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Optimal Integration of the Components of the Global Network of Gravitational-Wave Antennas

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Abstract—This paper considers the problem of optimal integration of the components of the global network of laser gravitational-wave antennas in order to improve the detection efficiency and to better estimate the parameters of astrophysical gravitational-wave signals. A quasi-harmonic burst (chirp) that accompanies the merger of a relativistic binary star at the end of its evolution has been selected as a signal. The shape of such a signal is known up to a set of parameters to be estimated against the background of large coherent and stochastic noise. An alternative possibility of taking into account the coherent excitation phase of individual detectors (component integration by input) is analyzed in addition to the well-known method for filtering output signals by coincidence in time (component integration by output). Statistical detection characteristics for both modes are calculated. The method typical for problems of distinguishing deterministic signals in radar systems is used. A significant increase in detection efficiency during the input integration of network components is shown.

Keywords: gravitational-wave radiation, gravitational-wave antennas, merger of relativistic binary stars, global network of gravitational-wave detectors.

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INTRODUCTION

In September 2015, LIGO laser detectors for the first time sensationally detected a burst of gravitational-wave (GW) radiation from the merger of a relativistic binary star with black hole (BH) components [1]. Other similar signals have been detected since then (until the end of 2017). A total of approximately five events have been recorded to date [2, 3]. The detection of the GW170814 signal from two merging black holes ($M{\sim}30$ M) from a distance of ∼540 Mpc by three detectors (including the Virgo interferometer) [4] elevated the research to a qualitatively different level. It made it possible to to reduce the area of localization of the source on the sky by an order of magnitude, to ∼60 square degrees. Finally, the GW signal from the neutron star (NS) merger was detected. It coincided with the GRB170817A gamma-ray burst (with a delay of 1.7 s) [5] and confirmed the hypothesis of generation of short gamma bursts as a result of the binary NS merger. All these facts make it possible to speak about that the real appearance of the new GW channel of astrophysical information and the heuristic meaning of multichannel astronomy, i.e., the strategy of parallel observation of transients on detectors of different physical nature.

In this case, the problem of constructing an optimal network of GW antennas acquires new relevance, both in terms of their geographical location and in terms of the nature of their interconnection and pickup and interpretation of their common signals against the background of local and global noise. The geographic location of laser GW antennas in the Einstein Telescope configuration with an axially symmetric (cylindrical) sum pattern [6] was optimized in [7, 8] by numerical Monte Carlo simulations with Markov chains. The optimization criteria were the most accurate polarization values of the GW signal, the angular localization of the source on the celestial sphere, and the parameters of its chirp form. At the current stage, the creation of a European–Asian network of four antennas in the Northern Hemisphere is occurring: VIRGO in Italy, KAGRA in Japan, Indigo in India, and the planned antenna in Novosibirsk [9]; there is discussion of taking them into these works.

Two approaches are known regarding the nature of the relationship of the individual network components and the specifics of the pickup (processing)

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of the multichannel signal. The first classic Weberian analysis of coincidences, in which all links of the network operate independently and their outputs are examined (processed) for the presence of coincidences [10], i.e., simultaneous emissions that can be generated by a common global perturbation (conventionally, "component integration by output"), and the second one, which takes the phase coherence of the input excitations of the network components into account. The signals of each link are processed according to this coherence. Next, an integral observable of the network is generated (synthesized) according to the maximum likelihood estimate. The statistics of this observable is used to detect GW perturbations (conventionally, "component integration by output"). Theoretically, such an approach was considered in [11], where an example of a simple network of two detectors was calculated in addition to the general description. It was shown that with a given (selected) error of the first kind (probability of the case) the probability of correct detection turns out to be much higher in the mode of integration by input. However, this is true in the case of a moderate SNR (SNR). For large SNR values, both approaches are equivalent in principle, i.e., they should yield the same results.

Note that the modern understanding of detection by the "coincidence circuit" is not limited only to recording the simultaneous power overshoot on individual detectors. It is also required that the noise threshold by mutual power (cross-correlation) is exceeded, which means an additional restriction on the selection of signal coincidences (similarity of pulse shapes) [5]. In other words, there is a so-called "precision coincidence selection" (criteria of excess power and cross power).

This quality of precision selection of coincidences should be preserved in the approach (operation mode) which is called "integration by input" above. Here, however, it should be remembered that a consistent and correlation reception is used for detecting chirp signals when the input "signal + noise" passes through a filter with a frequency response proportional to the spectrum of the signal that is assumed to be known. A noise spike at the output of such a filter will also have a spectrum similar to that of the "chirp" (for the white input noise model), but it will have to be distinguished from the case of the arrival of a real GW signal accompanying the binary merger. For this reason, it is reasonable to formulate the detection problem in the case of the integration of network components by input as the problem of distinguishing external influences (coherent for all network detectors) from signals of the same shape but generated by the local noise background of a individual link. In the general filtering theory, this corresponds to the so-called "minimax criterion" that leads to a majorizing estimate of the probability of a correct detection of a GW signal [12, 13]. This approach is used in this paper, which is a fundamental difference from [11] cited above.

The paper has the following structure. First, the detection of GW signals from the merger of relativistic binaries by a network of GW antennas is investigated within the maximum likelihood criterion when they are integrated by input (Sections 2, 3, and 4). Section 5 presents the algorithm for such detection in the non-Bayesian formulation, when the signal parameters are deterministic but unknown. Next, as an alternative, the problem of detection when the outputs of individual network components are optimally integrated is investigated (Section 6). Then, a typical case of filtering GW perturbations using the coincidence circuit is considered (Section 7). Finally, the relative efficiency of various detection circuits is evaluated in Section 8. The results and the planned Euro-Asian network of four detectors including the Russian link near Novosibirsk are discussed in the concluding section [9].

1. NETWORK DETECTION AS A PROBLEM OF DISTINGUISHING COHERENT AND STOCHASTIC PERTURBATIONS

We use the results of the theory of detecting vector signals against the background of Gaussian noise in deriving the network detection algorithm [14–16].

Let a vector random process be observed using an antenna system of l elements

$$\mathbf{y}(t) = \left\| \begin{matrix} y_1(t) \\ \vdots \\ y_l(t) \end{matrix} \right\| = ||y_1(t) \ldots y_l(t)||^{\mathrm{T}}$$
$$= \lambda \mathbf{s}(t) + (1-\lambda)\boldsymbol{\eta}(t) + \mathbf{n}(t), \quad -\infty \leqslant t \leqslant \infty, \tag{1}$$

where $\lambda = (0, 1)$ is the so-called state parameter that formalizes the presence (absence) of perturbations, $\mathbf{s}(t)$ is the useful GW signal, $\boldsymbol{\eta}(t)$ is the quasi-deterministic noise, $\mathbf{n}(t)$ is the additive Gaussian background noise. In the further analysis, we will assume that the Gaussian background is uncorrelated in time (white noise) and by individual elements of the antenna system (in space). Therefore, the correlation matrix of the random process $\mathbf{n}(t)$ is simplified

$$\mathbf{K}(t,\tau) = \left\langle \mathbf{n}(t)\mathbf{n}^{\mathrm{T}}(\tau) \right\rangle = N_0 \mathbf{I} \delta(t-\tau),$$

where $\mathbf{I}$ is the identity matrix. In this case, the logarithm of the conditional likelihood ratio can be represented as follows [15]

$$\ln \Lambda(\mathbf{y}|\mathbf{s}) = \frac{1}{N_0} \int_9^T \mathbf{y}^{\mathrm{T}}(t)\mathbf{s}(t)dt - \frac{1}{2}q_s, \quad (2a)$$

$$\ln \Lambda(\mathbf{y}|\boldsymbol{\eta}) = \frac{1}{N_0} \int_9^T \mathbf{y}^{\mathrm{T}}(t)\boldsymbol{\eta}(t)dt - \frac{1}{2}q_\eta, \quad (2b)$$

where

$$q_s = \frac{1}{N_0} \int_0^T \mathbf{s}^{\mathrm{T}}(t)\mathbf{s}(t)dt, \quad q_\eta = \frac{1}{N_0} \int_0^T \boldsymbol{\eta}^{\mathrm{T}}(t)\boldsymbol{\eta}(t)dt$$

are SNRs for the signal and quasi-determined noise, and T is the observation interval. General formulas (2a), (2b) are further refined by substituting a specific waveform of the astrophysical GW perturbation and associated noise bursts.

2. CHIRP SIGNALS

Only astrophysical GW bursts that accompany the merger of the components of the relativistic binary star at the end of its evolution have been successfully detected to date. The shape of such a GW signal $\mathbf{s}(t) = ||s_1(t) \dots s_l(t)||^{\mathrm{T}}$ has been analyzed in detail in the literature and can be found in monographs. In particular, one of the first is the monograph [17] in which the spiral phase of the binary star was calculated in the Newtonian approximation, and the loss of rotational energy was calculated using the GRT quadrupole formula for GW radiation. Post-Newtonian trajectory corrections are omitted here. Such a simplified shape of a GW chirp signal, which is a quasi-harmonic oscillation with an increasing amplitude and carrier frequency by its type, is sufficient for the purposes of this paper, so that

$$s_i(t) = af_i(t_c - t; M, \varphi), \quad i = \overline{1, l}, \quad (3a)$$

where t_c is the time of merger (coalescence) , which means the end of the spiral phase of the GW burst, M is the so-called chirp mass, $a = M/d$ is the scale factor: d is the distance to the source calculated by GW luminosity

$$f_i(t; M, \varphi) = \gamma_i A(t; M) \cos\left[\Psi(t; M) + \psi_i + \varphi\right],$$
$$A(t; M) = \left[\frac{5}{256}\frac{M}{t}\right]^{1/4},$$
$$\Psi(t; M) = -2\left[\frac{t}{5M}\right]^{5/8}, \quad t \geqslant 0,$$

γ_i and ψ_i are the coefficients that determine the radiation pattern and φ is the initial phase.

It is necessary to introduce the delay Δt_i, which depends on the position of the cell for each of them to specify the reception of signal (3a) by an individual network cell. This correction should be formally added to the characteristic time stamp of the signal $t_c = t_1 + \Delta t_i$, for $i = 2, 3, \dots l$, where t_1 is the label of the first cell from which the delay is calculated. Then, the reaction of an individual component (cell) of the network will depend on its coordinates.

In the further analysis, we also use the signal shape $f_i(t; M, \psi)$ in the complex record

$$f_i(t; M, \psi) = \mathrm{Re}\left[\widetilde{\gamma}_i \widetilde{s}(t; M) \exp\{j\varphi\}\right], \quad (3b)$$

where

$$\widetilde{\gamma}_i = \gamma_i \exp\{j\psi_i\}, \widetilde{s}(t; M)$$
$$= aA(t; M) \exp\{j\Psi(t; M)\}. \quad (3c)$$

In the non-Bayesian formulation, the chirp mass from the bank of signal patterns $M \in \mathbf{M}$ is considered as an unknown but nonrandom discrete parameter.

Let us now define local noises. When selecting the form of quasi-deterministic noises $\boldsymbol{\eta}(t) = ||\eta_1(t) \dots \eta_l(t)||^{\mathrm{T}}$ under conditions of expected uncertainty, let us use the minimax approach according to which in the worst case the forms of gravitational $s_i(t)$ and nongravitational $\eta_i(t)$ perturbations are the same:

$$\eta_i(t) = a_i \mathrm{Re}\left[\widetilde{\gamma}_i \widetilde{s}(t_i - t; M_i) \exp\{j\varphi_i\}\right],$$
$$M_i \in \mathbf{M}. \quad (3d)$$

The i index marks an individual network detector, thus emphasizing the individuality of i noise in each of its links.

3. GENERAL FORMULATION OF THE DISTINGUISHING CRITERION

The solution $\lambda = 1$ (there is a GW signal) in the Bayesian formulation is taken in accordance with the likelihood ratio algorithm if the following condition is met:

$$\frac{\Lambda(\mathbf{y}|\lambda = 1)}{\Lambda(\mathbf{y}|\lambda = \mathbf{0})} \geqslant \frac{P(\lambda = 1)}{P(\lambda = 0)}, \quad (4a)$$

where $\Lambda(\mathbf{y}|\lambda)$ is the unconditional likelihood ratio in the state λ, and $P(\lambda)$ are a priori probabilities of the presence and absence of a useful GW signal. The minimax approach is associated with the hypothesis of equality of a priori probabilities $P(\lambda = 1) = P(\lambda = 0) = 1/2$, which leads to a majorizing estimate of the probability of missing a GW signal [12, 13].

The probability of distinguishing between deterministic GW signals and sporadic nongravitational noise against a white Gaussian background described by statistical errors of the first and second kind of α, β is defined by the following equation [15] (see also Appendix)

$$\alpha = \beta = 1 - \Phi\left(\frac{1}{2}\sqrt{q}\right), \tag{4b}$$

where α and β are probabilities of false positive (type I error) and signal skip (type II error), and the parameter q is given by the following formula

$$q = \frac{1}{N_0}\int_0^T [\mathbf{s}(t) - \boldsymbol{\eta}(t)]^{\mathrm{T}}[\mathbf{s}(t) - \boldsymbol{\eta}(t)]dt. \tag{4c}$$

In the non-Bayesian formulation, the solution $\lambda = 1$ is taken if the following condition is met:

$$\ln \Lambda^*(\mathbf{y}|\lambda = 1) \geqslant \ln \Lambda^*(\mathbf{y}|\lambda = 0), \tag{4d}$$

where $\Lambda^*(\mathbf{y}|\lambda)$ is the logarithm of the likelihood ratio in which the unknown signal parameters and noise in the states $\lambda = 1, 0$ are replaced by their maximum likelihood estimates (the parameters $t_c, t_1, \ldots, t_l$ and $\varphi, \varphi_1, \ldots, \varphi_l$ are considered nonenergy and irrelevant).

4. DISTINGUISHING CRITERION FOR THE INPUT INTEGRATION OF NETWORK COMPONENTS

In practice, the observer has only implementations of output random processes of individual components (cells) of the network. They should be inverted to the input using the known transfer function of each receiver. It will result in input vector process (1). After this operation, the observer will be able to consider the network detection algorithm (see Section 2) under the optimal input integration of network components.

Let us now specify general relations (2a), (2b) by substituting the waveform of the chirp signal $\mathbf{s}(t)$ and the quasi-deterministic noise $\boldsymbol{\eta}(t)$ in them. For $\lambda = 1$ (there is a GW signal) we arrive at the following expression

$$\ln \Lambda(\mathbf{y}|\lambda = 1) = \frac{1}{N_0}\left[a\int_{-\infty}^{\infty}\sum_{i=1}^{l} y_i(t) f_i(t_c - t; M, \varphi)dt - \frac{1}{2}a^2 E_s(M)\right]. \tag{5a}$$

The signal energy is introduced above

$$E_s(M) = \sum_{i=1}^{l}\int_{-\infty}^{\infty} f_i^2(\cdot)dt.$$

Next, the main unknown signal parameters, the amplitude (scale) factor and the initial phase (a, φ), in the non-Bayessian approach should be replaced by their likelihood estimates, which are found by setting the corresponding partial derivatives of Eq. (5a) equal to zero. This procedure on the scale factor $\widehat{a}$ yields

$$\ln \Lambda(\mathbf{y}|\lambda = 1; \widehat{a}) = \frac{1}{2N_0 E_s(M)}\Bigg[\mathrm{Re}\ \exp(j\varphi) \times \int_{-\infty}^{\infty}\sum_{i=1}^{l} y_i(t)\widetilde{\gamma}_i\widetilde{s}(t_c - t; M)dt\}\Bigg]^2. \tag{5b}$$

Formula (5b) includes a random process (total for the antenna network components) obtained by matched filtering of the general GW signal by individual network links, which will be further denoted as $z(t; M)$ for brevity (the formula is given in mathematical form of a physically unrealizable filter, which also has integration by negative time values)

$$z(t; M) = \int_{-\infty}^{\infty}\sum_{i=1}^{l} y_i(\tau)\widetilde{\gamma}_i\widetilde{s}(t - \tau; M)d\tau.$$

As a result, we obtain a compact form.

$$\ln \Lambda(\mathbf{y}|\lambda = 1; \widehat{a}) = \frac{1}{2N_0 E_s(M)}\left[\mathrm{Re}\{\exp(j\varphi)z(t_c; M)\}\right]^2,$$

which should be optimized by substituting the likelihood value of the parameter of the initial phase $\widehat{\varphi}$. As a result we obtain:

$$\ln \Lambda(\mathbf{y}|\lambda = 1; \widehat{a}, \widehat{\varphi}) = \frac{1}{2N_0 E_s(M)}|z(t_c; M)|^2.$$

Optimization by the parameters of the chirp signal is not carried out here. In practice, it is replaced by a set of discrete filters matched with chirp signals corresponding to the selected interval values $M + \Delta M, t_c + \Delta t_c$. Consequently, in the non-Bayesian approach, the optimized logarithm of the likelihood ratio has the following form if there is a GW signal and a diskette bank of matched filtering patterns

$$\ln \Lambda^*(\lambda = 1) = \max_{M \in \mathbf{M}, -\infty < t < \infty} \frac{1}{2N_0 E_s(M)}|z(t; M)|^2. \tag{5c}$$

In this formula, the subscript with variations by the parameters of the chirp signal means the selection of a diskette matched filter that gives the maximum response.

The likelihood ratio logarithm in the absence of a GW signal but in the presence of a quasi-deterministic perturbation individual in each link of the network of detectors with its random values of amplitude and phase (3d) can be found in a similar way. Then, for $\ln \Lambda^*(\mathbf{y}|\lambda = 0)$ we obtain the following

$$\ln \Lambda^*(\mathbf{y}|\lambda = 0) = \sum_{i=1}^{l} \max_{\substack{M \in \mathbf{M}, \\ -\infty < t < \infty}} \frac{1}{2N_0 E(M_i)} |z_i(t; M)|^2. \qquad (5d)$$

In this expression, in contrast to (5c), there is a summation over individual network detectors for each of which the result of matched filtering $z_i(t; M)$ is determined by its individual pattern depending on the local quasi-deterministic noise. At the same time, we have the following:

$$z_i(t; M) = \int_{-\infty}^{\infty} y_i(\tau) \widetilde{\gamma}_i \widetilde{s}(t - \tau; M) d\tau.$$

Formulas (5c) and (5d), when substituted into decision rule (4d), make it possible to judge the effect of a GW signal on a network of detectors with an optimal component integration by input.

5. DISTINGUISHING UNDER OPTIMAL INTEGRATION OF OUTPUTS

We briefly consider the traditional version of "output integration" in its optimal formulation in order to compare different methods for integrating the antenna network components.

Let $y_i(t)$ be a random process at the output of an individual component of the antenna system, and $\widehat{\mathbf{x}}$ is the vector of signal parameters or quasi-deterministic noise. The parameter vector is individual for each link in the state $\lambda = 0$

$$\widehat{\mathbf{x}}_i = ||\widehat{a}_i \widehat{t}_i \widehat{M}_i \widehat{\varphi}_i||^{\mathrm{T}},$$

as before, "caps" mean the maximum likelihood estimates of unknown values a_i, t_i, M_i, φ_i. In the state $\lambda = 1$, the parameter vector is common for the entire network $\widehat{\mathbf{x}} = ||\widehat{\mathbf{x}}_1 \ldots \widehat{\mathbf{x}}_l||^{\mathrm{T}}$.

Maximum likelihood estimates for large SNRs can be represented as true with a small fluctuation deviation. In particular, we write the following for the parameter vector of an individual link

$$\widehat{\mathbf{x}}_i \simeq \mathbf{x}_i + \boldsymbol{\xi}_i, \qquad (6a)$$

where the true value vector is $\mathbf{x}_i = ||a_i t_i M_i \varphi_i||^{\mathrm{T}}$, and the vector of deviations $\boldsymbol{\xi}_i = ||\delta a_i \delta t_i \delta M_i \delta \varphi_i||^{\mathrm{T}}$ is represented as Gaussian noise with a zero mean value and a correlation matrix

$$\mathbf{K}_i = \left\langle \boldsymbol{\xi} \boldsymbol{\xi}^{\mathrm{T}} \right\rangle = \mathbf{I}_i^{-1}, \qquad (6b)$$

inverse to the Fisher information matrix $\mathbf{I}_i$ [15].

By analogy with Eq. (1), we use the logic of the distinguishing problem to the parameter vector $\widehat{\mathbf{x}}$. Then, an approximate equation can be considered under conditions of a priori uncertainty

$$\widehat{\mathbf{x}} \simeq \lambda \mathbf{x}_{\mathrm{s}} + (1 - \lambda) \mathbf{x}_\eta + \boldsymbol{\xi},$$

where $\mathbf{x}_{\mathrm{s}}$ is the vector of signal parameters, and $\mathbf{x}_\eta$ is the vector of parameters of quasi-determined noise. The components of these vectors depend on the state parameter λ.

Optimal distinguishing between gravitational and nongravitational signals in the case of integration of network components by their outputs is carried out according to the condition similar to (4d) but applied to the parameter vector $\widehat{\mathbf{x}}$ instead of the input random process $\mathbf{y}(t)$, i.e., the solution $\lambda = 1$ is taken if the following condition is met

$$\ln \Lambda^*(\widehat{\mathbf{x}}|\lambda = 1) \geqslant \ln \Lambda^*(\widehat{\mathbf{x}}|\lambda = 0), \qquad (6c)$$

where $\ln \Lambda^*(\widehat{\mathbf{x}}|\lambda)$ is the conditional likelihood ratio of the random process $\widehat{\mathbf{x}}$ when replacing the unknown parameters by maximum likelihood estimates.

Taking the standard method for calculating the likelihood ratio of vector Gaussian signals [15, 16] into account, it is possible to reduce (6c) to the condition for detecting the GW signal against the background of quasi-determined noise

$$\widehat{\lambda} = 1 : \sum_{i=1}^{l} \widehat{\mathbf{x}}_i^{\mathrm{T}} \mathbf{K}_i^{-1} (\widehat{\mathbf{x}}_{\mathrm{s}i} - \widehat{\mathbf{x}}_{\eta i}) \geqslant 0, \qquad (6d)$$

where $\mathbf{K}_i^{-1} = \mathbf{I}_i$,

$$\widehat{\mathbf{x}}_{\mathrm{s}i} = ||\widehat{a}_i \widehat{t}_{\mathrm{c}} \widehat{M} \widehat{\varphi}||^{\mathrm{T}}$$

is the maximum likelihood estimate of the parameter vector $\mathbf{x}_{\mathrm{s}i}$.

Expression (6d) determines the optimal algorithm for the selection of GW signals when integrating GW antennas by output, which, however, requires a correlation matrix of the network (6b).

6. DISTINGUISHING BY COINCIDENCE CIRCUIT

Historically, the simplest principle for detecting GW effects from observations of the output signals of GW detectors was the principle of "detection of coincidences" on spaced instruments. Coincidence was understood as simultaneity of output signals in the form of short pulse bursts. In fact, this means the integration of network components by output using only one main parameter characterizing the signal, i.e., the time of its arrival.

Let us consider the best integration of the components of the network of GW detectors according to the coincidence circuit. In accordance with the methodology adopted in this paper, we obtain the following for the maximum likelihood estimate of the time of arrival of perturbations

$$y_i(t) \to \widehat{\mathbf{t}}, \widehat{\mathbf{t}} = ||\widehat{t}_1 \dots \widehat{t}_l||^{\mathrm{T}} \simeq ||t_1 \dots t_l||^T + \delta\mathbf{t},$$

where $t_1, \dots, t_l$ are the true values of the times of occurrence of perturbations, and $\delta\mathbf{t} = ||\delta t_1 \dots \delta t_l||^{\mathrm{T}}$ is the estimate deviation vector.

Therefore, we obtain the following to distinguish between the arrivals of GW signals and noise

$$\widehat{\mathbf{t}} \simeq \lambda ||t_{\mathrm{c}} \dots t_{\mathrm{c}}||^{\mathrm{T}} + (1-\lambda)||t_1 \dots t_l||^{\mathrm{T}} + \delta\mathbf{t}.$$

In the integration of the network components according to the coincidence circuit, the solution $\lambda = 1$ is taken if

$$\sum_{i=1}^{l} \frac{1}{\sigma_i^2} \widehat{t}_i (\widehat{t}_{\mathrm{c}} - \widehat{t}_i) \geqslant 0, \tag{7}$$

where $\sigma_i^2 = \langle \delta t_i^2 \rangle$,

$$\widehat{t}_{\mathrm{c}} = \sum_{i=1}^{l} \frac{\widehat{t}_i}{\sigma_i^2} \left[\sum_{i=1}^{l} \frac{1}{\sigma_i^2} \right]^{-1} \tag{8}$$

is the maximum likelihood estimate of the coalescence time.

Expressions (4) and (5) determine the optimal algorithm for processing information by coincidence within the traditional approach for the antenna system of $l = 2$ components

$$\lambda = 1 : |\widehat{t}_1 - \widehat{t}_2| \leqslant \tau_{\mathrm{r}}, \tag{9}$$

where τ_{r} is the resolution time (coincidence window).

7. RELATIVE EFFICIENCY OF CIRCUITS FOR INTEGRATION

We considered two circuits for integrating components of a network of GW detectors above : by inputs (shared aperture) and by outputs. They are basically equivalent in their optimal versions, i.e., they should have a comparable detection efficiency of GW perturbations. However, the optimal integration by outputs is complicated by the need to know the mutual correlation matrix of the noise of the individual components used to filter the signal, which is difficult in practice. Therefore, a coincidence circuit is used in its precision version (criteria of excess power and cross power). Let us show that aperture synthesis (input integration) has a noticeable advantage over the coincidence circuit.

Following the method described in [15], we introduce the coefficient $\varkappa$ of the relative efficiency of the coincidence circuit.

$$\varkappa = \frac{P_{\mathrm{e}}}{P_{\mathrm{e,c}}} \leqslant 1, \tag{7}$$

where P_{e} and $P_{\mathrm{e,c}}$ are the probabilities of an erroneous solution in aperture synthesis and in the coincidence circuit, respectively. We use formulas (A3) and (A4) from Appendix for input integration from which

$$P_{\mathrm{e}} = 1 - \Phi\left(\frac{1}{2}\sqrt{q}\right), \tag{8}$$

where

$$q = \frac{1}{N_0} \sum_{i=1}^{l} \int_{-\infty}^{\infty} [s_i(t) - \eta_i(t)]^2 \, dt$$

$$= \frac{1}{N_0} \sum_{i=1}^{l} [B_{\mathrm{s},i}(0) + B_{\eta,i}(0) - 2B_{\mathrm{s}\eta,i}(0)]. \tag{9}$$

Here, there are autocorrelation functions of the signal $B_{\mathrm{s},i}(0)$ and noise $B_{\eta,i}(0)$ (in each link of the network) as well as their mutual correlation function at coinciding times $B_{\mathrm{s}\eta,i}(0)$:

$$B_{\mathrm{s}\eta,i}(\tau) = \int_{-\infty}^{\infty} s_i(t)\eta_i(t+\tau)dt.$$

Similarly, it can be shown that the corresponding characteristics (error probability) for the coincidence circuit are defined by the following formulas:

$$P_{\mathrm{e,c}} = 1 - \Phi\left(\frac{1}{2}\sqrt{q_{\mathrm{c}}}\right) \tag{10}$$

for

$$q_{\mathrm{c}} = \sum_{i=1}^{l} \frac{(t_{\mathrm{c}} - t_i)^2}{\sigma_i^2}. \tag{11}$$

Formulas (8)–(11) make it possible to estimate the coefficient $\varkappa$. The latter depends on the parameters of gravitational and nongravitational signals but also on the variances σ_i^2, which are difficult to calculate analytically for such a complex signal structure as a chirp. Therefore, below we estimate the relative efficiency for the case where GW signals and noise have the same parameters (including the times of occurrence) that differ only in the initial phases, see (3a), (3c).

In this representation, the phases of perturbations of individual links have random deviations $\delta\varphi_i$ from the phase of the GW chirp signal φ (noise $\eta_i(t)$ is incoherent)

$$\varphi_i = \varphi + \delta\varphi_i, \quad i = \overline{1, l}.$$

It is possible to find the correlation functions entering into formula (9) using representations of signal (3a) and noise (3c) taking the fact into account that the chirp phase is a nonenergy parameter.

$$B_{\mathrm{s},i}(0) = B_{\eta,i}(0),$$

$$B_{\mathrm{s}\eta,i}(0) = B_{\mathrm{s},i}(0)\cos\delta\varphi_i, \qquad (12)$$

where

$$B_{\mathrm{s},i}(0) = \int_{-\infty}^{\infty} |s_i|^2(t)dt.$$

Substituting (12) into Eq. (9) makes it possible to specify the parameter q for calculating the error in aperture synthesis.

Now it is possible to estimate the relative efficiency of the coincidence circuit. From formula (11) it follows that the coincidence of signal and noise bursts ($t_{\mathrm{c}} \approx t_i$) makes the parameter q_{c} small, which entails a higher error probability $P_{\mathrm{e,c}} \simeq (1/2)$ (10) in the proposition "there is signal" $\lambda = 1$. At the same time, we have the following for the efficiency coefficient estimation

$$\varkappa \simeq 2\left[1 - \Phi\left(\frac{1}{2}\sqrt{q}\right)\right]. \qquad (13)$$

We then find the following:

$$q \simeq \rho_{\mathrm{s}} \sum_{i=1}^{l} \sin^2\delta\varphi_i,$$

where

$$\rho_{\mathrm{s}} = \frac{1}{N_0}\int_{-\infty}^{\infty} \mathbf{s}^{\mathrm{T}}(t)\mathbf{s}(t)dt = \frac{B_{\mathrm{s}}(0)}{N_0}$$

is the SNR. The parameter q increases and the relative efficiency of the coincidence circuit tends to zero (13) for large values of $\rho_{\mathrm{s}} \gg 1$. In fact, the coincidence circuit gives a false positive, "there is a signal," in its absence. This occurs because it does not take phase relations into account between input perturbations. Here, aperture synthesis has a distinct advantage.

CONCLUSIONS

The main problem considered in this paper was the comparison of the possible options for integrating the components of the global network of GW antennas in order to estimate their comparative efficiency with respect to the selection, detection, and estimation of the parameters of the astrophysical GW signals. The chirp structure that describes GW radiation at the spiral phase of the merging of a relativistic binary star was chosen as a signal model. Two key methods for complex processing of information supplied by network components were investigated: the optimal summation of the responses brought to the input of each detector ("aperture synthesis") and the immediate joint processing of their output data (in particular, the "coincidence circuit"). It is obvious that both methods should lead to the same results for large SNRs with optimal processing. However, the effectiveness of these methods can differ significantly when measured at the threshold detector sensitivity. Although the analysis carried out in this paper cannot be considered exhaustive and strongly consistent, it was possible to use it to show the advantages of aperture synthesis compared to the traditional coincidence circuit method.

If network components are integrated by input, the optimal observable is formed as the output of the multichannel correlator, which is matched with the reference GW signal which is the same for all links (3b). A coherent accumulation of the observable occurs. The likelihood ratio in the presence of a signal depends on its phase ϕ (5b), which does not appear in generalized form (5c) being replaced by its maximum likelihood estimate.

On the contrary, the optimal observable is the sum of individual correlators in the channels of the individual reference signals with random phases (5d) in the absence of a signal and in the presence of only quasi-deterministic noise (3d). In this case, there is no phased accumulation.

Network component integration by input and coherent accumulation basically improve (maximize) the SNR, which in turn leads to more precise estimation of the parameters of the received signal (current frequency, rate of its change, and, ultimately, the chirp mass). In the first approximation, these parameters can only receive a rough estimate via the Cramer–Rao inequality [15]. As far as estimation accuracy of the source localization is concerned, it is mainly determined by the spatial scale of the network and depends little on the type of component integration.

It would be interesting to illustrate these results based on the example of a specific network, e.g., the European–Asian network of four GW-interferometers mentioned in the Introduction: VIRGO, KAGRA, including sites in India (Maharashtra, $\Phi\, 19^\circ 43'$ N, $\lambda\, 77^\circ 09'$ E) and in Novosibirsk ($\Phi\, 55^\circ 02'$ N, $\lambda\, 82^\circ 56'$ E). Note that the addition of the last two sites greatly narrows the areas of localization of the detected sources of GW signals along the meridians of the spherical geocentric coordinate system. Such calculations are being conducted and will be presented in the following paper.

APPENDIX

The problem of distinguishing between two signals can be reduced to the problem of detecting their difference signal [15]. In fact, let us rewrite Eq. (1) in the following form:

$$\Delta\mathbf{y}(t) = \mathbf{y}(t) - \boldsymbol{\eta}(t) = \lambda\Delta\mathbf{s}(t) + \mathbf{n}(t),$$

$$-\infty < t < \infty,$$

where

$$\Delta\mathbf{s}(t) = \mathbf{s}(t) - \boldsymbol{\eta}(t). \tag{A1}$$

It is apparent that the problem of distinguishing between deterministic signals $\mathbf{s}(t)$ and $\boldsymbol{\eta}(t)$ is statistically equivalent to detecting the useful signal $\Delta\mathbf{s}(t)$ (2) against the background of additive Gaussian noise $\mathbf{n}(t)$.

The solution $\lambda = 1$ is taken if the following condition is met

$$\ln \Lambda(\Delta\mathbf{y}|\lambda = 1) \geqslant C,$$

where $\Lambda(\Delta\mathbf{y}|\lambda)$ is the likelihood ratio of a random Gaussian process $\Delta\mathbf{y}(t)$ in the λ state, and C is the threshold level depending on the selected quality criteria.

The probabilities of type I α and type II β errors normally determined as follows

$$\alpha = P\left\{\ln \Lambda(\Delta\mathbf{y}|\lambda = 1) \geqslant C|\lambda = 0\right\};$$
$$\beta = P\left\{\ln \Lambda(\Delta\mathbf{y}|\lambda = 1) \leqslant C|\lambda = 0\right\},$$

are written for Gaussian background noise using the probability integral

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} \exp\left\{-\frac{z^2}{2}\right\} dz$$

as

$$\alpha = 1 - \Phi\left(\frac{1}{2}\sqrt{q} + \overline{c}\right),$$

$$\beta = 1 - \Phi\left(\frac{1}{2}\sqrt{q} - \overline{c}\right).$$

Here,

$$q = \frac{1}{N_0} \int\limits_{-\infty}^{\infty} \Delta\mathbf{s}^{\mathrm{T}}(t)\Delta\mathbf{s}(t)dt \tag{A2}$$

is the SNR parameter and $\overline{c} = C/\sqrt{q}$ is the normalized threshold.

Threshold C defined as

$$C = \ln \frac{P(\lambda = 0)}{P(\lambda = 1)}$$

depends on the selected quality criteria.

We also assume that type I and type II errors are equal within the ideal observer's criterion, i.e., $P(\lambda = 1) = 1 - P(\lambda = 0)$, which leads to the threshold $C = 0$. The threshold within the Neyman–Pearson criterion [15] is determined by the chosen type I error probability value $C = C(\alpha)$.

In this paper, we limit ourselves to the ideal observer criterion assuming the known a priori probabilities $P(\lambda = 1)$ and $P(\lambda = 0) = 1 - P(\lambda = 1)$. In the case of this approach, the probability of an erroneous solution P_{e} is

$$P_{\mathrm{e}} = P(\lambda = 1)\beta + P(\lambda = 0)\alpha, \tag{A3}$$

i.e., the error probability turns out to be as follows under the minimax condition $P(\lambda = 1) = P(\lambda = 0) = 0.5$

$$P_e = 1 - \Phi\left(\frac{1}{2}\sqrt{q}\right).$$

It is also useful to present the vector representation of SNR q (A2) in the scalar form:

$$q = \frac{1}{N_0} \sum_{i=1}^{l} \int\limits_{-\infty}^{\infty} \Delta s_i^2(t)dt$$
$$= \frac{1}{N_0} \sum_{i=1}^{l} \int\limits_{-\infty}^{\infty} \left[s_i(t) - \eta_i(t)\right]^2 dt. \tag{A4}$$

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