdots & s[2] \\ \vdots & \vdots & & \vdots \\ s[2p-1] & s[2p-2] & \cdots & s[p] \end{pmatrix} \begin{pmatrix} a[1] \\ a[2] \\ \vdots \\ a[p] \end{pmatrix} = -\begin{pmatrix} s[p+1] \\ s[p+2] \\ \vdots \\ s[2p] \end{pmatrix} \tag{3}$$

and $a[0] = 1$. The values of the damping factor α_k and frequency ω_k are calculated using the following expression:

$$\alpha_k = \ln|z_k|, \quad \omega_k = \arctan\left[\frac{\mathrm{Im}(z_k)}{\mathrm{Re}(z_k)}\right].$$

As a result of the substitution of the values $z_1, z_2, \ldots, z_p$, the set (1) is converted to a set of linear equations, which is solved relative to complex parameters $h_1, h_2, \ldots, h_p$. Then, for each of them, one can calculate the initial amplitude A_k and θ_k as follows:

$$A_k = |h_k|, \quad \theta_k = \arctan\left[\frac{\mathrm{Im}(h_k)}{\mathrm{Re}(h_k)}\right]. \tag{4}$$

The Prony's method outlined above applies only to discrete signals. In order to describe a continuous function by it, the discretization of the function needs to be implemented on the approximation interval. For a description of the behavior of an arbitrary function in an infinitely small neighborhood of a given point, one must discretize with an infinitely small sampling interval. A numerical solution in this case is difficult, because it requires the calculation of appropriate limits. Instead, one can use the value of the derivative function at the point, as will be shown in the next section.

3. MODIFICATION OF THE PRONY'S METHOD FOR APPROXIMATION OF CONTINUOUS FUNCTIONS

The following observation underlies the proposed method of approximation of continuous functions. If analytic function $f(x)$ is represented as a sum of a finite number of complex exponentials, the sequence of its derivatives at an arbitrary point x_0 can be represented as a sum of the same number of complex exponentials. For the proof, it is sufficient to consider the derivative of function $f(x)$ of order n at a point x_0:

$$f^{(n)}(x) = \left(\sum_{k=1}^{p} h_k z_k^{x-x_0}\right)^{(n)} = \sum_{k=1}^{p} h_k (\alpha_k + j\omega_k)^n z_k^{x-x_0}, \quad f^{(n)}(x_0) = \sum_{k=1}^{p} h_k y_k^n, \tag{5}$$

where $f^{(n)}(x) = d^n f(x) / dx^n$ and $y_k^n = (\alpha_k + j\omega_k)^n = e^{n(\ln|y_k| + j\arg(y_k))}$ is a complex exponential with the damping coefficient $\ln|y_k|$ and normalized angular frequency $\arg(y_k)$.

Thus, the solution can be obtained from the set (1) by calculating the derivative of the function $f(x)$ at the point x_0 and by replacing the variables:

$$\begin{pmatrix} y_1^0 & y_2^0 & \cdots & y_p^0 \\ y_1^1 & y_2^1 & \cdots & y_p^1 \\ \vdots & \vdots & & \vdots \\ y_1^{p-1} & y_2^{p-1} & \cdots & y_p^{p-1} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_p \end{pmatrix} = \begin{pmatrix} f(x_0) \\ \dot{f}(x_0) \\ \vdots \\ f^{(p-1)}(x_0) \end{pmatrix}, \tag{6}$$

where $\dot{f}(x) = df(x)/dx$. The search for a solution of the set (6) is performed in the same way as for set (1), except that, with allowance for the replacement of the variables, the damping coefficients α_k and frequencies ω_k are calculated as follows:

$$\alpha_k = \mathrm{Re}(y_k), \quad \omega_k = \mathrm{Im}(y_k). \tag{7}$$

Therefore, the proposed approximation method is the following action sequence.

1. Solving the set of linear equations for the coefficients $a[1], a[2], \ldots, a[p]$:

$$\begin{pmatrix} f^{(p-1)}(x_0) & f^{(p-2)}(x_0) & \cdots & f^{(0)}(x_0) \\ f^{(p)}(x_0) & f^{(p-1)}(x_0) & \cdots & f^{(1)}(x_0) \\ \vdots & \vdots & & \vdots \\ f^{(2p-2)}(x_0) & f^{(2p-3)}(x_0) & \cdots & f^{(p-1)}(x_0) \end{pmatrix} \begin{pmatrix} a[1] \\ a[2] \\ \vdots \\ a[p] \end{pmatrix} = -\begin{pmatrix} f^{(p)}(x_0) \\ f^{(p+1)}(x_0) \\ \vdots \\ f^{(2p-1)}(x_0) \end{pmatrix}. \tag{8}$$

2. Finding the complex exponentials $y_1, y_2, \ldots, y_p$ by calculating the roots of the characteristic polynomial (2).

3. Substituting the values $y_1, y_2, \ldots, y_p$ into set (6) and solving it with respect to $h_1, h_2, \ldots, h_p$.

4. Calculating the desired parameters A_k, θ_k, α_k, and ω_k by formulas (4) and (7) respectively.

As a result, we will find an approximation of the original function $f(x)$ in the neighborhood of the point x_0 by the complex exponentials:

$$\overline{f}(x) = \sum_{k=1}^{p} h_k z_k^{x-x_0}.$$

If the function $f(x)$ is the sum of p damping complex exponentials, for exact fitting of parameters, it is required to find a $2p - 1$ derivative at point x_0, while the exact approximation will be provided throughout the whole domain of function $f(x)$ regardless of the choice of x_0.

4. COMPARISON OF THE MODIFIED PRONY'S METHOD WITH THE TAYLOR SERIES APPROXIMATION

A comparison of the proposed method with approximation using Taylor series is performed below. The Taylor series is as follows:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x),$$

where $n!$ is the factorial of n, $f^{(n)}(x_0)$ is the nth derivative of $f(x)$ at the point x_0, and $R_n(x)$ is the residual term of the Taylor formula. When using a finite number of derivatives, the approximation of the function $f(x)$ is a polynomial of finite degree:

$$\overline{\overline{f}}(x) = \sum_{n=0}^{2p-1} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \tag{9}$$

When using (9) at the point x_0, the equality of values of functions $f(x_0)$ and $\overline{\overline{f}}(x_0)$ and all $2p - 1$ derivatives is provided, while in the modified Prony's method, the equality of values of the functions and only of the first $p - 1$ derivatives is provided. The remaining p derivatives are used to calculate the coefficients of the linear prediction (8). Owing to the use of the predictions, often one can get a good approximation of

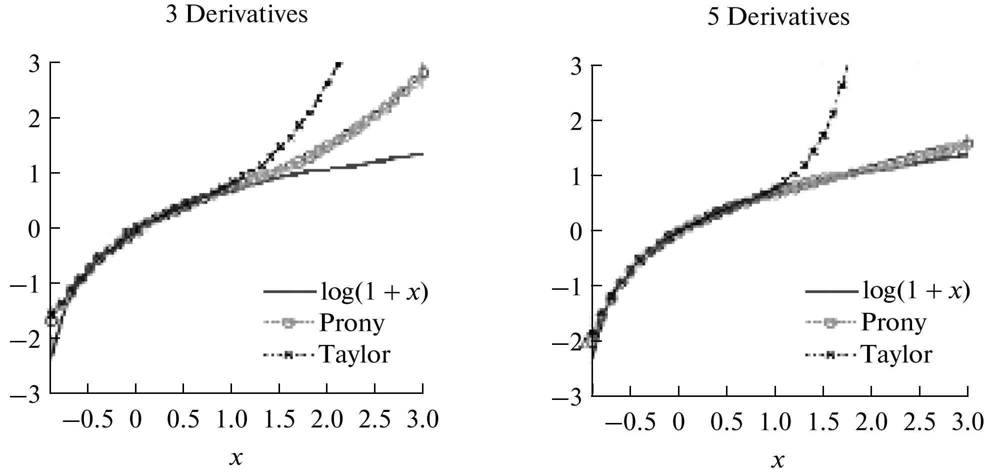

Fig. 2. Approximation of function $f(x) = \log(1 + x)$, $x_0 = 0$.

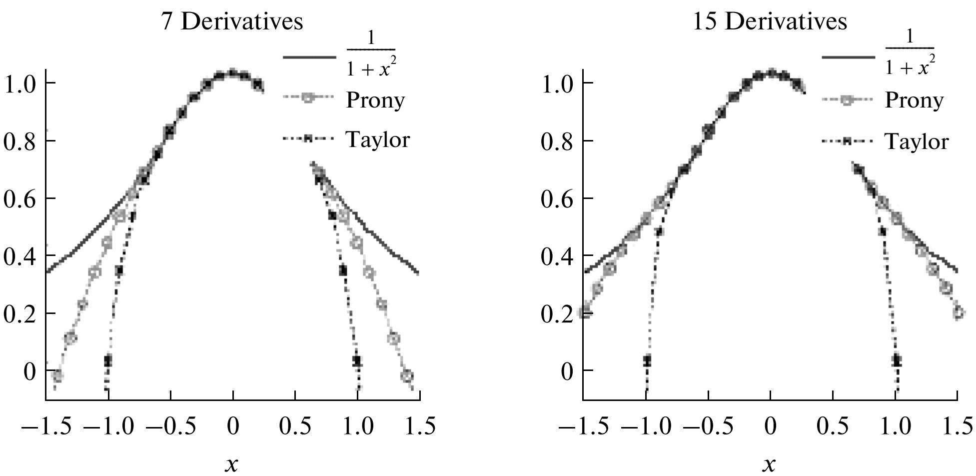

Fig. 3. Approximation of function $f(x) = \frac{1}{1+x^2}$, $x_0 = 0$.

the initial function (including the nonperiodic) compared to the Taylor polynomial. Figure 2 shows an example of approximation of functions $f(x) = \log(1 + x)$, $x_0 = 0$. Notice that for $x > 1$ the approximation using the Taylor series becomes worse with increasing number of derivatives, whereas the approximation by complex exponentials tends to $f(x)$.

Figure 3 shows an example of approximation of functions $f(x) = \frac{1}{1+x^2}$, $x_0 = 0$. One can see that for $|x| > 1$ the compared methods have the same characteristics.

The advantage of the proposed method becomes evident for functions containing periodic components. Figure 4 shows an example of approximation of function $f(x) = \log(1 + x) + \cos(6x)$, $x_0 = 0$. One can see that the parameters of the exponentials are estimated sufficiently precisely to execute the extrapolation of the function for four periods. The above example shows that the proposed approximation method is well applicable for continuous quasiperiodic signals.

5. ESTIMATION OF THE INSTANTANEOUS PARAMETERS OF A REAL-VALUED SINGLE-COMPONENT PERIODIC SIGNAL

In digital signal processing, the task of estimating the changing parameters of a real-valued single-component periodic signal arises fairly often and it is a separate case having practical value in modeling of speech signals [5]. A single-component periodic signal means a signal that can be represented at any point

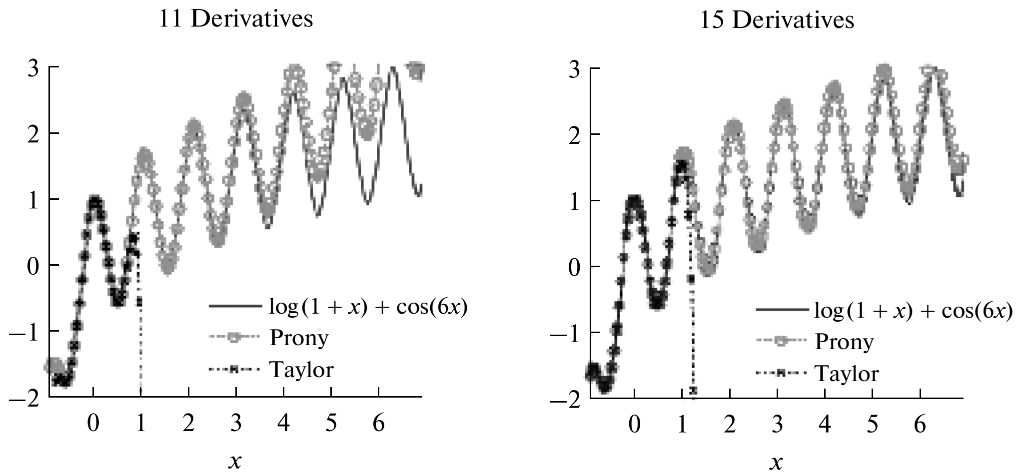

Fig. 4. Approximation of function $f(x) = \log(1+x) + \cos(6x)$, $x_0 = 0$.

in time as one damped sinusoid (in contrast to (5), where the sum of complex damped periodic functions is used):

$$f(x) = Ae^{\alpha n}\cos(\omega x + \theta). \tag{10}$$

The most popular approach to the solution of this task is the energy separation algorithm (ESA) based on the Teager–Kaiser nonlinear energy operator (TEO). The operator is as follows [5]:

$$\Psi[f(x)] = \dot{f}^2(x) - f(x)\ddot{f}(x). \tag{11}$$

As is shown in [6], this operator provides a more accurate estimate of an energy signal noisy by additive noise compared to the squaring operator

$$S[f(x)] = x^2(x).$$

In view of the fact that, for a periodic signal with constant amplitude and frequency $f(x) = A\cos(\omega x + \theta)$, the following relations are correct

$$\Psi[f(x)] = A^2\omega^2, \quad \Psi[\dot{f}(x)] = A^2\omega^4,$$

the frequency and the absolute value of the amplitude can be obtained as follows:

$$\omega = \sqrt{\frac{\Psi[\dot{f}(x)]}{\Psi[f(x)]}}, \tag{12}$$

$$|A| = \frac{\Psi[f(x)]}{\sqrt{\Psi[\dot{f}(x)]}}. \tag{13}$$

Expressions (12) and (13) form the core of the energy separation algorithm for continuous signals. To process discrete signals, the discrete approximation of the operator TEO [7] is used

$$\Psi\left[s[n]\right] = s^2[n] - s[n-1]s[n+1]. \tag{14}$$

In the article [10] several appropriate algorithms called DESA (Discrete Energy Separation Algorithm) for discrete signals are deduced. They make it possible to estimate the instantaneous frequency and amplitude. The DESA-1 algorithm uses two adjacent samples to calculate the derivatives of the first order and reduces to calculation of the following expressions:

$$\omega[n] \approx \arccos\left(1 - \frac{\Psi\left[y[n]\right] + \Psi\left[y[n+1]\right]}{4\Psi\left[s[n]\right]}\right), \quad |A[n]| \approx \sqrt{\frac{\Psi\left[s[n]\right]}{1 - \left(1 - \frac{\Psi\left[y[n]\right] + \Psi\left[y[n+1]\right]}{4\Psi\left[s[n]\right]}\right)^2}},$$

where $y[n] = s[n] - s[n-1]$. The algorithm allows estimating the parameters of periodic signals with a frequency not exceeding the Nyquist frequency.

In the DESA-2 algorithm, the first-order derivatives are calculated using the difference between the sampling with indices $n + 1$ and $n - 1$. The DESA-2 algorithm reduces to computing the following expressions:

$$\omega[n] \approx \frac{1}{2}\arccos\left(1 - \frac{\Psi[y[n]]}{2\Psi[s[n]]}\right), \quad |A[n]| \approx \frac{2\Psi[s[n]]}{\sqrt{\Psi[y[n]]}},$$

where $y[n] = s[n + 1] - s[n - 1]$. In the DESA-2 algorithm, the range of estimation of the frequency is limited to half the Nyquist frequency owing to the selected method of calculation of the derivative.

The above DESA algorithms are very popular owing to the low computational complexity and the lack of the need to use the Hilbert transformation and complex numbers. DESA algorithms are used in various applications such as demodulation [8], the separation of sound and speech sources [9], and noise reduction.

In [10], it is shown that, if the signal is a decaying sinusoid, the damping coefficient can be determined using the following expression:

$$\alpha = -\frac{\gamma_3[f(x)]}{2\Psi[f(x)]},$$

where $\gamma_3[f(x)] = f(x)f^{(3)}(x) - \dot{f}(x)\ddot{f}(x)$ is the energy operator of the third order. In the case of a discrete signal, one can use its approximation

$$\gamma_{3a}[s[n]] = \Psi[s[n]] - \Psi[s[n-1]].$$

We estimate the instantaneous parameters of a single-component periodic signal using a method derived in Section 3. If $f(x)$ is a real damped sinusoid, then one can find all the desired parameters according to its value and three derivatives at point x_0. As a result, we generate the following algorithm:

1. Calculation of the derivatives of the signal $\dot{f}, \ddot{f}, f^{(3)}$.

2. Calculation of the coefficients of the characteristic polynomial (2) of the set (8)

$$a[1] = \frac{f \cdot f^{(3)} - \dot{f} \cdot \ddot{f}}{\dot{f}^2 - f \cdot \ddot{f}} = \frac{\gamma_3[f]}{\Psi[f]}, \tag{15}$$

$$a[2] = \frac{\ddot{f}^2 - \dot{f} \cdot f^{(3)}}{\dot{f}^2 - f \cdot \ddot{f}} = \frac{\Psi[\dot{f}]}{\Psi[f]}. \tag{16}$$

3. Calculation of the roots of polynomial (2)

$$y_{1,2} = \frac{1}{2}\left(-a[1] \pm \sqrt{a^2[1] - 4a[2]}\right) = -\frac{\gamma_3[f]}{2\Psi[f]} \pm \sqrt{\frac{\gamma_3^2[f]}{4\Psi^2[f]} - \frac{\Psi[\dot{f}]}{\Psi[f]}}. \tag{17}$$

4. Calculation of initial complex amplitude

$$h = \frac{fy_2 - \dot{f}}{y_2 - y_1} = \frac{1}{2}\left(x + \frac{\frac{\gamma_3^2[f]}{2\Psi[f]} f + \dot{f}}{\sqrt{\frac{\gamma_3^2[f]}{4\Psi^2[f]} - \frac{\Psi[\dot{f}]}{\Psi[f]}}}\right). \tag{18}$$

5. Calculation of desired parameters of damped sinusoid

$$\alpha = \operatorname{Re}(y_1) = -\frac{\gamma_3[f]}{2\Psi[f]}, \tag{19}$$

$$\omega = \operatorname{Im}(y_1) = \sqrt{\frac{\Psi[\dot{f}]}{\Psi[f]} - \frac{\gamma_3^2[f]}{4\Psi^2[f]}}, \tag{20}$$

$$A = 2|h|, \tag{21}$$

$$\theta = \arctan\left[\frac{\operatorname{Im}(h)}{\operatorname{Re}(h)}\right]. \tag{22}$$

It is easy to show that, if $f(x)$ is a continuous sinusoid, i.e., when $\gamma_3[f] = 0$, the obtained expressions for estimating the instantaneous frequency and amplitude are identical to the ESA. In addition, it should be noted that the expression for estimating the damped coefficient (19) is identical to the expression given in [10] and the expression for estimating the instantaneous frequency (20) can be obtained from the case of decaying sinusoid considered in [5].

We perform the discretization of expressions (15)–(22). For this, in the first step of the algorithm, we use an approximation of the derivatives of the discrete signal $s[n]$, $\dot{s}[n]$, $\ddot{s}[n]$, and $s^{(3)}[n]$ by means of finite differences. Similar to the algorithms DESA-1 and DESA-2, we will separately consider two cases, namely, using a two-point or three-point differentiator.

In the first case, the differentiator is $d_1[n] = s[n] - s[n-1]$. Calculating the convolution of the sequence of its coefficients, we obtain the following expressions for the pulse h_{1-3} and frequency $H_{1-3}(e^{j\omega})$ characteristics of the differentiators of all desired orders:

$$h_1 = [1;-1],\quad H_1(e^{j\omega}) = (1-e^{-j\omega})e^{j\frac{\omega}{2}} = 2j\sin\left(\frac{\omega}{2}\right); \tag{23}$$

$$h_2 = [1;-2;1],\quad H_2(e^{j\omega}) = \left[2j\sin\left(\frac{\omega}{2}\right)\right]^2; \tag{24}$$

$$h_3 = [1;-3;3;-1],\quad H_3(e^{j\omega}) = \left[2j\sin\left(\frac{\omega}{2}\right)\right]^3. \tag{25}$$

According to the properties of linear system with constant parameters, if the input signal is a complex exponential $s[n] = e^{j\omega n}$, the frequency response of the system is a complex multiplier joining its input and output, i.e.,

$$d_{1-3}[n] = s[n]H_{1-3}(e^{j\omega}). \tag{26}$$

It follows from expressions (5) and (26) that the difference of the frequency characteristics of the differentiators of ideal $H_{1-3}^{\text{ideal}}(e^{j\omega}) = (j\omega)^{1-3}$ will result in error of estimation of the instantaneous frequency. Using relations (20), (23)–(25), we obtain an expression for the estimation of the instantaneous frequency based on calculation error of the derivatives:

$$\omega = 2\arcsin\left(\mathrm{Im}(y_1)/2\right). \tag{27}$$

Thus, by applying a two-point differentiator to approximate the first three derivatives of the signal and calculating the estimate of the instantaneous frequency using expression (27), we obtain the discrete estimation algorithm of an instantaneous sinusoidal parameter (for brevity, it is denoted as "DIPA-1"—*Discrete Instantaneous Prony Algorithm*). DIPA-1 uses four serial samples of the analyzed signal.

In the second case.

We apply the three-point differentiator $d_1[n] = s[n+1]/2 - s[n-1]/2$. We obtain the following pulse and frequency characteristics denoted by h_{1-3} and $H_{1-3}(e^{j\omega})$, respectively:

$$h_1 = \left[\frac{1}{2};0;-\frac{1}{2}\right],\quad H_1(e^{j\omega}) = \left(\frac{1}{2}-\frac{e^{-2j\omega}}{2}\right)e^{j\omega} = j\sin(\omega); \tag{28}$$

$$h_2 = \left[\frac{1}{4};0;-\frac{1}{2};0;\frac{1}{4}\right],\quad H_2(e^{j\omega}) = [j\sin(\omega)]^2; \tag{29}$$

$$h_3 = \left[\frac{1}{8};0;-\frac{3}{8};0;\frac{3}{8}0;-\frac{1}{8}\right],\quad H_3(e^{j\omega}) = [j\sin(\omega)]^3. \tag{30}$$

Note that $s[n]$, $\dot{s}[n]$, $\ddot{s}[n]$, and $s^{(3)}[n]$ are estimated independently of one another, since they are formed by disjoint sets of the signal samples. In theory, this somewhat improves the algorithm stability to additive noise. Using (20), (28)–(30), we obtain the corresponding expression for the instantaneous frequency adjustment:

$$\omega = \arcsin\left(\mathrm{Im}(y_1)\right).$$

As a result, we obtain an algorithm for estimation of parameters (denoted as "DIPA-2") which uses seven consecutive samples of the signal. As for DESA-2 owing to the selected method of calculating the derivative, the range of estimation of the instantaneous frequency is limited to half the Nyquist frequency.

We will show that the proposed algorithm DIPA-2 is more stable to additive white noise compared with the DESA algorithms. It is known that a discrete TEO approximation is sensitive to broadband noise [5]; however, the influence of noise can be greatly reduced by applying the a low-pass filter to the TEO output [7]. The idea is based on the fact that the TEO output typically has a narrower frequency band in comparison with the signal. We write the outputs of the differentiators of all orders obtained by the three-point symmetric difference according to (28)–(30):

$$d_1[n] = \frac{1}{2}s[n+1] - \frac{1}{2}s[n-1], \tag{31}$$

$$d_2[n] = \frac{1}{4}s[n+2] - \frac{1}{2}s[n] + \frac{1}{4}s[n-2], \tag{32}$$

$$d_3[n] = \frac{1}{8}s[n+3] - \frac{3}{8}s[n+1] + \frac{3}{8}s[n-1] - \frac{1}{8}s[n-3]. \tag{33}$$

Using (11) and (31)–(33), we obtain an alternative discrete TEO approximation:

$$\tilde{\Psi}(s[n]) = (d_1[n])^2 - s[n]d_2[n] = \frac{1}{4}\Psi(s[n-1]) + \frac{1}{2}\Psi(s[n]) + \frac{1}{4}\Psi(s[n+1]), \tag{34}$$

$$\tilde{\Psi}(d_1[n]) = (d_2[n])^2 - d_1[n]d_3[n] = \frac{1}{4}\Psi(d_1[n-1]) + \frac{1}{2}\Psi(d_1[n]) + \frac{1}{4}\Psi(d_1[n+1]). \tag{35}$$

From (34) and (35), it is shown that the output of operator $\tilde{\Psi}$ is actually the output of original operator Ψ (14) filtered by a low-pass filter with coefficients $\left[\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right]$.

6. RESULTS OF THE EXPERIMENT

This section provides a practical comparison of known algorithms for the estimation of the instantaneous amplitude and frequency of a single-component signal with the algorithms obtained in this work DIPA-1 and DIPA-2. Among the known algorithms, the following ones are used: DESA-1; DESA-2 [5]; classic four-point Prony's method [1], which is denoted as Prony; and combined five-point Prony's method [11], which is denoted as Prony.m. For comparison, synthetic quasiperiodic signals with known instantaneous parameters generated by means of the expression

$$s[n] = \left[1 + k\cos\left(\frac{\pi}{100}n\right)\right]\cos\left[\frac{\pi}{5}n + 20\lambda\sin\left(\frac{\pi}{100}n\right)\right],$$

where $(k,\lambda) \in \{(0.05i, 0.05j)\colon i = 1,\ldots,10;\ j = 1,\ldots,5\}$ and $n = 1,\ldots,400$, are used. The same discrete functions were used in [11] to estimate the precision of the algorithms for estimating the instantaneous frequency practically. The test sequence consists of 50 signals with different coefficients of amplitude and frequency modulation ranging from 5 to 50%. White noise of varying intensity is added to the synthesized signal, resulting in four additional test sequences with different signal-to-noise ratios. The average values of the absolute error of the estimate of the instantaneous frequency and amplitude are calculated for each sequence. Errors are classified as serious (hereinafter denoted GE—gross errors) or small (hereinafter denoted FE—fine errors). The total percentage of gross error is calculated as

$$\mathrm{GE}(\%) = \frac{N_{\mathrm{GE}}}{N_s} \times 100,$$

where N_{GE} is the number of measurements with deviations of more than ±20% from the true values, and N_s is the total number of measurements. The fine errors are normalized to the true values and are averaged over the total number:

$$\mathrm{FE}_p = \frac{1}{N_{\mathrm{FE}}}\sum_{n=1}^{N_{\mathrm{FE}}} \frac{\left|P^{\mathrm{true}}(n) - P^{\mathrm{est}}(n)\right|}{P^{\mathrm{true}}(n)} \times 100,$$

Estimation of the accuracy of the algorithms

	GE frequency, %		FE frequency, %		GE amplitude, %		FE amplitude, %	
Median filtering	no	yes	no	yes	no	yes	no	yes
Pure quasiperiodic signals								
DESA1 (5)	**0**	**0**	0.19	0.18	**0**	**0**	0.25	0.21
DESA2 (5)	**0**	**0**	0.22	0.20	**0**	**0**	0.28	0.23
Prony (4)	**0**	**0**	0.22	0.22	**0**	**0**	0.99	0.99
Prony.m (5)	**0**	**0**	**0.08**	**0.08**	–	–	–	–
DIPA1 (4)	**0**	**0**	0.38	0.21	0.36	**0**	9.86	3.21
DIPA2 (7)	**0**	**0**	0.15	0.15	**0**	**0**	**0.19**	**0.17**
Quasiperiodic signals with noise, signal/noise ratio is 40 dB								
DESA1 (5)	0.01	**0**	1.5	0.68	**0**	**0**	1.71	0.90
DESA2 (5)	**0**	**0**	2.04	1.09	0.02	**0**	2.70	1.26
Prony (4)	2.49	0.02	3.48	1.56	3.12	0.20	4.12	1.97
Prony.m (5)	8.40	0.89	4.85	2.72	–	–	–	–
DIPA1 (4)	2.76	0.20	3.79	1.84	6.49	0.12	9.68	3.72
DIPA2 (7)	**0**	**0**	**0.58**	**0.36**	**0**	**0**	**0.59**	**0.46**
Quasiperiodic signals with noise, signal/noise ratio is 30 dB								
DESA1 (5)	2.70	0.08	4.13	2.02	2.64	0.54	4.90	2.69
DESA2 (5)	4.95	0.99	5.50	3.14	10.2	1.95	6.28	3.44
Prony (4)	19.1	5.98	6.63	4.18	24.3	7.74	7.24	4.52
Prony.m (5)	33.3	16.6	8.08	5.54	–	–	–	–
DIPA1 (4)	21.0	6.41	7.19	4.64	23.7	4.48	8.90	4.59
DIPA2 (7)	**0.05**	**0**	**1.80**	**0.96**	**0.01**	**0**	**1.77**	**1.22**
Quasiperiodic signals with noise, signal/noise ratio is 20 dB								
DESA1 (5)	25.8	7.81	7.44	5.30	31.6	13.4	8.38	5.86
DESA2 (5)	37.3	14.5	8.93	6.48	48.6	20.2	8.92	6.84
Prony (4)	57.0	37.9	8.86	6.82	60.3	33.4	9.17	7.81
Prony.m (5)	69.8	57.2	10.0	8.01	–	–	–	–
DIPA1 (4)	59.8	31.8	9.33	7.62	46.5	14.2	8.81	7.46
DIPA2 (7)	**3.60**	**0.30**	**4.90**	**2.94**	**1.42**	**0.37**	**5.50**	**3.80**
Quasiperiodic signals with noise, signal/noise ratio is 15 dB								
DESA1 (5pt)	45.8	23.1	8.68	7.30	53.7	30.3	9.08	7.69
DESA2 (5pt)	58.3	34.7	9.50	7.97	66.2	40.9	9.92	8.43
Prony (4pt)	74.3	56.3	9.24	7.85	73.9	47.1	9.70	9.28
Prony.m (5pt)	82.5	76.5	10.23	9.93	–	–	–	–
DIPA1 (4pt)	76.3	51.2	9.52	8.56	57.6	27.7	9.46	8.88
DIPA2 (7pt)	**12.6**	**2.70**	**6.99**	**4.88**	**11.6**	**4.25**	**8.35**	**6.27**

where N_{FE} is the number of fine errors, P^{true} is the true value of the parameter, P^{est} is the parameter value obtained as a result of the estimation, and P means the corresponding parameter (instantaneous frequency or amplitude).

In [5], it is proposed to use a median filtering of output parameters to enhance the stability to noise of the DESA algorithm. To estimate the possibilities of this approach in the experiment, the five-point median filtering is also used for each method. The obtained error values are listed in the table.

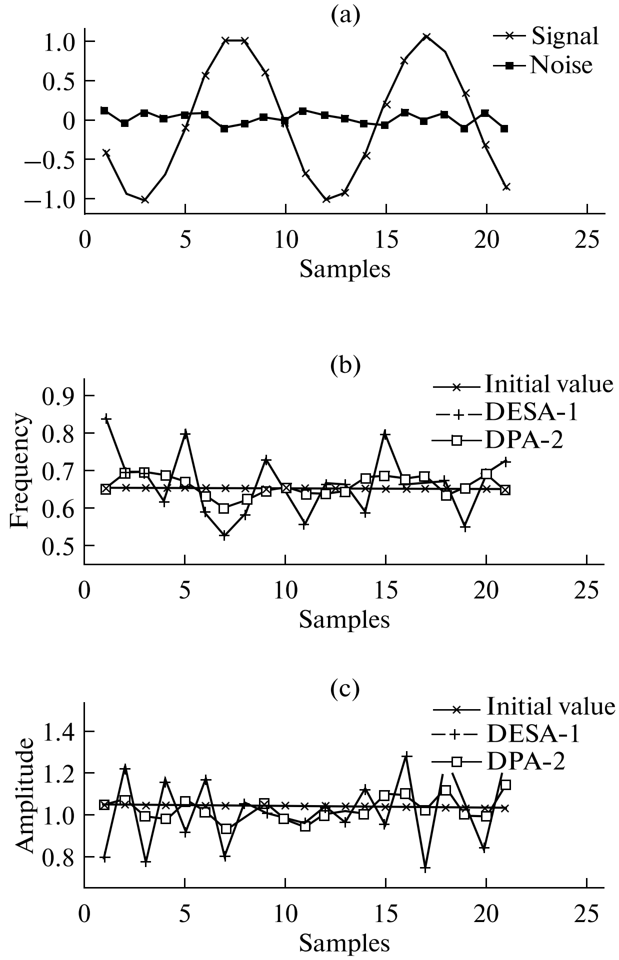

Fig. 5. Estimate of the instantaneous frequency and amplitude, signal/noise ratio of 20 dB: (a) initial signal and additive noise; (b) estimate of the instantaneous frequency; (c) estimate of the instantaneous amplitude.

For pure quasiperiodic signals, all algorithms show closely related accuracy; however, the five-point Prony.m algorithm provides the most accurate estimate of the instantaneous frequency, while DIPA-2 provides the most accurate estimate of amplitude. The estimate of the instantaneous frequency using DIPA-2 is more accurate than that by means of DESA. This shows that the proposed algorithm has a higher time resolution, in spite of the expansion of the analysis window (seven points instead of five).

If noise is added to the signal, the accuracy of the known algorithms based on the Prony's method (Prony and Prony.m) sharply degrades compared with DESA. In addition, the proposed algorithm DIPA-1 behaves in the same way. However, the seven-point algorithm DIPA-2 has a significant advantage for all signal-to-noise ratios ≤30 dB. Median filtering expands the analysis window of each algorithm by four points (i.e., DESA becomes nine-point). However, the overall accuracy of all methods with median filtering is worse than the seven-point DIPA-2 algorithm without a median filtering. One can make an empirical conclusion that filtering of the TEO output in DIPA-2 is more productive than filtering of the output parameters implemented in [5]. A brief example of estimation of parameters of the test signal is shown in Fig. 5.

7. CONCLUSIONS

A method of approximation of analytic functions by damped complex exponentials is described in the work. The method is based on the use of a modified Prony's method, which in contrast to the original one performing the approximation by points, performs the approximation by the derivatives of the function at a given point. In comparison with a Taylor series, the modified Prony's method provides a better description of quasiperiodic functions, using the same number of derivatives. The advantage of this method is that the calculated parameters have a direct frequency interpretation and can be used for time–frequency analysis of a signal.

This method of approximation of analytic functions was used to derive the algorithm for estimating the instantaneous parameters of a damped sinusoid. For this purpose, a special case of the limited signal approximation with a limited number of derivatives was considered. The discrete versions of the algorithm by approximating the derivative of a discrete signal by finite differences were obtained. The impact of the error introduced by the approximation was taken into consideration. The relation of the obtained algorithm to a known algorithm of energy separation was shown. The stability of the algorithm to additive noise was investigated. By comparison with known discrete algorithms for estimating instantaneous sinusoidal parameters, it was shown that the proposed algorithm is more stable to additive white noise.

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