

THEORETICAL AND MATHEMATICAL  
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## Analysis of a Rectangular Waveguide with Allowance for Losses in the Walls

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**Abstract**—We propose a model of an infinite waveguide of constant rectangular cross section with losses in the walls which are described by the Schukin–Leontovich boundary conditions. The waveguide is analyzed using the non-complete Galerkin method. We use the standard basis for waveguide with ideally conducting walls supplemented with functions providing precise fulfillment of the boundary conditions. The eigen modes of the waveguide in the THz range are calculated and dispersion characteristics are obtained.

**Keywords:** waveguide systems, Schukin–Leontovich boundary conditions, incomplete Galerkin method.

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### FORMULATION OF THE PROBLEM

We consider a waveguide with a constant rectangular cross-section  $V = \{(x, y) \in S : x \in (0, a), y \in (0, b), z \in \mathbb{R}\}$ , where  $z$  is the waveguide axis.

Electromagnetic field inside the waveguide is described by the Maxwell equations

$$\begin{aligned} \operatorname{curl} \mathbf{H} &= ik\mathbf{E}, \\ \operatorname{curl} \mathbf{E} &= -ik\mathbf{H}, \\ \operatorname{div} \mathbf{E} &= 0, \\ \operatorname{div} \mathbf{H} &= 0 \end{aligned} \quad (1)$$

with the Schukin–Leontovich [1] boundary conditions on the side boundary of the waveguide,

$$[\mathbf{n}, \mathbf{E}]_{\partial V} = Z_s [\mathbf{n}, [\mathbf{n}, \mathbf{H}]]_{\partial V}. \quad (2)$$

Here,  $\mathbf{n}$  is the unit normal to the  $\partial V$  boundary. For the rectangular area this conditions can be presented as

$$\begin{aligned} E_x &= Z_s H_z, & E_z &= -Z_s H_x, & y &= 0, \\ E_x &= -Z_s H_z, & E_z &= Z_s H_x, & y &= b, \\ E_y &= Z_s H_z, & E_z &= -Z_s H_y, & x &= 0, \\ E_y &= -Z_s H_z, & E_z &= Z_s H_y, & x &= a, \end{aligned} \quad (3)$$

where  $Z_s$  is the impedance of the material of the waveguide walls.

To find the general solution of the problem (1)–(3) we use the incomplete Galerkin method. Let  $\mathbf{G}_{nm}^{(e1)}$ ,  $\mathbf{G}_{nm}^{(e2)}$ ,  $\mathbf{G}_{nm}^{(h3)}$  be the basis functions of the electric type [2] of an ideal waveguide,  $0 < n \leq N$ ,  $0 < m \leq M$ .

$\mathbf{G}_{nm}^{(h1)}$ ,  $\mathbf{G}_{nm}^{(h2)}$ ,  $\mathbf{G}_{nm}^{(e3)}$  are the basis functions of the magnetic type [2] of an ideal waveguide,  $0 < n \leq N$ ,  $0 < m \leq M$ ,  $n + m > 0$ .

Let us introduce additional basis functions  $\mathbf{G}_{nm}^{(ex)}$ ,  $\mathbf{G}_{nm}^{(ey)}$ ,  $\mathbf{G}_{nm}^{(ezx)}$ ,  $\mathbf{G}_{nm}^{(ezy)}$ ,  $0 < n \leq N$ ,  $0 < m \leq M$ ,  $n + m > 0$  with a nonzero tangential component of the electric field on boundary  $\partial S$ , which provide fulfillment of the Schukin–Leontovich conditions [3].

Let us denote the summation over multi-index  $(n, m)$  for the cases of electric- and magnetic-type fields as

$$\begin{aligned} \sum_{n=0}^N \sum_{m=0}^M q_{nm} &= \sum_{(e)} q_{nm}, \\ \sum_{n+m>0}^{N,M} q_{nm} &= \sum_{(h)} q_{nm}. \end{aligned}$$

We assume electromagnetic fields in the cross-section of a waveguide in the form

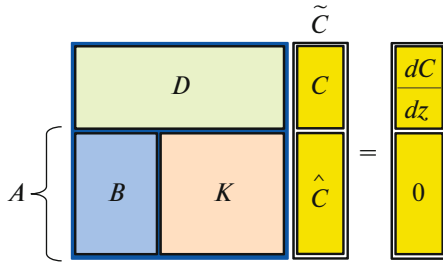
$$\mathbf{H}_z = \sum_{(h)} W_{nm}^{(h1)}(z) \mathbf{G}_{nm}^{(h1)}, \quad (4)$$

$$\begin{aligned} \mathbf{E}_\perp &= \sum_{(e)} W_{nm}^{(e2)}(z) \mathbf{G}_{nm}^{(e2)} + \sum_{(h)} W_{nm}^{(e3)}(z) \mathbf{G}_{nm}^{(e3)} \\ &+ \sum_{(h)} W_{nm}^{(ex)}(z) \mathbf{G}_{nm}^{(ex)} + \sum_{(h)} W_{nm}^{(ey)}(z) \mathbf{G}_{nm}^{(ey)}, \end{aligned} \quad (5)$$

$$\begin{aligned} \mathbf{E}_z &= \sum_{(e)} W_{nm}^{(e1)}(z) \mathbf{G}_{nm}^{(e1)} + \sum_{(h)} W_{nm}^{(ezx)}(z) \mathbf{G}_{nm}^{(ezx)} \\ &+ \sum_{(h)} W_{nm}^{(ezy)}(z) \mathbf{G}_{nm}^{(ezy)}, \end{aligned} \quad (6)$$

$$\mathbf{H}_\perp = \sum_{(e)} W_{nm}^{(h3)}(z) \mathbf{G}_{nm}^{(h3)} + \sum_{(h)} W_{nm}^{(h2)}(z) \mathbf{G}_{nm}^{(h2)}. \quad (7)$$

By substituting the field representation (4)–(7) in the Maxwell equations (1) and boundary conditions



**Fig. 1.** The structure of the matrix of the system of differential-algebraic equations.

(3) and using the properties of the constructed basis [3] one obtains a differential-algebraic system of linear equations with respect to unknown coefficients  $W_{nm}$ .

$$\begin{aligned} \frac{dW_{nm}^{(e2)}}{dz} &= -ik \left( 1 - \frac{\kappa_{nm}^2}{k^2} \right) W_{nm}^{(h3)} \\ &+ \frac{1}{b} \sum_{m'=0}^M \frac{2\pi m}{\kappa_{nm} b} [(-1)^{m'} - 1] W_{nm'}^{(ezy)} \\ &+ \frac{1}{a} \sum_{n'=0}^N \frac{2\pi n}{\kappa_{nm} a} [(-1)^{n'} - 1] W_{n'm}^{(ezx)}, \end{aligned} \quad (8)$$

$$\frac{dW_{nm}^{(h3)}}{dz} = -ik W_{nm}^{(e2)}, \quad (9)$$

$$\begin{aligned} \frac{dW_{nm}^{(h2)}}{dz} &= -ik \left( 1 - \frac{\kappa_{nm}^2}{k^2} \right) W_{nm}^{(e3)} \\ &+ \frac{\kappa_{nm} Z_s}{ik} \left\{ -\frac{2}{b} \sum_{m'=0}^M [(-1)^{m+m'} + 1] W_{nm'}^{(h1)} \right. \\ &\left. + \frac{2}{a} \sum_{n'=0}^N [(-1)^{n+n'} + 1] W_{n'm}^{(h1)} \right\}, \end{aligned} \quad (10)$$

$$\begin{aligned} \frac{dW_{nm}^{(e3)}}{dz} &= -ik W_{nm}^{(h2)} + \frac{1}{b} \sum_{m'=0}^M \frac{\pi n}{\kappa_{nm} b} [(-1)^{m'} - 1] W_{nm'}^{(ezy)} \\ &- \frac{1}{a} \sum_{n'=0}^N \frac{\pi m}{\kappa_{nm} a} [(-1)^{n'} - 1] W_{n'm}^{(ezx)}, \end{aligned} \quad (11)$$

$$\begin{aligned} W_{nm}^{(h1)} &= \frac{\kappa_{nm}}{ik} W_{nm}^{(e3)} - \frac{Z_s}{ik} \left\{ \frac{2}{b} \sum_{m'=0}^M [(-1)^{m+m'} + 1] W_{nm'}^{(h1)} \right. \\ &\left. + \frac{2}{a} \sum_{n'=0}^N [(-1)^{n+n'} + 1] W_{n'm}^{(h1)} \right\}, \end{aligned} \quad (12)$$

$$W_{nm}^{(e1)} = -\frac{\kappa_{nm}}{ik} W_{nm}^{(h3)}, \quad (13)$$

$$W_{nm}^{(ezx)} = Z_s \left( \frac{\pi n}{\kappa_{nm} a} W_{nm}^{(h2)} - \frac{\pi m}{\kappa_{nm} b} W_{nm}^{(h3)} \right), \quad (14)$$

$$W_{nm}^{(ezy)} = Z_s \left( \frac{\pi n}{\kappa_{nm} a} W_{nm}^{(h2)} + \frac{\pi m}{\kappa_{nm} b} W_{nm}^{(h3)} \right), \quad (15)$$

$$W_{nm}^{(ex)} = -\frac{Z_s \kappa_{nm}}{ik} W_{nm}^{(e3)}, \quad (16)$$

$$W_{nm}^{(ey)} = -\frac{Z_s \kappa_{nm}}{ik} W_{nm}^{(h3)}, \quad (17)$$

where  $\kappa_{nm} = \sqrt{\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2}$ . The system of equation (8)–(7) can be transformed into a system of differential equations.

Introduce column  $\tilde{C} = (W^{(e2)}, W^{(h3)}, W^{(h2)}, W^{(e3)}, W^{(e1)}, W^{(h1)}, W^{(ex)}, W^{(ey)}, W^{(exz)}, W^{(eyz)})^T$  with all sought coefficients, column  $C = (W^{(e2)}, W^{(h3)}, W^{(h2)}, W^{(e3)})^T$  with the coefficients under the differentiation sign in respect to  $z$  in (9)–(18) and column  $\hat{C} = (W^{(e1)}, W^{(h1)}, W^{(ex)}, W^{(ey)}, W^{(exz)}, W^{(eyz)})^T$ .

Thus,  $\tilde{C} = (C^T, \hat{C}^T)^T$ . Let  $P$  and  $\tilde{P}$  be the height of columns  $C$  and  $\tilde{C}$ , respectively.

The system differential equations (8)–(11) in the introduced notations has the form

$$\frac{dC}{dz} = DC, \quad (18)$$

where  $D$  is a  $P \times \tilde{P}$  matrix.

Algebraic equations (12)–(17) can be written in the form

$$A\tilde{C} = 0, \quad (19)$$

where  $A$  is matrix  $(\tilde{P} - P) \times \tilde{P}$ . Then, system (19) can be written in the form

$$BC + K\tilde{C} = 0, \quad (20)$$

where  $A = [B, K]$ ,  $K$  is a square matrix of  $(\tilde{P} - P) \times (\tilde{P} - P)$  (Fig. 1).

Let us express  $\hat{C}$  via  $C$ :

$$\hat{C} = -K^{-1}BC. \quad (21)$$

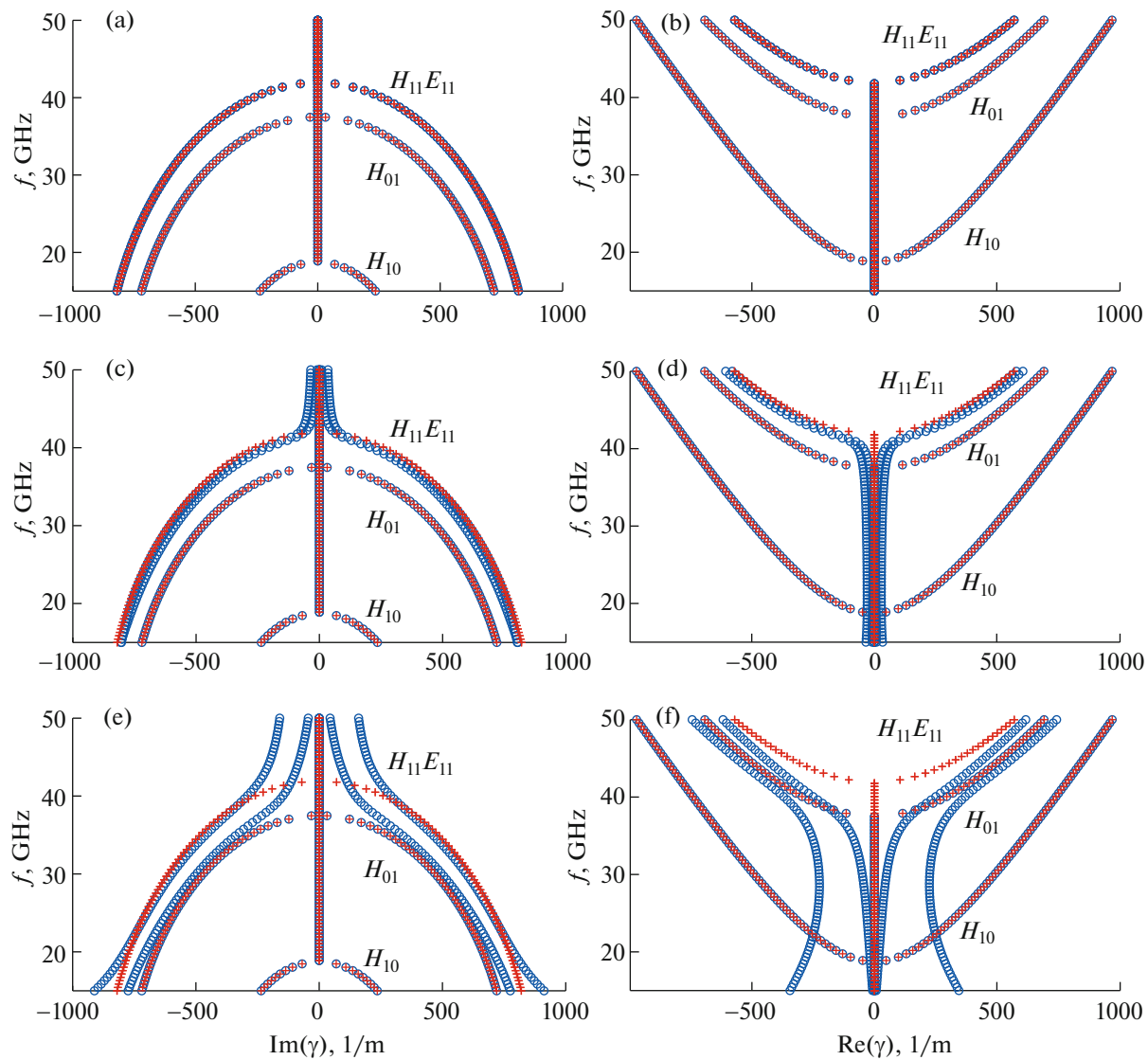
Thus, a complete column with unknowns  $\tilde{C} = (C, \hat{C})^T$  can be expressed in terms of a column of unknowns of a system of homogeneous differential equations (SHDE)  $C$ ,

$$\tilde{C} = QC = \begin{pmatrix} I \\ -K^{-1}B \end{pmatrix} C, \quad (22)$$

where  $I$  is a unit matrix of dimension  $P \times P$ , and  $-K^{-1}B$  is a matrix of dimension  $(\tilde{P} - P) \times P$ .

Substitution of (22) into (18) gives SHDE with respect to column  $C$  with a square matrix  $T$ ,

$$\frac{dC}{dz} = DC\tilde{C} = DQC = TC. \quad (23)$$



**Fig. 2.** The dispersion characteristics of (a, b) an ideal infinite waveguide obtained (+) analytically and (circles) using the proposed method at  $Z_s = 0$ ; (c, d) (+) ideal infinite waveguide and (circles) a waveguide with losses at an impedance value of  $Z_s = 0.05(1 - i)$ ; (e, f) (+) ideal infinite waveguide and (circles) waveguide with losses at an impedance value of  $Z_s = 0.25(1 - i)$ .

Calculation of the eigen-vectors and eigen-values of matrix  $T$  gives the set of eigen modes of the waveguide. Consequent application of the algorithm for a selected frequency range  $k \in [k_1, k_2]$  yields the dispersion characteristics of the waveguide.

## NUMERICAL EXPERIMENT

We calculated the dispersion characteristics of a rectangular waveguide with a cross-section of 10 cm by 20 cm. At  $Z_s = 0$  system (23) transforms to a system of equations for an ideal waveguide and the dispersion characteristics agree with those calculated analytically (Figs. 2a, 2b).

With a nonzero impedance the dispersion characteristics become distorted and attenuation of the mode emerges (Figs. 2c–2f).

## CONCLUSIONS

A mathematical vector model of a waveguide with a rectangular cross-section has been developed. An algorithm for its calculation on a computer has been created and the dispersion characteristics were obtained. The proposed algorithm also allows calculations of ladder-type waveguide systems, which are widely used in designing klystron systems.

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