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Mathematical Modeling of Impedance Waveguide Systems

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Abstract—A method for constructing eigenmodes of an infinite waveguide of a constant rectangular cross section with low losses in the walls, which are described by the Shchukin–Leontovich boundary conditions, is discussed. The dispersion characteristics of these waveguides are constructed.

Keywords: waveguide systems, Shchukin–Leontovich conditions, incomplete Galerkin method.

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INTRODUCTION

The problem of calculating complex waveguide systems with impedance walls [1] described by the Shchukin–Leontovich boundary conditions is a natural generalization of the problem for the case of an ideally conducting boundary when the impedance of the walls is zero. With a zero impedance, a complete orthonormal basis is constructed in a transverse cross section of the waveguide by using the eigenfunctions of the section for the Laplace operator; in this basis, the transverse components of the electromagnetic field are expanded [2, 3]. The issue of the presence of hybrid modes arises during the transition to impedance walls; therefore, the electromagnetic field can no longer be divided into fields of electric and magnetic types and also the classical basis no longer satisfies the new boundary conditions. Thus, for a problem with losses, one has to use various approximative methods or proceed to a generalized formulation of the problem, in which the boundary conditions are also satisfied in a general sense [4].

In [5], a method to accurately account for losses in walls was proposed. This method consists in the construction of a special basis that allows one to satisfy boundary conditions exactly. The new basis is the classical basis upgraded by adding to it supplementary elements that provide the fulfillment of the boundary conditions, in which case the coefficients at these additional basis elements in the field expansion are solutions of algebraic equations. The coefficients at the standard basis functions solve a system of linear ordinary differential equations with a rigid matrix. Its calculation requires one to apply special methods, for example, the method of directed orthogonalization [6]. This allows one to increase the stability of the algorithm but does not solve the problem completely.

In this paper, we construct a system of basic functions with low impedance, which are a generalization of the classical basis for a regular waveguide. They satisfy Maxwell's equations accurately and obey boundary conditions with a sufficiently high accuracy and thereby permit us to avoid the occurrence of rigid matrix problems.

1. PROBLEM FORMULATION

An infinite waveguide with a constant rectangular cross section $S = \{x \in (-a, a), y \in (-b, b)\}$ is considered; $\mathbf{z}$ is the waveguide axis. The electromagnetic field inside the waveguide is described by the system of Maxwell's equations:

$$\text{curl}\, E = ikH, \tag{1}$$

$$\text{curl}\, H = -ikE. \tag{2}$$

System (1), (2) is combined with the Shchukin–Leontovich boundary conditions [4] on the side wall:

$$[n, E] = -W\,[n, [n, H]]\,, \tag{3}$$

where $\mathbf{n}$ is the outer normal to the boundary ∂S and W is the surface impedance. For a rectangular waveguide, these conditions take the form

$$E_x = -WH_z, \quad E_z = WH_x, \quad y = b, \tag{4}$$

$$E_x = WH_z, \quad E_z = -WH_x, \quad y = -b, \tag{5}$$

$$E_y = WH_z, \quad E_z = -WH_y, \quad x = a, \tag{6}$$

$$E_y = -WH_z, \quad E_z = WH_y, \quad x = -a. \tag{7}$$

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2. THE SYSTEM OF MAXWELL'S EQUATIONS

Problem (1)–(3) for the waveguide of constant rectangular cross section has symmetry about the axes $\mathbf{x}$ and $\mathbf{y}$. Each projection of the field can be both even and odd in $\mathbf{x}$ and in $\mathbf{y}$, which allows us to distinguish four types of solutions, namely,

$$\begin{cases} E_x = C^{(ex)}\cos(px)\sin(qy)\exp(i\gamma z), \\ E_y = C^{(ey)}\sin(px)\cos(qy)\exp(i\gamma z), \\ E_z = C^{(ez)}\sin(px)\sin(qy)\exp(i\gamma z), \\ H_x = C^{(hx)}\sin(px)\cos(qy)\exp(i\gamma z), \\ H_y = C^{(hy)}\cos(px)\sin(qy)\exp(i\gamma z), \\ H_z = C^{(hz)}\cos(px)\cos(qy)\exp(i\gamma z), \end{cases} \tag{8a}$$

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$$\begin{cases} E_x = C^{(ex)}\sin(px)\sin(qy)\exp(i\gamma z), \\ E_y = C^{(ey)}\cos(px)\cos(qy)\exp(i\gamma z), \\ E_z = C^{(ez)}\cos(px)\sin(qy)\exp(i\gamma z), \\ H_x = C^{(hx)}\cos(px)\cos(qy)\exp(i\gamma z), \\ H_y = C^{(hy)}\sin(px)\sin(qy)\exp(i\gamma z), \\ H_z = C^{(hz)}\sin(px)\cos(qy)\exp(i\gamma z), \end{cases} \tag{8c}$$

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where p, q, and γ are now complex numbers, unlike the lossless case.

Without loss of generality, we consider the solution of form (8a). Solutions with other parity types are constructed in the same way, and all further reasoning for them can be repeated without changes.

The set of solutions (8) contains the basis functions of the electric and magnetic types of an ideal waveguide, which will be convenient to study the limit transition $W \to 0$. Due to the symmetry with respect to rotation by π around the axis, the boundary conditions (5) and (7) are automatically satisfied if conditions (4) and (6) are fulfilled.

Without loss of generality, we look for waves that propagate in the positive direction of the axis z. This leads to the condition

$$\operatorname{Re}\gamma \geq 0, \quad \operatorname{Im}\gamma \geq 0. \tag{9}$$

Waves that propagate in the negative direction are symmetrical to the former waves with respect to rotation by π around the axis x or y.

Substituting (8) into equations (1) and (2) for the z-component of the curl, we can express $C^{(ez)}$ and $C^{(hz)}$ in terms of of the transverse field coefficients:

$$C^{(ez)} = -\frac{ip}{k}C^{(hy)} + \frac{iq}{k}C^{(hx)}, \tag{10}$$

$$C^{(hz)} = -\frac{ip}{k}C^{(ey)} + \frac{iq}{k}C^{(ex)}. \tag{11}$$

After substituting (8), (10), and (11) into the expressions for the transverse components of the curl from (1) and (2), we obtain

$$M\mathbf{C} = k\gamma\mathbf{C}, \tag{12}$$

$$M(p,q,k) = \begin{pmatrix} 0 & 0 & pq & k^2-p^2 \\ 0 & 0 & q^2-k^2 & -pq \\ -pq & p^2-k^2 & 0 & 0 \\ k^2-q^2 & pq & 0 & 0 \end{pmatrix}, \tag{13}$$

$$\mathbf{C} = \begin{pmatrix} C^{(ex)} \\ C^{(ey)} \\ C^{(hx)} \\ C^{(hy)} \end{pmatrix}. \tag{14}$$

The eigenvectors and eigenvalues of problem (12) can be written as functions of parameters p, q, and k:

$$\mathbf{C}_1 = \begin{pmatrix} \frac{pq}{k\sqrt{k^2-p^2-q^2}} \\ \frac{q^2-k^2}{k\sqrt{k^2-p^2-q^2}} \\ 1 \\ 0 \end{pmatrix},$$

$$\mathbf{C}_2 = \begin{pmatrix} \frac{k^2-p^2}{k\sqrt{k^2-p^2-q^2}} \\ \frac{-pq}{k\sqrt{k^2-p^2-q^2}} \\ 0 \\ 1 \end{pmatrix}, \tag{15}$$

$$\mathbf{C}_3 = \begin{pmatrix} \frac{pq}{k\sqrt{k^2-p^2-q^2}} \\ \frac{k^2-q^2}{k\sqrt{k^2-p^2-q^2}} \\ 1 \\ 0 \end{pmatrix},$$

$$\mathbf{C}_4 = \begin{pmatrix} \frac{p^2-k^2}{k\sqrt{k^2-p^2-q^2}} \\ \frac{-pq}{k\sqrt{k^2-p^2-q^2}} \\ 0 \\ 1 \end{pmatrix}. \qquad (16)$$

Vectors $\mathbf{C}_1$ and $\mathbf{C}_2$ correspond to the eigenvalue $\gamma = \sqrt{k^2 - p^2 - q^2}$, while $\mathbf{C}_3$ and $\mathbf{C}_4$ correspond to $\gamma = -\sqrt{k^2 - p^2 - q^2}$.

Since we are seeking the solutions for waves propagating along the positive direction of the z-axis, we should leave the pair of solutions $\mathbf{C}_1$ and $\mathbf{C}_2$ with the corresponding eigenvalue γ. Any linear combination of eigenvectors (15) is also an eigenvector with the same eigenvalue γ:

$$C = S_1\mathbf{C}_1 + S_2\mathbf{C}_2 = VS, \qquad (17)$$

where $S = (S_1, S_2)^{\mathrm{T}}$ and $V = (\mathbf{C}_1, \mathbf{C}_2)$ is a 4×2 matrix composed of two columns $\mathbf{C}_1$ and $\mathbf{C}_2$.

Thus, given the parameters p, q, and k, the set of solutions satisfying Maxwell's equations (8) takes the form

$$C(p, q, k) = V(p, q, k)S, \qquad (18)$$

where S is an arbitrary column of height 2.

3. BOUNDARY CONDITIONS

We proceed to the consideration of the boundary conditions. As has been stated above, due to symmetry, only four equations (4), (6) out of the eight boundary conditions remain.

Substitute (8), (10), and (11) into (4) and (6):

$$BC = 0, \qquad (19)$$

$$B(p, q, k) = \begin{pmatrix} B_1(p, q, k) & \Theta \\ \Theta & B_2(p, q, k) \end{pmatrix}, \qquad (20)$$

where Θ is a 2×2 zero matrix,

$$B_1(p, q, k) = \begin{pmatrix} \sin qb + \frac{iqW}{k}\cos qb & -\frac{ipW}{k}\cos qb \\ -\frac{iqW}{k}\cos pa & \sin pa + \frac{ipW}{k}\cos pa \end{pmatrix}, \qquad (21)$$

$$B_2(p, q, k) = \begin{pmatrix} \frac{iq}{k}\sin qb - W\cos qb & -\frac{iq}{k}\sin pa \\ -\frac{iq}{k}\sin pa & \frac{ip}{k}\sin pa - W\cos pa \end{pmatrix}. \qquad (22)$$

Substituting (18) into (19), we obtain an overdetermined system of equations with respect to the column S as follows:

$$BC = BVS = QS = 0, \qquad (23)$$

where $Q = BV$ is a 4×2 matrix.

Thus, the original problem reduces to finding such p, q and a nontrivial column S for which (23) is satisfied.

In the general case, such p and q do not exist. In fact, we divide matrix Q into two square 2×2-blocks:

$$Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \qquad (24)$$

We require that the determinants of these blocks are equal to zero

$$f_1(p, q) = \det Q_1(p, q) = 0, \qquad (25)$$

$$f_2(p, q) = \det Q_2(p, q) = 0. \qquad (26)$$

This system of two equations in p and q will provide a linear dependence of a pair of lines in each of the blocks. Consequently, dependent lines of each block can be thrown out of system (23), which results in the square matrix

$$Q'(p, q)S = 0. \qquad (27)$$

In order for nontrivial solution (27) to exist, the determinant of the latter matrix should also vanish, which gives the third equation for p and q:

$$f_3(p, q) = \det Q'(p, q) = 0. \qquad (28)$$

In the general case, system (25), (26), and (28) is inconsistent.

We will seek a solution that satisfies Maxwell's equations (1), (2) exactly and boundary conditions (3) approximately. Let us consider two methods of its construction: by minimizing the residual and using the perturbation theory.

4. RESIDUAL MINIMIZATION

Define the residual $\delta(p,q)$ of system (19):

$$Q(p,q)S = \delta(p,q). \tag{29}$$

Instead of solving the original problem (23), we seek p, q, and column S, $||S|| = 1$, such that the residual norm $||\delta||$ attains its minimum. Here, $||\cdot||$ is understood as the standard norm in spaces of columns of the proper height. We consider $||\delta||^2$:

$$||\delta(p,q)||^2 = S^* Q^*(p,q) Q(p,q) S, \tag{30}$$

$$U(p,q) = Q^*(p,q) Q(p,q), \tag{31}$$

where matrix U is Hermitean. Let $0 \le \lambda_1 \le \lambda_2$ be its eigenvalues and let S_1 and S_2 be the corresponding normalized eigenvectors $||S_1|| = ||S_2|| = 1$. Then

$$\min_{||S||=1} ||\delta(p,q)||^2 = \lambda_1(p,q). \tag{32}$$

Thus, the problem of finding the minimum of the residual norm $||\delta||$ has been reduced to finding the minimum of the eigenvalues of matrix $U(p,q)$. Using p, q, and S already found, we find $\gamma = \sqrt{k^2 - p^2 - q^2}$ and C by formula (18).

Since W in the Shchukin–Leontovich boundary conditions is assumed to be small, we can search for p and q in the neighborhood of p and q for an ideal waveguide.

5. CONSTRUCTION OF SOLUTIONS USING PERTURBATION THEORY

For a small impedance $|W| \ll 1$, the values of p and q can be found in a neighborhood of unperturbed p and q of an ideal waveguide. Therefore, we look for p and q in the form of an expansion series in the small parameter W:

$$p = p_0 + \frac{W}{a} p_1 + O(W^2),$$
$$q = q_0 + \frac{W}{b} q_1 + O(W^2), \tag{33}$$

where $p_0 = \dfrac{\pi n}{2a}$ and $q_0 = \dfrac{\pi m}{2b}$ correspond to the modes of an ideal waveguide.

We substitute expansion (33) into (19) and decompose matrix B of boundary conditions into a series in W:

$$B = B^{(0)}(p_0, q_0) + W B^{(1)}(p_0, q_0, p_1, q_1) + O(W^2). \tag{34}$$

Matrix $B^{(0)}(p_0, q_0)$ describes ideal boundary conditions and becomes zero if p and q of an ideal waveguide are substituted in it. Thus,

$$B = W B^{(1)}(p_0, q_0, p_1, q_1) + O(W^2), \tag{35}$$

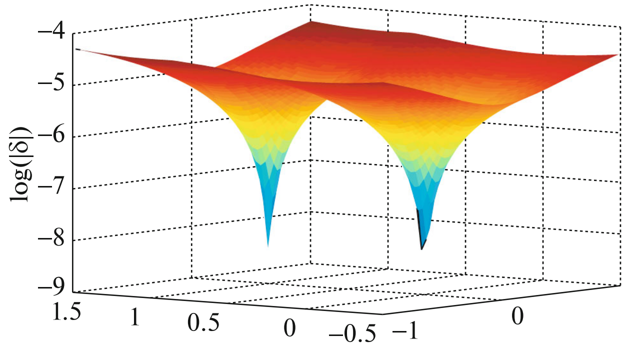

Fig. 1. Minima of the residual $\delta(p,q)$; $W = 0.001(1-i)$, $n = 1$, $m = 1$.

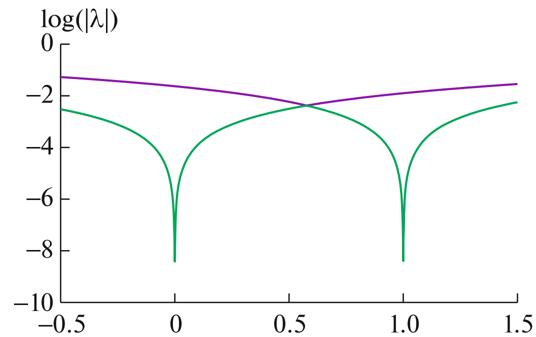

Fig. 2. $\lambda_1(p,q)$ and $\lambda_2(p,q)$ along the line joining the minima.

where

$$B^{(1)} = \begin{pmatrix} B_1^{(1)} & \Theta \\ \Theta & B_2^{(1)} \end{pmatrix}, \tag{36}$$

$$B_1^{(1)} = \begin{pmatrix} q_1 + \frac{iq_0}{k} & -\frac{ip_0}{k} \\ -\frac{iq_0}{k} & p_1 + \frac{ip_0}{k} \end{pmatrix},$$
$$B_2^{(1)} = \begin{pmatrix} \frac{iq_0 q_1}{k} - 1 & -\frac{ip_0 q_1}{k} \\ -\frac{iq_0 p_1}{k} & \frac{ip_0 p_1}{k} - 1 \end{pmatrix}. \tag{37}$$

The condition on the boundary (19) in the first order of expansion in W takes the form

$$B^{(1)}(p_0, q_0, p_1, q_1)\mathrm{C} = 0. \tag{38}$$

Since matrix $B^{(1)}$ is block-diagonal, the determinants of both blocks must be equal to zero for the existence of a nontrivial solution containing nonzero electric and magnetic fields:

$$\begin{cases} \det\left(B_1^{(1)}(p_1, q_1)\right) = 0, \\ \det\left(B_2^{(1)}(p_1, q_1)\right) = 0. \end{cases} \tag{39}$$

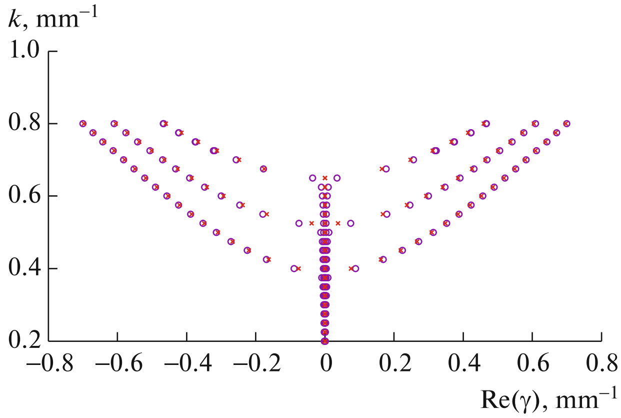

Fig. 3. Re(γ): ideal waveguide (blue), waveguide with losses $W = 0.01(1 - i)$ (red).

As a result, we obtain a system of equations for the first-order corrections p_1 and q_1 as:

$$\begin{cases} p_1 q_1 + \frac{ip_0}{k} q_1 + \frac{iq_0}{k} p_1 = 0, \\ \frac{ip_0}{k} p_1 + \frac{iq_0}{k} q_1 = 1. \end{cases} \tag{40}$$

System (40) can be reduced to a quadratic equation and, therefore, it has two solutions for the corrections p_1 and q_1.

6. NUMERICAL EXPERIMENT

During mathematical simulation, the properties of the basis functions obtained by the two methods described above were investigated.

At high frequencies, i.e., for large values $k > k_{\text{crit}}(n, m) = \sqrt{p_0^2 + q_0^2}$, the pairs of p and q obtained by a numerical method, i.e., by solving (32), and obtained from system (40) by the method of the perturbation theory are consistent with each other at high accuracy. For small k, a discrepancy in the values found is observed, where the corrections obtained by the perturbation theory give the values of the residual (29) several times larger than the minima found numerically.

Thus, for the analysis of running weakly decaying modes, the perturbation method must be used, while the minimization method must be used for strongly decaying modes.

Figure 1 shows the complex plane passing through two local minimums. The z-axis represents the values of $\lambda_1(p, q)$ on a logarithmic scale. The value of peaks can be used to estimate the residual order. In this case, the order of the minimum of λ_1 is approximately $\exp(-27) \approx 2 \times 10^{-12}$.

Figure 2 shows $\lambda_1(p, q)$ and $\lambda_2(p, q)$ on the line containing two minima.

It is seen in Fig. 2 that two minima correspond to two different eigenvalue branches.

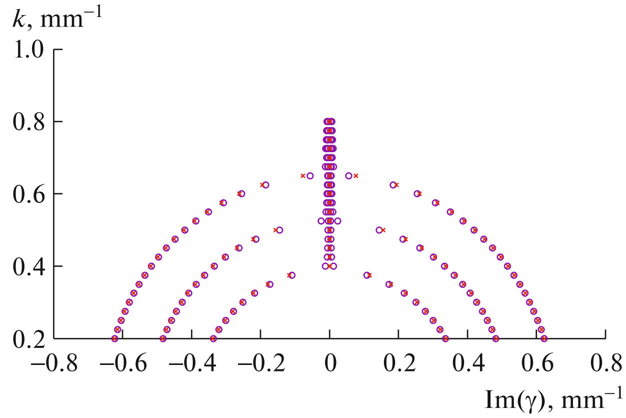

Fig. 4. Im(γ): ideal waveguide (blue colored), waveguide with losses $W = 0.01(1 - i)$ (red colored).

Dispersion characteristics of an infinite rectangular waveguide are constructed. In Figs. 3 and 4, the real and imaginary parts of the modes corresponding to $n = 1$ and $m = 1$ are presented.

A comparison of the calculated dispersion characteristics with those obtained by the method from [1] has been made. In the case of a small impedance, the agreement of results is observed (Figs. 5 and 6).

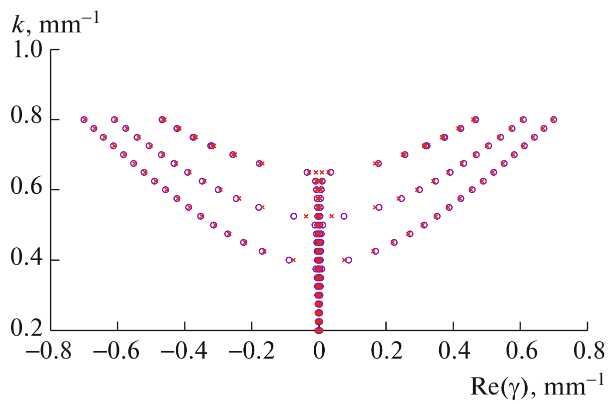

Fig. 5. Re(γ), dispersion characteristics obtained by the method [1] (blue) and by the residual minimization (red); $W = 0.01(1 - i)$.

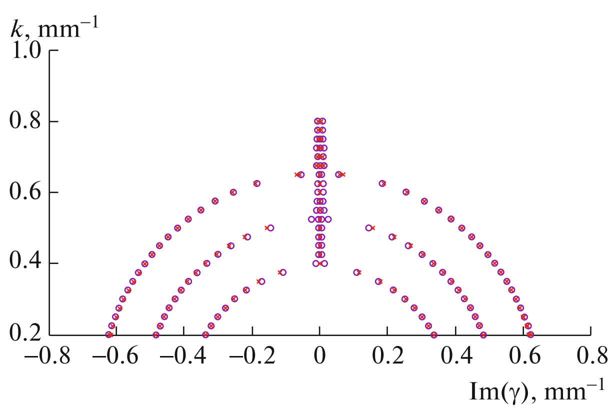

Fig. 6. Im(γ), dispersion characteristics obtained by the method [1] (blue) and by the residual minimization (red); $W = 0.01(1 - i)$.

CONCLUSIONS

Two methods for constructing solutions that satisfy Maxwell's equations exactly and the Shchukin-Leontovich conditions approximately with a small impedance are considered: by minimizing the residual and using the perturbation theory. At high frequencies, both methods give high accuracy, while at low frequencies it is preferable to apply the numerical algorithm. Both methods for the corresponding range of frequencies yield good agreement with the method in [1]. Since the waveguide modes with losses are distinguished independently of each other by the proposed method, the use of these modes to describe fields in sections of constant cross-section for waveguides of complex shape will significantly reduce the rigidity of the resulting matrix problems.

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