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SHORT COMMUNICATIONS

The Parameters of a Point Source on the Gravity Field of a Sphere

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Abstract—This article describes a method for calculation of the parameters of a point source from the gravity field measured on a sphere. Conclusions about the conditions for the practical application of the method developed are drawn on the basis of a comparison of the results of spherical source coordinates and mass estimation by a field given on a plane and on a sphere.

Keywords: gravity, inverse problem, forward problem, iti-method

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INTRODUCTION

In recent years, new models of the gravity field of the Earth (Tapley et al., 2004) and the Moon (Kato et al., 2010; Zuber et al., 2013) with a high degree of accuracy (up to ±0.01 μGal) and detail (from 1° to 1′) (Save et al., 2016; Wiese et al., 2016) have been created based on the results of satellite missions (GRACE, GOCE, GRAIL, and KAGYA),.

Such models contain information about high-frequency anomalies of the gravity field; they can be used not only for solving planetary problems of gravimetry, but also for geological study of density inhomogeneities of the lithosphere and upper mantle.

To solve this problem, one should adapt mathematical methods for solving the inverse gravimetric problem, which have been widely developed for a plane, to apply them to a spherical surface. In particular, one of the most important areas is the development of methods of the of localization of singular points for a gravity field specified on a sphere, in order to determine the coordinates and the mass of sources of a gravity field located inside the sphere.

In gravimetry the inflection-tangent-intersection method (ITI) is widely used for solving the problem of determining the depth and mass of a point source of a field specified on a plane. The main efforts in solving gravimetric problems, taking the Earth's sphericity into account, are focused on solving forward problems (Bulychev et al., 1998; Kuznetsov et al., 2017; Starostenko et al., 1986) in a spherical coordinate system. Express methods for estimating the parameters of the anomaly sources, i.e., methods for solving inverse problems, are largely neglected. Thus, adaptation of mathematical methods for solving inverse gravimetric problem developed for plane to fields defined on a spherical surface would to some extent fill this gap.

THE MATHEMATICAL FOUNDATION OF THE METHOD

To determine the parameters of geological objects that create gravity anomalies, these objects are often approximated by bodies of a regular shape. Thus, many geological objects of more or less isometric form can be approximated by a spherical (point) source. The greater the distance is from the body to the attracted point the better approximation is (Mironov, 1980). The classical method for determining the parameters of a spherical object (center coordinates and mass) using a gravity field specified on a plane is described in nearly all textbooks on gravimetry (Mironov, 1980). The simplest form is known in gravimetry as the ITI method, where the depth of a spherical source h is determined as a distance $x_{1/2}$ from the of the gravity field anomaly extremum to the point where the amplitude of the anomaly is two times lower than the maximum value (Bloch, 2009; Bulychev et al., 2017)

$$h = 1.31x_{1/2}. \tag{1}$$

We consider the case of deducing the parameters of a spherical mass from a gravity field defined on a spherical surface. In a spherical coordinate system, the position of a point is determined by three coordinates (R, θ, λ), where R is the distance from the origin to the point, and θ and λ are the zenith and azimuth angles, respectively.

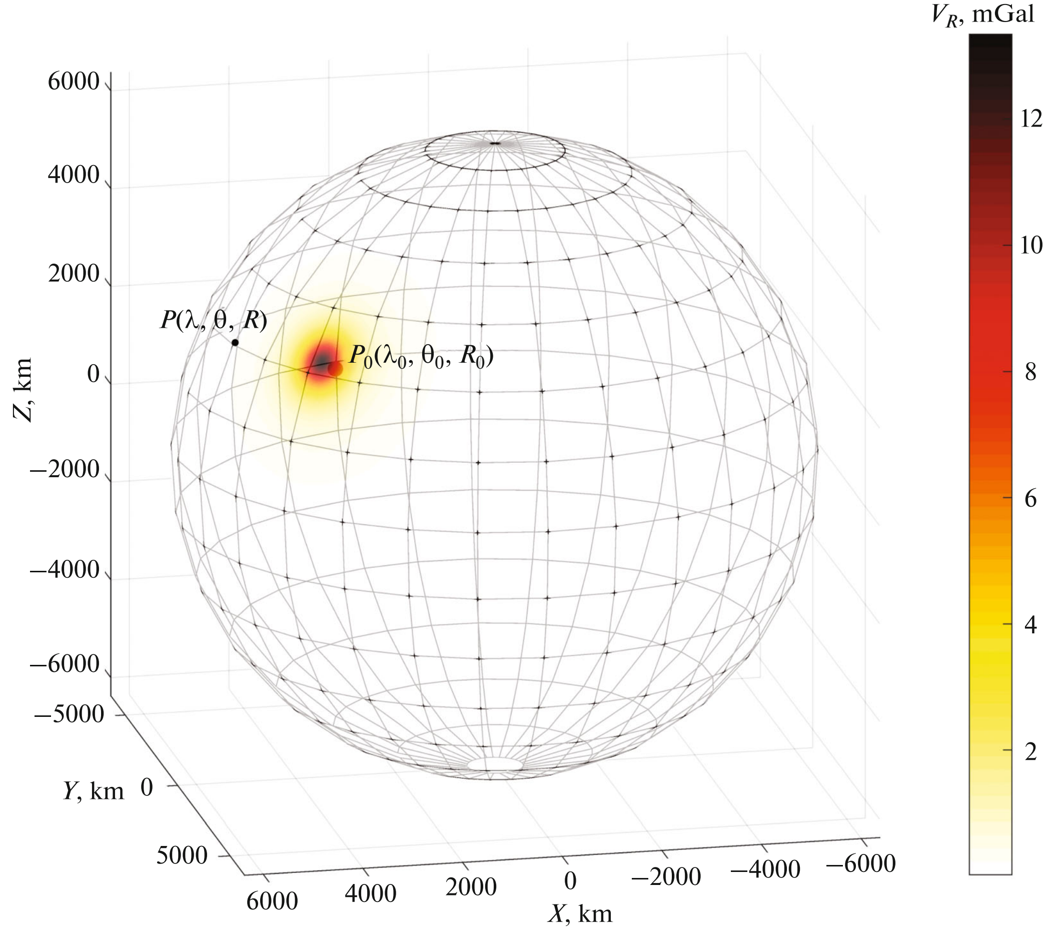

Fig. 1. The gravitational field of a point source on a sphere.

Let the center of mass of the gravity field source (spherical mass) be inside a sphere with radius R at some point with coordinates P_0 (R_0, θ_0, λ_0) and let its gravity field be known at points on the surface of the sphere P (R, θ, λ) (Fig. 1). Let us determine the mass and the position of the center of mass of the source through the known values of the gravity field, that is, the gravity potential and the vertical (radial) component of the attractive force.

In a spherical coordinate system, the gravity potential of a spherical mass M located at the point P_0 (R_0, θ_0, λ_0), at the arbitrary point P (R, θ, λ) is expressed as follows:

$$V(P) = \mathrm{G}\frac{M}{R_{PP_0}} = \mathrm{G}\frac{M}{\sqrt{R^2 + R_0^2 - 2RR_0\cos\psi}}, \quad (2)$$

$$\cos\psi = \sin\theta_0\sin\theta\cos(\lambda_0 - \lambda) + \cos\theta_0\cos\theta, \quad (3)$$

where G is the gravitation constant.

Differentiating the expression for the potential along the direction towards the center of the sphere, we obtain an expression for the vertical (radial) component of the attractive force of the spherical mass at point P:

$$V_R(P) = -V'(P) = \mathrm{G}M\frac{R - R_0\cos\psi}{(R^2 + R_0^2 - 2RR_0\cos\psi)^{3/2}}. \quad (4)$$

On a sphere of a given radius, the gravity potential and the attractive force reach an extremum at a point with a latitude and longitude equal to the latitude and longitude of a point mass, respectively. Therefore, at the extremum point, $\theta_{max} = \theta_0$ and $\lambda_{max} = \lambda_0$: $\cos\psi = 1$ and the expressions for the gravity potential and gravity at the extremum point are:

$$V_{max} = \mathrm{G}M\frac{1}{R - R_0}, \quad (5)$$

$$V_{R_{max}} = \mathrm{G}M\frac{1}{(R - R_0)^2}. \quad (6)$$

The distance from the origin to the point mass R_0 can be determined from the ratio of the corresponding components of the gravity action of the sphere (gravity potential, gravity) at the point P to the maximum value (n, k):

$$n = \frac{V(P)}{V_{\max}} = \frac{R - R_0}{\sqrt{R^2 + R_0^2 - 2RR_0 \cos\psi}}, \quad (7)$$

$$k = \frac{V_R(P)}{V_{R_{max}}} = \frac{(R - R_0 \cos\psi)(R - R_0)^2}{(R^2 + R_0^2 - 2RR_0 \cos\psi)^{3/2}}. \quad (8)$$

Expression (7) is reduced to the equation of the second degree with respect to R_0, which has two roots, that is, those greater and less than R, where only the smaller one satisfies the condition that the source be inside the sphere:

$$R_0^2 - 2RR_0 \frac{1 - n^2 \cos\psi}{1 - n^2} + R^2 = 0, \quad (9)$$

$$R_0 = \frac{k\sqrt{(1-\cos\psi)(2-k^2(1+\cos\psi))} + k^2\cos\psi - 1}{k^2 - 1} R. \quad (10)$$

Equation (8) reduces to a sixth degree algebraic equation for R_0:

$$\begin{aligned} &k_6 R_0^6 + k_5 R_0^5 + k_4 R_0^4 + k_3 R_0^3 \\ &+ k_2 R_0^2 + k_1 R_0 + k_0 = 0, \end{aligned} \quad (11)$$

$$k_6 = k^2 - \cos 2\psi,$$

$$k_5 = R(4\cos^2\psi - 6k^2\cos\psi + 2\cos\psi),$$

$$k_4 = R^2(12k^2\cos^2\psi - 6\cos^2\psi + 3k^2 - 8\cos\psi - 1),$$

$$\begin{aligned} k_3 = R^3(&-8k^2\cos^3\psi + 4\cos^2\psi \\ &- 12k^2\cos\ \psi + 12\cos\ \psi + 4), \end{aligned}$$

$$k_2 = R^4(12k^2\cos^2\psi - \cos^2\psi + 3k^2 - 8\cos\psi - 6),$$

$$k_1 = R^5(-6k^2\cos\psi + 2\cos\psi + 4),$$

$$k_0 = R^6(k^2 - 1).$$

Equation (11) is solved by numerical methods (for example, by the method of tangents) with a given accuracy. Among the six possible roots of the equation, the only desired solution is the real value whose value is less than R.

The mass of the sphere is determined taking the found position of the center of the sphere into account:

$$M = \frac{V_{\max}(R - R_0)}{G}, \quad (12)$$

$$M = \frac{V_{R_{\max}}(R - R_0)^2}{G}. \quad (13)$$

THE ALGORITHMIZATION METHOD AND TEST CALCULATIONS

To calculate the parameters of a point source from the gravity field defined on a sphere, the following algorithm is implemented in Matlab. Input data are a grid file of an anomalous gravity potential or field in a geographic coordinate system, which cover a spherical surface partially or completely and the radius of the sphere where the field is defined. In the automatic mode, the field extremum is selected. Based on formulas (9)–(13), the depth of the source and its mass are calculated. An increase in the accuracy of the calculations is achieved by repeatedly solving equations (9) and (11) for all points where the field is defined (except for the extremum point). The results of the program are the depth and mass of the source. Optionally, the density or radius of the source can be calculated depending on the specified parameter.

The results of the depth estimates of a known point source using the proposed algorithm are presented: (a) for theoretical (exact) values of elements of the gravity field and (b) for theoretical values with "added" noise. The values of the gravity field on the plane were also calculated using a cylindrical projection of the field defined on a spherical surface; the depth of the point source was estimated using the well-known ITI method.

For the test calculations, three spherical surfaces were selected, which corresponded to a small celestial body (for example, the Moon) with a radius of 2000 km, an average body (for example, the Earth) with a radius of 6371 km and a large (for example, Neptune) with a radius of 25000 km. Accurate (theoretical) values of the gravity field of a point source located at depths of 30, 100, 250, and 500 km from the surface of the spheres (Fig. 1) were set on each sphere over the entire surface with a resolution of 0.5° × 0.5°. The source weight was 5×10^{14} kg (assuming a density of 0.12 g/cm^3, the radius is 9.98 km); the latitude and longitude of the source were 30° and 40°, respectively.

Table 1 shows the depths of a point source calculated in two ways: taking sphericity into account and on a plane. To solve the inverse problem on the plane, the spherical coordinates were transformed into the Miller cylindrical projection (Fig. 2) and the calculations were performed using equation (1).

Obviously, the depth of the source center of mass calculated without sphericity is overestimated. The error increases with the depth of the center of mass and with a decrease in the radius of the sphere where the anomaly is defined. When sphericity is taken into account, the results are consistent with theoretical values, provided that errors in measuring the gravity field are excluded.

To assess the sensitivity of the described method to interference, white noise with levels of 5, 10, and 15% of the maximum field value was added to the gravity field of the source located at a depth of 500 km for all

Table 1. A comparison of the results of determining the depth of a point source with and without sphericity.

Sphere radius, km	True depth, km	Depth on inverse problem		Relative divergence,%
		on a sphere, km	on a plane, km	
2000	30	30	35	17
	100	100	119	19
	250	250	328	31
	500	500	713	43
6371	30	30	34	13
	100	100	117	17
	250	250	297	19
	500	500	637	27
25000	30	30	33	10
	100	100	115	15
	250	250	292	17
	500	500	595	18

Table 2. A comparison of the results of determining the depth of a point source with and without sphericity

Sphere radius	Source depth, km	Depth at noise level			Relative error at noise level		
		5%	10%	15%	5%	10%	15%
2000	30	31.6	27.4	34.0	5.4	8.7	13.2
	100	94.3	90.0	112.1	5.7	10.0	12.1
	250	236.5	227.1	220.7	5.4	9.2	11.7
	500	469.7	445.4	570.2	6.1	10.9	14.0
6371	30	28.2	32.9	33.1	6.1	9.7	10.4
	100	93.7	89.7	113.7	6.3	10.3	13.7
	250	234.5	276.9	217.6	6.2	10.7	13.0
	500	463.5	565.7	574.0	7.3	13.1	14.8
25000	30	31.9	27.1	33.2	6.2	9.6	10.8
	100	106.1	111.1	87.7	6.1	11.1	12.3
	250	233.1	222.5	215.2	6.7	11.0	13.9
	500	453.3	415.7	589.5	9.3	16.9	17.7

spheres. The source parameters were determined by the values of the radial component of the attractive force, which differ from the maximum value by a factor of less than 2. Table 2 presents the results of estimated depth of a point source obtained from "noisy" data.

From these data it follows that growth of the noise level causes an increase in the error of the calculated

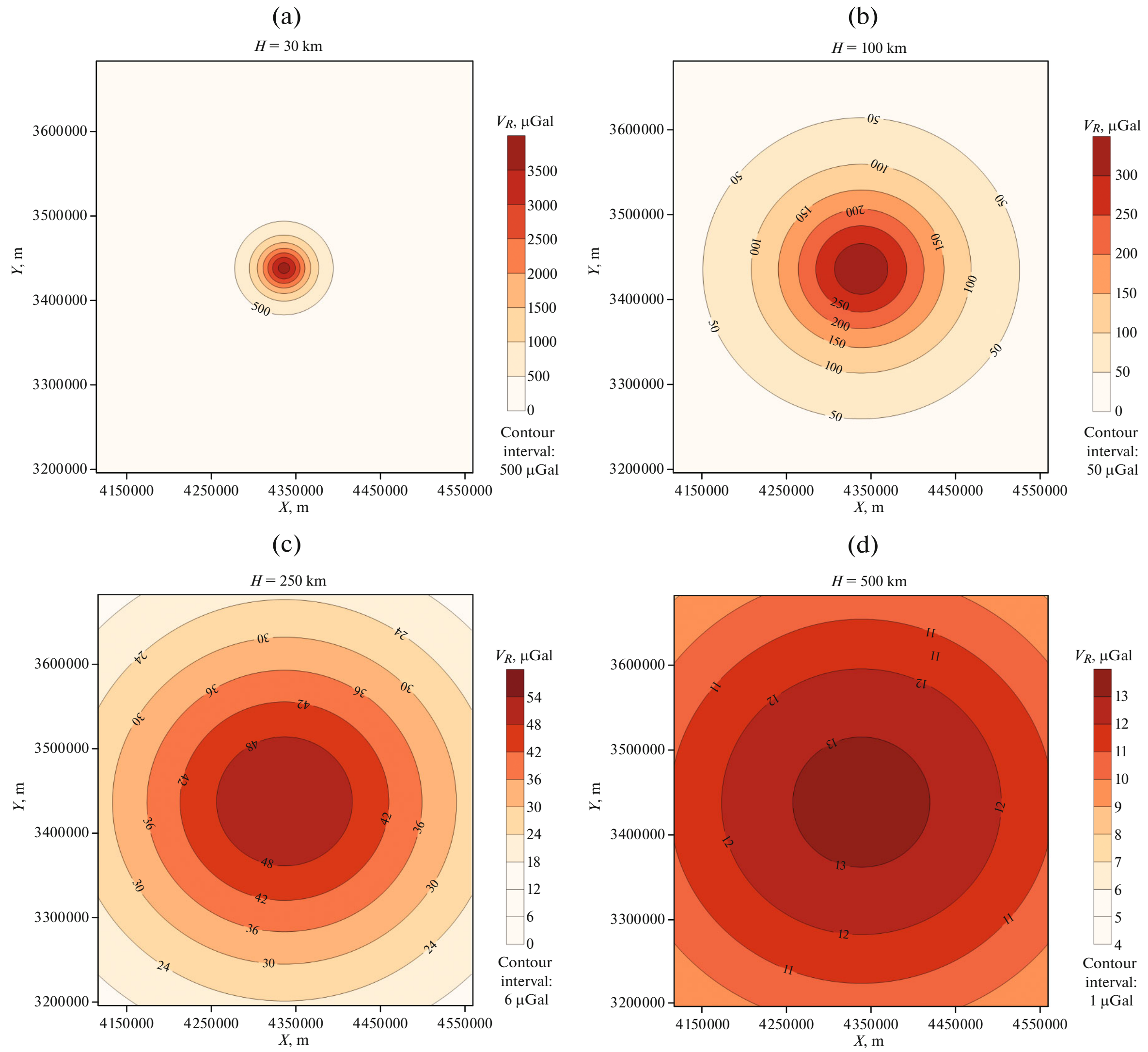

Fig. 2. The attractive force of a point mass located at a depth: (a), 30 km, (b), 100 km, (c), 250 km, (d) 500 km on the surface of a sphere with a radius of 6371 km.

parameters to a minor extent; the contribution of the noise component increases slightly with an increase in the radius of the sphere. The relative error of the depth calculated for the sphere with a radius of 6371 km with a noise level of 15% does not exceed 15%. In terms of gravity prospecting, such a result is quite good: the accuracy of determining the depths on a plane using ITI is usually estimated at 10–20%. Thus, we can conclude that this method, despite the absence of additional procedures for filtering the original data, is quite resistant to noise.

CONCLUSIONS

An equation for determining the position of a point source inside a sphere from the distribution of the anomalous gravity potential or attractive force on the sphere surface was proposed. According to the comparison of the parameters of anomalous bodies determined with and without sphericity it was shown that ignoring sphericity can lead to significant errors. The robustness of the method to interference increases with the radius of the sphere. For Earth conditions, 15% interference leads to an error in determining the source depth within 15%.

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