ISSN 0027-1349, Moscow University Physics Bulletin, 2020, Vol. 75, No. 5, pp. 409–414. © Allerton Press, Inc., 2020.
Russian Text © The Author(s), 2020, published in Vestnik Moskovskogo Universiteta, Seriya 3: Fizika, Astronomiya, 2020, No. 5, pp. 30–34.

THEORETICAL AND MATHEMATICAL PHYSICS

The Existence of a Boundary-Layer Stationary Solution to a Reaction–Diffusion Equation with Singularly Perturbed Neumann Boundary Condition

N. N. Nefedov[1*] and N. N. Deryugina[1]**

[1] *Department of Physics, Moscow State University, Moscow, 119991 Russia*

Received June 22, 2020; revised July 7, 2020; accepted July 14, 2020

Abstract—This paper considers an initial-boundary value problem for a reaction–diffusion equation with a singularly perturbed Neumann boundary condition in a closed, simply connected two-dimensional domain. From a physical point of view, the problem describes processes with an intensive flow through the boundary of a given area. The existence of a stationary solution is proved, its asymptotic is constructed, and the Lyapunov stability conditions for it are established. The asymptotics of the solution are constructed by the classical Vasilieva algorithm using the Lusternik–Vishik method. The existence and stability of the solution are proved using the asymptotic method of differential inequalities.

Keywords: singularly perturbed problems,reaction-diffusion, boundary layer, asymptotic methods, differential inequalities.

DOI: 10.3103/S0027134920050185

INTRODUCTION

Boundary value problems for parabolic and elliptic equations describe various processes in physics, chemistry, biophysics, and in other applied problems. The monograph by Pao [1] considers various types of elliptic problems and describes their possible applications. An extensive monograph [2] also focuses on the physical application of the corresponding mathematical models. A number of interesting physics applications can be found in [3, 4]. In applications these equations are widely used to describe excitable spatio-temporal media and are called reaction-diffusion-advection equations. They are used both to study autowaves (similar to the well-known Belousov–Zhabotinsky reaction) and to describe stationary structures that determine the dynamics of the process. Parabolic equations of this type act as mathematical models in nonlinear wave theory and are called Burgers-type equations (see [5, 6] and references therein). Reaction-diffusion equations can describe processes with intense sources; in this case, a small parameter appears at the differential operator. A distinctive feature of solving such problems is the transition layers [7, 8]. Solutions to a number of problems have a boundary layer, that is, a sharp transition near the boundary of the considered area.

This paper considers a problem with a singular second-type boundary condition. The work develops and generalizes the results of [9, 10] for a new type of problems, which is of interest from both theoretical and practical points of view.

1. PROBLEM SETUP

We consider a reaction–diffusion equation with a singularly perturbed Neumann boundary condition in a closed, simply connected two-dimensional domain D, bounded by a sufficiently smooth boundary ∂D. Below, we assume that the boundary ∂D is specified parametrically: $x = \varphi(\theta)$, $y = \psi(\theta)$, where $0 \leq \theta \leq \Theta$ is a parameter, and as it increases from 0 to Θ, point $(\varphi(\theta), \psi(\theta))$ passes through each point of the boundary ∂D:

$$\begin{cases} \mathcal{N}u := \varepsilon^2 \Delta u - \dfrac{\partial u}{\partial t} - f(u, x, \varepsilon) = 0, \\ x = (x_1, x_2) \in D, \quad t > 0, \\ u(x, 0, \varepsilon) = u_{\text{init}}(x), \quad (x) \in \bar{D}, \\ \left.\dfrac{\partial u}{\partial n}\right|_{\partial D} = \dfrac{h(x,t)}{\varepsilon}, \quad x \in \partial D, \quad t > 0. \end{cases} \tag{1}$$

The derivative in the boundary condition is taken along the inward normal to ∂D. From a physical

*E-mail: nefedov@phys.msu.ru
**E-mail: derunat@gmail.com

point of view, the problem describes processes with an intense flux across the boundary of a given area. This work has the goal of investigating the stationary solution of problem (1). This solution is a solution to the elliptic boundary value problem

$$\begin{cases} \mathcal{L}u := \varepsilon^2 \Delta u - f(u, x, \varepsilon) = 0, \\ \left.\dfrac{\partial u}{\partial n}\right|_{\partial D} = \dfrac{h(x)}{\varepsilon}, \\ (x) \in D. \end{cases} \tag{2}$$

The right side of the boundary condition contains a small parameter $\varepsilon > 0$, which corresponds to the presence of intense sources at the boundary ∂D. The presence of a small parameter makes the Neumann boundary condition singular and results in a more complex boundary layer in the solution.

The purpose of this work was to investigate the existence of a solution with a boundary layer for problem (2) and its Lyapunov stability.

We require the fulfillment of the following conditions:

(A0) $f(u, x, \varepsilon)$ everywhere in $\bar{D}$, $h(x)$ on the boundary ∂D and $\varphi(\theta)$, $\psi(\theta)$ at $0 \le \theta \le \Theta$ are sufficiently smooth functions.

(A1) Degenerate equation $f(u, x, 0) = 0$ has a root $u = u_0(x)$: $f_u\big(u_0(x), x, 0\big) > 0$, $x \in \bar{D}$.

2. CONSTRUCTING THE ASYMPTOTICS

2.1. The Local Coordinates

To describe the solution near ∂D let us introduce the local coordinates (r, θ) in its sufficiently small neighborhood, where r is the distance from a given point inside this neighborhood to a point on the boundary ∂D with the coordinates $(\varphi(\theta), \psi(\theta))$ along the normal to ∂D. It is known that if the boundary is sufficiently smooth (the functions $\varphi(\theta)$ and $\psi(\theta)$ have Hoelder continuous derivatives), then in a sufficiently small neighborhood of the boundary there is a bijection between the initial coordinates (x_1, x_2) and local coordinates (r, θ), given by the formulas:

$$\begin{cases} x_1 = \varphi(\theta) - r\dfrac{\psi_\theta}{\sqrt{\varphi_\theta^2 + \psi_\theta^2}}, \\ x_2 = \psi(\theta) + r\dfrac{\varphi_\theta}{\sqrt{\varphi_\theta^2 + \psi_\theta^2}}. \end{cases}$$

The unit tangent vector $\mathbf{k}$ and the unit normal vector $\mathbf{n}$ to ∂D are given by the formulas

$$\mathbf{k} = \begin{pmatrix} \dfrac{1}{\sqrt{1 + \psi_\theta^2/\varphi_\theta^2}} \\ \dfrac{\psi_\theta/\varphi_\theta}{\sqrt{1 + \psi_\theta^2/\varphi_\theta^2}} \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} \dfrac{-\psi_\theta}{\sqrt{\varphi_\theta^2 + \psi_\theta^2}} \\ \dfrac{\varphi_\theta}{\sqrt{\varphi_\theta^2 + \psi_\theta^2}} \end{pmatrix}.$$

Passing to the new variables, we obtain the following expression for the problem operator:

$$\mathcal{L} = \varepsilon^2 \left(\frac{\partial^2}{\partial r^2} + H_\theta \frac{\partial H_\theta}{\partial r} \frac{\partial}{\partial r} + \frac{1}{H_\theta} \frac{\partial}{\partial \theta} \Big(\frac{1}{H_\theta} \Big) \frac{\partial}{\partial \theta} + \frac{1}{H_\theta^2} \frac{\partial^2}{\partial \theta^2} \right),$$

where H_θ are H_r are the Lame parameters:

$$H_r = \sqrt{\Big(\frac{\partial x_1}{\partial r}\Big)^2 + \Big(\frac{\partial x_2}{\partial r}\Big)^2} = 1,$$

$$H_\theta = \sqrt{\Big(\frac{\partial x_1}{\partial \theta}\Big)^2 + \Big(\frac{\partial x_2}{\partial \theta}\Big)^2}.$$

Let us introduce the stretched variable $\xi = \dfrac{r}{\varepsilon}$. Then, expanding the coefficients of the partial derivatives in series in powers of ε, we obtain for the problem operator:

$$\mathcal{L} = \frac{\partial^2}{\partial \xi^2} + \varepsilon \frac{\partial}{\partial \xi} \frac{\varphi_{\theta\theta}\psi_\theta - \psi_{\theta\theta}\varphi_\theta}{\sqrt{\psi_\theta^2 + \varphi_\theta^2}} + \sum_{i=2}^{\infty} \varepsilon^i \mathcal{L}_i,$$

where $\mathcal{L}_i$ are linear differential operators containing partial derivatives $\dfrac{\partial}{\partial \theta}$ and $\dfrac{\partial^2}{\partial \theta^2}$. Since the local coordinate r is introduced as a distance along the inward normal to ∂D, the boundary condition operator in local and stretched variables takes the form

$$\varepsilon \frac{\partial}{\partial n} = \varepsilon \frac{\partial}{\partial r} = \frac{\partial}{\partial \xi}.$$

2.2. The General Form of the Asymptotics

Let us construct the asymptotics of the solution using the standard algorithm of the boundary function method (see [11, 12]), according to which the sought function is represented as the sum

$$u(x, \varepsilon) = \bar{u}(x, \varepsilon) + \Pi(\xi, \theta, \varepsilon). \tag{3}$$

Nonlinearity is presented in a similar form:

$$f(u, x, \varepsilon) = \bar{F} + \Pi f(u, \xi, \theta, \varepsilon) = f(\bar{u}(x, \varepsilon), x, \varepsilon) + f\left(\bar{u}\left(\varphi(\theta) - \varepsilon\xi \frac{\psi_\theta}{\sqrt{\varphi_\theta^2 + \psi_\theta^2}}, \psi(\theta) + \varepsilon\xi \frac{\varphi_\theta}{\sqrt{\varphi_\theta^2 + \psi_\theta^2}}, \varepsilon\right) + \Pi(\xi, \theta, \varepsilon), \varphi(\theta) - \varepsilon\xi \frac{\psi_\theta}{\sqrt{\varphi_\theta^2 + \psi_\theta^2}}, \psi(\theta) + \varepsilon\xi \frac{\varphi_\theta}{\sqrt{\varphi_\theta^2 + \psi_\theta^2}}, \varepsilon\right)$$

$$- f\left(\bar{u}\left(\varphi(\theta) - \varepsilon\xi\frac{\psi_\theta}{\sqrt{\varphi_\theta^2+\psi_\theta^2}}, \psi(\theta) + \varepsilon\xi\frac{\varphi_\theta}{\sqrt{\varphi_\theta^2+\psi_\theta^2}}, \varepsilon\right), \varphi(\theta) - \varepsilon\xi\frac{\psi_\theta}{\sqrt{\varphi_\theta^2+\psi_\theta^2}}, \psi(\theta) + \varepsilon\xi\frac{\varphi_\theta}{\sqrt{\varphi_\theta^2+\psi_\theta^2}}, \varepsilon\right).$$

In these representations $\bar{u}(x,\varepsilon)$ is the regular part of the asymptotics that describes the function u far from the boundary ∂D and $\Pi(\xi,\theta,\varepsilon)$ is the boundary layer part of the asymptotics that describes the solution near the boundary ∂D. Both terms are represented as series in powers of ε:

$$\bar{u}(x,\varepsilon) = \bar{u}_0(x) + \varepsilon\bar{u}_1(x) + \ldots$$
$$\Pi(\xi,\theta,\varepsilon) = \Pi_0(\xi,\theta) + \varepsilon\Pi_1(\xi,\theta) + \ldots.$$

Let us expand $\varepsilon\bar{F}$ and Πf into series in powers of ε using the notation:

$$\begin{cases} \bar{f}(x) = f\big(\bar{u}_0(x), x, 0\big), \\ \tilde{f}(\xi,\theta) = f\big(\bar{u}_0(\varphi(\theta),\psi(\theta)) \\ +\Pi_0(\xi,\theta), \varphi(\theta), \psi(\theta), 0\big) \\ = f\big(\bar{u}_0(0,\theta) + \Pi_0(\xi,\theta), 0, \theta, 0\big), \end{cases}$$

and let us use similar designations for the derivatives of these functions (either old or new coordinates can be used to represent the boundary layer part of the nonlinearity, whichever is more convenient for the specifics of the problem).

2.3. Regular Part

Substituting (3) into the original problem (2) and dividing this problem in a standard way into problems for the regular and boundary layer parts, we obtain a sequence of problems for determining the coefficients of the regular and boundary layer parts of the asymptotic approximation. For $\bar{u}_0(x)$ we obtain: $f(\bar{u}_0(x), x, 0) = 0$. Taking condition (A1) into account $\bar{u}_0(x) = u_0(x)$. The first order of the regular part function is determined from the equation: $\bar{f}_u(x)\bar{u}_1 = -\bar{f}_\varepsilon(x)$. The arbitrary order of the regular part function is determined from equations of the form: $\bar{f}_u(x)\bar{u}_k = F_k(x)$, where $F_k(x)$ is a function known at each step that depends on the coefficients of the regular part of the asymptotic approximation of previous orders.

2.4. Boundary Layer Part

The problem for the zero order of the boundary layer part of the solution has the form:

$$\begin{cases} \dfrac{\partial^2\Pi_0}{\partial\xi^2} \\ = f\Big(\bar{u}_0\big(\varphi(\theta),\psi(\theta)\big) + \Pi_0, \varphi(\theta), \psi(\theta), 0\Big), \\ \dfrac{\partial\Pi_0}{\partial\xi}\Big|_{\xi=0} = h(0,\theta), \\ \Pi_0(\infty,\theta) = 0. \end{cases} \tag{4}$$

Problem (4) is an ordinary differential equation of the second order (θ is a parameter), which can be analyzed on the phase plane (Π_0, Π_0') (Fig. 1).

Point $(0,0)$ in the phase plane is a saddle point of rest. Problem (4) has a solution if the line $\Pi_0' = h(0,\theta)$ intersects the separatrix directed to the point $(0,0)$. An analysis of the phase plane shows that several such solutions are possible. Below, we show that if at $\xi = 0$ the value of (Π_0, Π_0'), $0 \le \theta \le \Theta$ lands in points A, B, C, D, then, moving from them along the separatrix to the point $(0,0)$, we can obtain a solution to problem (4). Let us formulate the conditions for selecting solutions:

(A2) Equation

$$\pm\left(2\int_0^s f(\bar{u}_0(\varphi(\theta),\psi(\theta)) + \sigma, \varphi(\theta), \psi(\theta), 0))d\sigma\right)^{1/2} = h(0,\theta)$$

has a root $s = s(\theta)$ for each fixed θ such that $f(\bar{u}_0(\varphi(\theta), \psi(\theta)) + s(\theta), \varphi(\theta), \psi(\theta), 0)) > 0$ at $s(\theta) > 0$ and $f(\bar{u}_0(\varphi(\theta),\psi(\theta)) + s(\theta), \varphi(\theta), \psi(\theta), 0)) < 0$ at $s(\theta) < 0$.

We note that the points indicated by squares in Fig. 1 for $\Pi_0(0,\theta) > 0$, correspond to solutions with a nonmonotonic boundary layer, since as the derivative Π_0 moves from these points along the separatrix to the saddle point $(0,0)$, it goes over zero and changes sign. The study of such types of solutions, as well as solutions that correspond to the points indicated in Fig. 1 by open circles, were not carried out in this work.

The problem for the first order of the boundary layer part has the form:

$$\begin{cases} \dfrac{\partial^2\Pi_1}{\partial\xi^2} - \tilde{f}_u(\xi,\theta)\Pi_1 = G_1(\xi,\theta), \\ \dfrac{\partial\Pi_1}{\partial\xi}\Big|_{\xi=0} = -\dfrac{\partial\bar{u}_0(0,\theta)}{\partial r}, \\ \Pi_1(\infty,\theta) = 0, \end{cases} \tag{5}$$

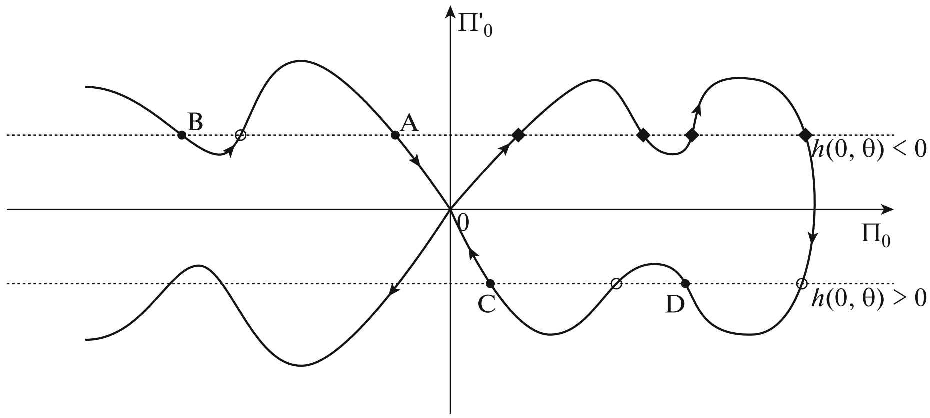

Fig. 1. An example of the phase plane of the problem (4) for a fixed θ.

where

$$G_1(\xi,\theta) = \left(\tilde{f}_u(\xi,\theta) - \bar{f}_u(x)\right)$$
$$\times\left(\bar{u}_1(\varphi(\theta),\psi(\theta)) - \xi\frac{\psi_\theta}{\sqrt{\varphi_\theta^2+\varphi_\theta^2}}\frac{\partial \bar{u}_0}{\partial x_1}\Big|_{(\varphi(\theta),\psi(\theta))}\right.$$
$$\left. + \xi\frac{\varphi_\theta}{\sqrt{\varphi_\theta^2+\varphi_\theta^2}}\frac{\partial \bar{u}_0}{\partial x_2}\Big|_{(\varphi(\theta),\psi(\theta))}\right)$$
$$- \tilde{f}_{x_1}(\xi,\theta)\xi\frac{\psi_\theta}{\sqrt{\psi_\theta^2+\varphi_\theta^2}}$$
$$+ \tilde{f}_{x_2}(\xi,\theta)\xi\frac{\varphi_\theta}{\sqrt{\psi_\theta^2+\varphi_\theta^2}} + \tilde{f}_\varepsilon(\xi,\theta)$$
$$+ \bar{f}_{x_1}(x)\xi\frac{\psi_\theta}{\sqrt{\psi_\theta^2+\varphi_\theta^2}} - \bar{f}(x)_{x_2}\xi\frac{\varphi_\theta}{\sqrt{\psi_\theta^2+\varphi_\theta^2}}$$
$$- \bar{f}(x)_\varepsilon - \frac{\partial \Pi_0}{\partial \xi}\frac{\varphi_{\theta\theta}\psi_\theta - \psi_{\theta\theta}\varphi_\theta}{\sqrt{\psi_\theta^2+\varphi_\theta^2}}.$$

The solution to this problem can be obtained explicitly:

$$\Pi_1(\xi,\theta) = \frac{z(\xi,\theta)}{\frac{\partial z}{\partial \xi}(0,\theta)}\left(-\frac{\partial \bar{u}_0(0,\theta)}{\partial r}\right.$$
$$\left. + \frac{1}{z(0,\theta)}\int\limits_0^\infty z(\chi,\theta)G_1(\chi,\theta)\partial\chi\right)$$
$$- z(\xi,\theta)\int\limits_0^\infty \frac{1}{z^2(\eta,\theta)}\int\limits_\eta^\infty z(\chi,\theta)G_1(\chi,\theta)\partial\chi\partial\eta,$$

where $z(\xi,\theta) = \dfrac{\partial \Pi_0}{\partial \xi}$.

Using the proposed scheme, we can construct an asymptotics of an arbitrary order (k):

$$U_k = \sum_{i=0}^{k}\varepsilon^i(\bar{u}_i(x) + \Pi_i(\xi)).$$

3. JUSTIFICATION OF THE ASYPMTOTICS

3.1. The Existence of a Solution

To prove the existence of a solution, let us use the scheme of the asymptotic method of differential inequalities (see, for example, [7, 18] and references therein). For this, we construct the upper and lower solutions of the problem (2) in the domain $\bar{D}$, $\beta(x,\varepsilon)$, and $\alpha(x,\varepsilon)$. For convenience, let us recall the definition of these functions.

(B1): $\alpha(x,\varepsilon) \le \beta(x,\varepsilon)$ for $x \in \bar{D}$.

(B2): $\mathcal{L}(\beta) \le 0 \le \mathcal{L}(\alpha)$ for $x \in D$.

(B3): $\varepsilon\dfrac{\partial\beta}{\partial n}\Big|_{\partial D} \le h(x) \le \varepsilon\dfrac{\partial\alpha}{\partial n}\Big|_{\partial D}$.

Let us build the upper and lower solutions as a modification of the $(n+1)$th order of the constructed asymptotics:

$$\begin{cases} \alpha(x,\varepsilon) = U_n + \varepsilon^{n+1}\Big(\bar{u}_{n+1}(x) - \gamma \\ +\Pi_{n+1}(\xi,\theta) + \Pi_\alpha(\xi,\theta)\Big), \\ \beta(x,\varepsilon) = U_n + \varepsilon^{n+1}\Big(\bar{u}_{n+1}(x) + \gamma \\ +\Pi_{n+1}(\xi,\theta) + \Pi_\beta(\xi,\theta)\Big). \end{cases} \quad (6)$$

Here $\gamma > 0$. Functions $\Pi_\alpha(\xi,\theta)$ and $\Pi_\beta(\xi,\theta)$ are determined from problems similar to (5):

$$\begin{cases} \dfrac{\partial^2\Pi_\alpha}{\partial\xi^2} - \tilde{f}_u(\xi,\theta)\Pi_\alpha = G_\alpha(\xi,\theta), \\ \dfrac{\partial\Pi_\alpha}{\partial\xi}\Big|_{\xi=0} = -\delta, \\ \Pi_\alpha(\infty,\theta) = 0, \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 \Pi_\beta}{\partial \xi^2} - \tilde{f}_u(\xi,\theta)\Pi_\beta = G_\beta(\xi,\theta), \\ \left.\dfrac{\partial \Pi_\beta}{\partial \xi}\right|_{\xi=0} = \delta, \\ \Pi_\beta(\infty,\theta) = 0, \end{cases}$$

where $G_\alpha(\xi,\theta) = -\gamma\big(\tilde{f}(\xi,\theta)_u - \bar{f}(x_1,x_2)_u\big) + Ae^{-\kappa\xi}$, $G_\beta(\xi,\theta) = -\gamma\big(\tilde{f}(\xi,\theta)_u - \bar{f}(x_1,x_2)_u\big) - Ae^{-\kappa\xi}$. It can be shown that for the function $\Pi_0(\xi,\theta)$ the exponential estimate $|\Pi_0(\xi,\theta)| \le Ce^{-\kappa\xi}$ is satisfied, where $C > 0$ and $\kappa > 0$. This implies the exponential estimate of the expression $\big(\tilde{f}(\xi,\theta)_u - \bar{f}(x)_u\big)$ and, therefore, it is possible to choose such numbers A and κ, that G_α is a positive and G_β is a negative exponentially decreasing function.

Functions $\Pi_\alpha(\xi,\theta)$ and $\Pi_\beta(\xi,\theta)$ are defined explicitly using formulas similar to the formula for $\Pi_1(\xi,\theta)$:

$$\Pi_\alpha(\xi,\theta) = \frac{z(\xi,\theta)}{\dfrac{\partial z}{\partial \xi}(0,\theta)}\delta + \frac{z(\xi,\theta)}{\dfrac{\partial z}{\partial \xi}(0,\theta)}\frac{1}{z(0,\theta)}\int\limits_0^\infty z(\chi,\theta)G_\alpha(\chi,\,\theta)d\chi - z(\xi,\theta)\int\limits_0^\xi \frac{1}{z^2(\eta,\theta)}\int\limits_\eta^\infty z(\chi,\theta)G_\alpha(\chi,\theta)d\chi d\eta,$$

$$\Pi_\beta(\xi,\theta) = \frac{z(\xi,\theta)}{\dfrac{\partial z}{\partial \xi}(0,\theta)}(-\delta) + \frac{z(\xi,\theta)}{\dfrac{\partial z}{\partial \xi}(0,\theta)}\frac{1}{z(0,\theta)}\int\limits_0^\infty z(\chi,\theta)G_\beta(\chi,\theta)d\chi - z(\xi,\theta)\int\limits_0^\xi \frac{1}{z^2(\eta,\theta)}\int\limits_\eta^\infty z(\chi,\theta)G_\beta(\chi,\theta)d\chi d\eta.$$

It is established in a standard way that exponential estimates hold for $\Pi_\alpha(\xi,\theta)$ and $\Pi_\beta(\xi,\theta)$. It is also easy to show that due to condition (A2) $\Pi_\alpha(\xi,\theta) < 0$, $\Pi_\beta(\xi,\theta) > 0$, which ensures the order condition of the upper and lower solutions of (B1).

Let us now check the fulfillment of condition (B2). We substitute the expressions for the upper and lower solutions of (6) into the original operator of the problem (2). We get:

$$\begin{cases} \mathcal{L}\alpha(x,\varepsilon) = \varepsilon^{n+1}\big(G_\alpha + \bar{f}_u(x)\gamma\big) + O(\varepsilon^{n+2}), \\ \mathcal{L}\beta(x,\varepsilon) = \varepsilon^{n+1}\big(G_\beta - \bar{f}_u(x)\gamma\big) + O(\varepsilon^{n+2}). \end{cases}$$

Due to condition (A1), the positivity of G_α and negativity of G_β condition (B2) is fulfilled.

To check the fulfillment of condition (B3), let us substitute (6) into the boundary operator of the problem (2)

$$\begin{aligned} \varepsilon\left.\frac{\partial \beta}{\partial n}\right|_{\partial D} &= h(x) - \varepsilon^n \left.\frac{\partial \Pi_\beta}{\partial \xi}\right|_{\xi=0} + O(\varepsilon^{n+1}) \\ &= h(x) - \varepsilon^n\delta + O(\varepsilon^{n+1}) < h(x), \\ \varepsilon\left.\frac{\partial \alpha}{\partial n}\right|_{\partial D} &= h(x) - \varepsilon^n \left.\frac{\partial \Pi_\alpha}{\partial \xi}\right|_{\xi=0} + O(\varepsilon^{n+1}) \\ &= h(x_1,x_2) + \varepsilon^n\delta + O(\varepsilon^{n+1}) > h(x). \end{aligned}$$

Therefore, for functions $\alpha(x,\varepsilon)$ and $\beta(x,\varepsilon)$ all of the conditions (B1)–(B3) are fulfilled. It follows from the results of works on comparison theorems (see, for example, [1]) that there exists a solution to problem (2) for which the following inequality holds: $\alpha(x,\varepsilon) \le u(x,\varepsilon) \le \beta(x_1,x_2,\varepsilon)$ at $x \in \bar{D}$. Thus, the following theorem has been proven.

Theorem 1. *Suppose that conditions* (A0)–(A2) *are fulfilled. Then for sufficiently small ε for the function $\Pi_0(\xi,\theta)$ chosen according to condition* (A2) *there exists a corresponding solution $u(x_1,x_2,\varepsilon)$ of problem* (2) *with a boundary layer near ∂D, for which the function $U_n(x,\varepsilon)$ is a uniform asymptotic approximation with accuracy ε^{n+1} at $x \in \bar{D}$.*

4. ASYMPTOTIC STABILITY OF THE SOLUTION

The method of differential inequalities also makes it possible to prove the Lyapunov asymptotic stability of the solution (3) as a stationary solution of the parabolic problem (1). For this, nonstationary upper and lower solutions of problem (1) are constructed in the form $\bar{\alpha}(x,t,\varepsilon) = u(x,\varepsilon) + e^{-\lambda(\varepsilon)t}\big(\alpha(x,\varepsilon) - u(x,\varepsilon)\big)$, $\bar{\beta}(x,t,\varepsilon) = u(x,\varepsilon) + e^{-\lambda(\varepsilon)t}\big(\beta(x,\varepsilon) - u(x,\varepsilon)\big)$, where $\lambda(\varepsilon) > 0$ is sufficiently small. It is clear that $\bar{\alpha} < \bar{\beta}$. It can be shown that $\mathcal{N}_\varepsilon\beta < 0$ and $\mathcal{N}_\varepsilon\alpha > 0$. The proof of similar inequalities is presented, for example, in [10, 17]. In this case a modified first order asymptotics, that is, $\alpha_1(x,\varepsilon)$ and $\beta_1(x,\varepsilon)$, can be taken as $\alpha(x,\varepsilon)$ and $\beta(x,\varepsilon)$. Thus, for the solutions described by Theorem 1, asymptotic Lyapunov stability occurs with an attraction domain of at least $[\alpha_1(x,\varepsilon), \beta_1(x,\varepsilon)]$.

Theorem 2. *Suppose that conditions* (A0)–(A2) *are fulfilled. Then, every solution* $u(x,\varepsilon)$ *of problem* (2), *the existence of which is supported by Theorem* 1, *is asymptotically Lyapunov stable with a stability domain of at least* $[\alpha_1(x,\varepsilon), \beta_1(x,\varepsilon)]$; *therefore,* $u(x,\varepsilon)$ *is the only solution to problem* (2) *in this area.*

CONCLUSIONS

This paper considers a new type of problem with a singularly perturbed boundary condition of the second kind. A formal asymptotics of an arbitrary order of accuracy is constructed, and conditions for the operator of the problem are obtained, ensuring the existence of a stable solution with a boundary layer. The existence and stability of the solution was proven using the scheme of the asymptotic method of differential solutions. The paper also highlights the conditions that allow the presence of a nonmonotonic boundary layer, which determines a possible direction for further research in this area.

FUNDING

This work was supported by the Russian Foundation for Basic Research (project no. 19-01-00327).

REFERENCES

1. C. Pao, *Nonlinear Parabolic and Elliptic Equations* (Springer, New York, 1993).
2. V. Volpert, *Elliptic Partial Differential Equations* (Birkhäuser, Boston, 2011).
3. K. V. Zhukovsky, Mosc. Univ. Phys. Bull. **73**, 45 (2018).
4. K. V. Zhukovsky, Mosc. Univ. Phys. Bull. **71**, 237 (2016).
5. O. V. Rudenko, Dokl. Math. **94**, 708 (2016).
6. N. N. Nefedov and O. V. Rudenko, Dokl. Math. **97**, 99 (2018).
7. N. Nefedov, Diff. Equat. **36**, 298 (2000).
8. N. Nefedov, N. Levashova, and A. Orlov, Mosc. Univ. Phys. Bull. **73**, 565 (2018).
9. V. Butuzov, N. Nefedov, L. Recke, and K. Schneider, Int. J. Bifurc. Chaos **24** (8) (2014).
10. N. Nefedov and E. I. Nikulin, Mosc. Univ. Phys. Bull. **75**, 116 (2020).
11. A. B. Vasil'eva and V. F. Butuzov, *Asymptotic Expansion of Solutions to Singularly Perturbed Equations* (Nauka, Moscow, 1973) [in Russian].
12. A. B. Vasil'eva and V. F. Butuzov, *Asymptotic Methods in the Theory of Singular Perturbations* (Vysshaya Shkola, Moscow, 1990) [in Russian].
13. A. B. Vasil'eva, V. F. Butuzov, and N. N. Nefedov, Tr. Mat. Inst. im. V.A. Steklova RAN **268**, 268 (2010).
14. N. Nefedov, L. Recke, and K. Schneider, J. Math. Anal. Appl. **405**, 90 (2013).
15. N. Nefedov and K. Sakamoto, Hiroshima Math. J. **33**, 391 (2003).
16. A. B. Vasil'eva, V. F. Butuzov, and N. N. Nefedov, Fundam. Prikl. Mat. **4**, 799 (1998).
17. N. T. Levashova, N. N. Nefedov, and A. O. Orlov, Comput. Math. Math. Phys. **59**, 573 (2019).
18. N. Nefedov, Lect. Notes Comput. Sci. **8236**, 62 (2013).
19. N. Nefedov and O. Rudenko, Dokl. Math. **97**, 99 (2018).

Translated by L. Trubitsyna