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Parametric Generation of Light in a Cavity: An Analytical Approximation

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Abstract—A new analytical approximate solution is suggested for the problem of nonlinear parametric generation of light in a cavity. This solution is much more accurate than the known ones. Two- and three-cavity lasing schemes are considered and criteria for their adequacy are ascertained. The accuracy of the results is confirmed by computer simulation.

Keywords: laser physics, nonlinear optics, parametric generation of light, optical cavity.

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INTRODUCTION

The onset of the laser age in the early 1960s gave a powerful pulse to the development of nonlinear optics, which, in turn, promoted new achievements in laser physics. Effective optical harmonic generators were designed, as well as optical parametric oscillators with smoothly variable radiation frequencies [1]. Parametric optical sources are used, in particular, for the creation of non-classical light conditions with unusual properties, where correlating pairs of photons are generated. They are used in experimental studies of the quantum nature of light, which is shown best in the Bell inequality, Zeno's paradox, and quantum interference (see, e.g., [2]). In addition, parametrically generated light can be used for standardless perfectly accurate calibration of photodetectors [3], non-demolition measurements of light intensity [4, 5], and data transmission protected from leakage with quantum cryptography [6, 7]. Parametrically generated light is an ideal source in highly accurate balanced optical measurements due to the strong correlation between signal and idler photons, since the level of the differential photocurrent turns out to be lower than the shot noise in the frequency range below the inverse photon-containment time in a cavity [8, 9]. A cavity is used because the efficiency of the parametric transformation of pumping is 10^{-8}–10^{-7}, i.e., only one of ten million or more of pumping photons decays to two (signal and idler) photons. The cavity allows an increase in the effective length of the nonlinear interaction due to multiple passes through the crystal.

There is no exact analytical solution of the problem of the description of fields in a cavity due to the complexity of the initial set of differential equations of parametric interaction and consideration of boundary conditions of refraction by mirrors. The simplest approximation (the so-called non-depleted pump approximation) consists in the neglect of the pump depletion, but it is very rough. A more accurate solution was suggested in [10, pp. 468, 469], where the pump depletion is considered, but the signal increase is ignored, which formally violates the principle of conservation of energy. Nevertheless, this approximation has been widely used up to the present time [11–15]. An attempt to develop a more accurate analytical solution was recently made in [16, 17] under the assumption that the amplitudes of both pump and generated light beams change linearly along the length of a nonlinear crystal. This provided a more accurate solution and allowed the authors to avoid the violation of the principle of conservation of energy. In addition, losses were taken into account in [17] and a strict criterion for the adequacy of the solutions suggested was developed.

Further studies of the peculiarities of light generation in a cavity allowed us to draw the conclusion that an analytical solution can be derived from a weaker assumption about linear changes only in the pump amplitude. This assumption is confirmed by many numerical experiments. The accuracy of the analytical approximation is much higher in this case. These results are discussed in this work.

1. BASIC EQUATIONS FOR THE DESCRIPTION OF PARAMETRIC INTERACTIONS

The parametric interaction of light in a nonlinear crystal is a process of the generation of pairs of photons in a medium with quadratic nonlinearity under the action of pumping photons. Three waves participate in the process, namely, signal (s), idler (i), and pumping (p) waves; their frequencies are connected by the relationship $\omega_p = \omega_s + \omega_i$.

The initial set of differential equations of parametric interactions has the form [10, pp. 468, 469]:

$$\begin{cases} \left(\frac{1}{u_s}\frac{\partial}{\partial t} + \frac{\partial}{\partial z} + \alpha_s\right) A_s = \beta_1 A_p A_i, \\ \left(\frac{1}{u_i}\frac{\partial}{\partial t} + \frac{\partial}{\partial z} + \alpha_i\right) A_i = \beta_2 A_p A_s, \\ \frac{\partial A_p}{\partial z} = -\beta_p A_s A_i, \end{cases} \quad (1)$$

where A are the complex amplitudes of the corresponding waves, α are the loss factors, β are the nonlinear coupling factors, u are the group velocities, and z is the longitudinal optical axis along which beams propagate.

The third equation in set (1) is written for monochromatic pumping, which shows that it is depleted mainly due to a nonlinear interaction but not dissipative loss.

This set of differential equations should be solved with allowance for reflection from mirrors in a cavity:

$$A(z=0;t) = rA(L;t-T), \quad (2)$$

where L is the total bypass length around the cavity, T is the total bypass time around the cavity, and r is the amplitude reflection coefficient of the exit mirror. The reflection coefficients of other mirrors are taken equal to unity.

Let us simplify the initial set (1) in the following way. We consider the stationary case of the steady-state conditions for the collinear interaction of all the three waves that propagate along the z axis and take the symmetry of signal and idler waves under equal loss factors $\alpha_s = \alpha_i$ and nonlinear coupling $\beta_1 = \beta_2$ factors into account. In this case, the two first equations of set (1) are identical. Thus,

$$\begin{cases} \frac{dA}{dz} + \alpha A = \beta A_p A, \\ \frac{dA_p}{dz} = -2\beta A^2. \end{cases} \quad (3)$$

Let us write

$$\begin{cases} dA = (\beta A_p - \alpha) A dz, \\ dA_p = -2\beta A^2 dz, \end{cases} \quad (4)$$

divide the second equation of the set by the first equation, and separate the variables:

$$2\beta A dA = (\alpha - \beta A_p) dA_p. \quad (5)$$

Integrating Eq. (5) with accounting for the second equation of set (3), we derive

$$\frac{dA_p}{dz} = (\beta A_p - 2\alpha) A_p + C, \quad (6)$$

where the constant $C = -2\beta A_0^2 - (\beta A_{p0} - 2\alpha) A_{p0}$ and 0 subscripts designate the initial amplitudes at $z = 0$.

Again, separate variables in Eq. (6):

$$\frac{dA_p}{(\beta A_p - 2\alpha) A_p + C} = dz. \quad (7)$$

Let us now add the term $-\alpha$ in the numerator within the differential sign and transform the denominator:

$$\frac{d(\beta A_p - \alpha)}{(\beta A_p - \alpha)^2 - \alpha^2 + C\beta} = dz. \quad (8)$$

Integrate Eq. (8):

$$\frac{1}{2C_1} \ln \left| \frac{\beta A_p - \alpha - C_1}{\beta A_p - \alpha + C_1} \right| = z + C_2. \quad (9)$$

Here the coefficients $C_1 = \sqrt{\alpha^2 - C\beta}$, $C_2 = \frac{1}{2C_1} ln \left| \frac{\beta A_{p0} - \alpha - C_1}{\beta A_{p0} - \alpha + C_1} \right|$, and, as before, $C = -2\beta A_0^2 - (A_{p0}\beta - 2\alpha) A_{p0}$.

We can derive the equation for A_p from Eq. (9):

$$A_p(z) = C_1 \left(\frac{2}{1 + e^{2C_1(z+C_2)}} - 1 \right) + \frac{\alpha}{\beta}. \quad (10)$$

Finally, according to the solution of Eq. (5),

$$A(z) = \sqrt{2(A_{p0} - A_p(z)) \left(A_{p0} + A_p(z) - \frac{2\alpha}{\beta} \right) + A_0^2}. \quad (11)$$

Strictly speaking, we should write $\pm$ in Eq. (10) because of the presence of the modulus in Eq. (9), but the solution with the sign "–" is inappropriate for modes of light generation in a cavity.

Let us note that a similar analytical solution was derived in [18] with neglect of the loss α.

Let us now describe a parametric interaction in a cavity.

2. TWO-CAVITY INTERACTION SCHEME

Let us first consider the case of double resonance (Fig. 2), when the pumping wave freely passes through mirrors, in contrast to the signal and idler waves. The boundary reflection conditions on mirrors are to be taken into account. The delay can be ignored, since we consider the stationary case of perfectly exact resonance of monochromatic waves in a stationary mode. All amplitudes A are real in this case.

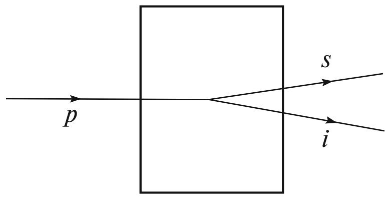

Fig. 1. The generation of a pair of photons in a nonlinear crystal.

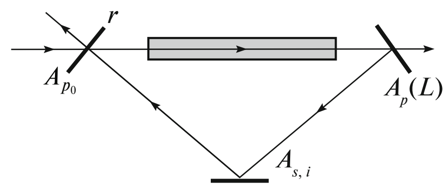

Fig. 2. *A* two-cavity interaction scheme: a pumping wave passes through mirrors without reflection, in contrast to signal and idler waves.

Numerical analysis of set (3) shows that the pump amplitude decreases linearly as z increases under real conditions of parametric light generation (Fig. 4). Therefore, taking the second equation of set (3) into account, one can take

$$A_p(z = L) = A_p(L) = A_{p0} - 2\beta A_0^2 L. \qquad (12)$$

Equation (11) with Eq. (12) implies the following relation:

$$\frac{A_0^2}{r^2} = 2(A_{p_0} - A_p(L))\left(A_{p_0} - A_p(L) - \frac{2\alpha}{\beta}\right) + A_0^2, \qquad (13)$$

where $A_p(L) = A_{p0} - 2\beta A_0^2 L$.

Let us substitute $A_p(L)$ in Eq. (13). Then

$$\frac{1 - r^2}{r^2 8\beta L} = A_{p0} - \beta A_0^2 L - \frac{\alpha}{\beta}. \qquad (14)$$

The non-negative solution of Eq. (14) has the form

$$A_0 = \sqrt{\frac{1}{r^2\beta L}\left(A_{p0} - \frac{\alpha}{\beta} - \frac{1 - r^2}{8r^2\beta L}\right)}. \qquad (15)$$

Since

$$A_{p0} - \frac{\alpha}{\beta} - \frac{1 - r^2}{8r^2\beta L} \geq 0 \qquad (16)$$

in the generation regime, then the threshold value of the pumping amplitude

$$A_{p0}^{th} = \frac{1}{\beta}\left(\alpha + \frac{1 - r^2}{8Lr^2}\right). \qquad (17)$$

In the general case of arbitrary z,

$$A_p(z) = C_1\left(\frac{2}{1 + exp(2C_1(z + C_2))} - 1\right) + \frac{\alpha}{\beta}, \qquad (18)$$

where C_1 is calculated like after Eq. (9), and A_0 is substituted from Eq. (15). At the same time, $A(z)$ is calculated by Eq. (11) with substitution of A_0 from Eq. (15) and $A_p(z)$ from Eq. (18).

The result is more accurate as compared to the well-known solutions [16, 17]. According to a computer experiment, this approximation is valid if the pumping power exceeds its threshold power

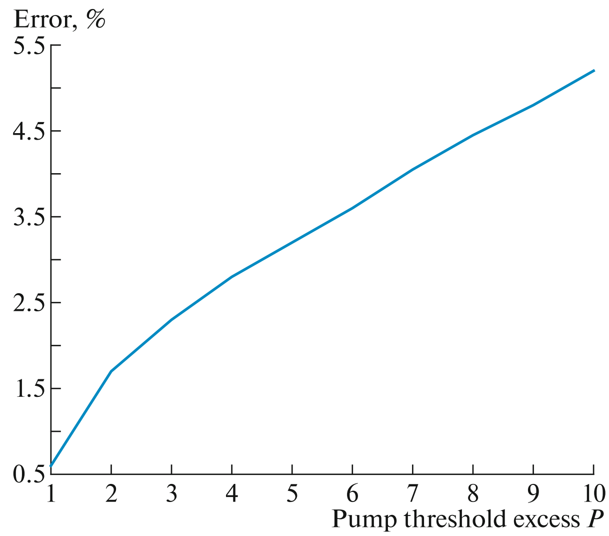

Fig. 3. The error of the theoretical approximation as a function of the parameter $P = \left(\frac{A_{p0}}{A_{p0}^{th}}\right)^2$ in the two-cavity case.

$P = \left(\frac{A_{p0}}{A_{p0}^{th}}\right)^2 \leq 10$. The error is less than 5% in this case (Fig. 3). In contrast to [16, 17], where the approximation of linear dependence of the amplitudes of all interaction waves on z was used, our assumption is weaker and provides a more accurate result. The calculation results are shown in Figs. 4 and 5. The analytical and numerical curves almost coincide in this case. This variant is not really usable, because there is no sense in waiting for the signal drop, as in Fig. 4, but we shown it to demonstrate the capabilities of our technique for describing not only a monotonous increase in the signal, but also its depletion.

3. THREE-CAVITY INTERACTION SCHEME

The mirrors were transparent for the pump in the two-cavity case. Now, all the three waves (s, i, p) pass round the cavity, like, e.g., in Fig. 6, and one more

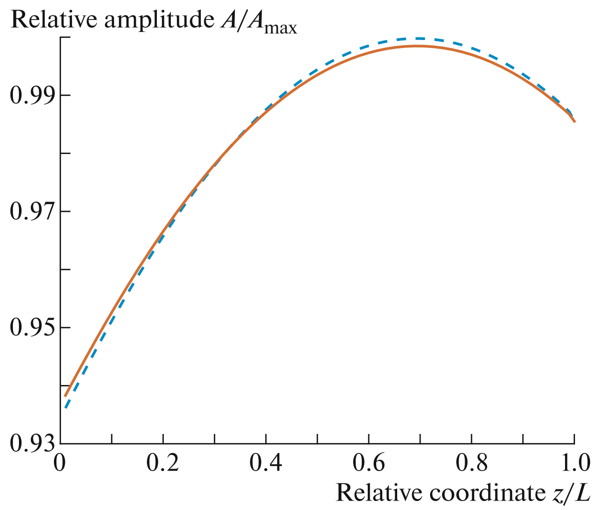

Fig. 4. The relative amplitude of idler and signal waves $\frac{A}{A_{max}}$ as a function of the relative coordinate z/L in the two-cavity interaction scheme at $\alpha L = 0.1$, $A_{p0}\beta L = 3.5$, and the reflection coefficient $r = 0.95$. The solid curve shows the result of numerical calculations; the dashed curve shows our analytical approximation.

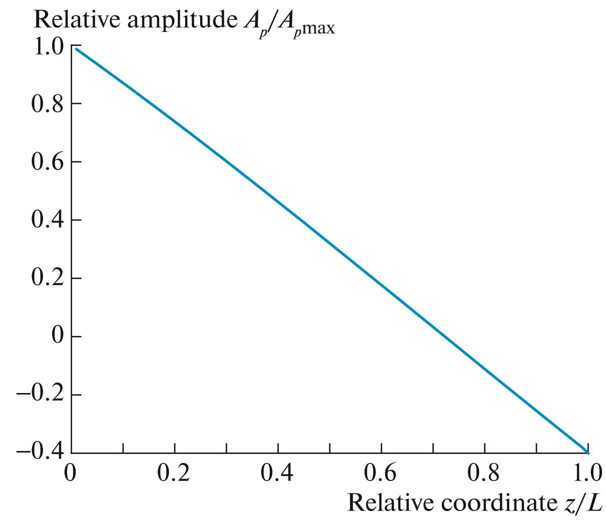

Fig. 5. The relative pumping amplitude $\frac{A_p}{A_{p\,max}}$ as a function of the relative coordinate z/L in the two-cavity interaction scheme under the same conditions as in Fig. 4.

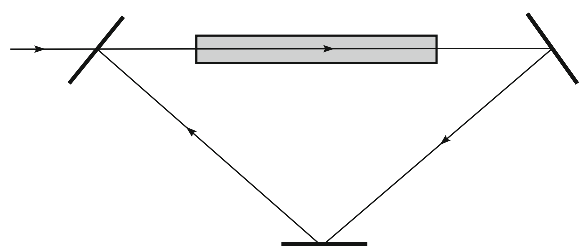

Fig. 6. The three-cavity interaction scheme: all the three waves reflect from mirrors.

equation is added that describes the boundary pumping conditions:

$$A_p(0;t) = r_p A_p(L;t-T) + A_p^0, \tag{19}$$

where $A_p^0 = A_p^{in}\sqrt{1-r_p^2}$, and A_p^{in} is the pump amplitude coming into the cavity from outside. As before, the delay is neglected.

According to numerical calculations, the pump amplitude also changes linearly in z under typical stationary generation conditions. This dependence is shown in Fig. 8.

Considering boundary conditions (2) and (19) and the first equation of set (3) and Eq. (11), we derive the set of algebraic equations

$$\begin{cases} rA(L) = A_0, \\ r_p A_p(L) = A_{p0} - A_p^0, \\ A_{p0} - 2\beta L A^2(L) = A_p(L), \\ A^2(L) = 2(A_{p0} - A_p(L))\left(A_p(L) + A_{p0} - \dfrac{2\alpha}{\beta}\right) + A_0^2. \end{cases} \tag{20}$$

Substituting $A_p(L)$ from the third equation of set (20) in the fourth one:

$$A^2(L) = 8\beta L A^2(L)\left(A_{p0} - \beta L A^2(L) - \frac{\alpha}{\beta}\right) + A_0^2 \tag{21}$$

and considering the first equation of this set, we derive

$$r^2 = 8\beta L r^2\left(A_{p0} - \beta L r^2 A_0^2 - \frac{\alpha}{\beta}\right) + 1. \tag{22}$$

Let us express A_0^2 from Eq. (21); considering the first equation from set (20), we have

$$A_0^2 = \frac{1}{\beta L r^2}\left(\frac{r^2-1}{8\beta L r^2} - \frac{\alpha}{\beta} + A_{p0}\right). \tag{23}$$

Here A_{p0} can be found from the second and third equations of set (20):

$$A_{p0}(1-r_p) = A_p^0 - 2\beta L r_p r^2 A_0^2. \tag{24}$$

The equation for A_0 can be derived from Eqs. (23) and (24):

$$A_0 = \sqrt{\frac{1}{\beta L r^2(1+r_p)}\left(A_p^0 - (1-r_p)\left(\frac{\alpha}{\beta} + \frac{r^2-1}{8\beta L r^2}\right)\right)}. \tag{25}$$

According to Eqs. (24) and (25),

$$A_{p0} = \frac{A_p^0\left(1 - \dfrac{2r_p}{1+r_p}\right) + \dfrac{2r_p(1-r_p)}{1+r_p}\left(\dfrac{\alpha}{\beta} + \dfrac{r^2-1}{8\beta L r^2}\right)}{1-r_p}. \tag{26}$$

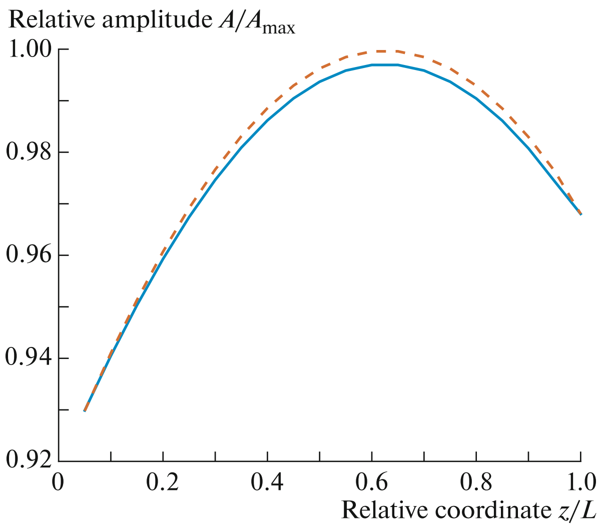

Fig. 7. The relative amplitude of idler and signal waves $\frac{A}{A_{max}}$ as a function of the relative coordinate z/L in the three-cavity interaction scheme at $\alpha L = 0.2$, $A_{p0}\beta L = 24$, and the reflection coefficients $r = 0.95$ and $r_p = 0.95$. The solid curve shows the numerical calculation result; the dashed curve shows our analytical approximation.

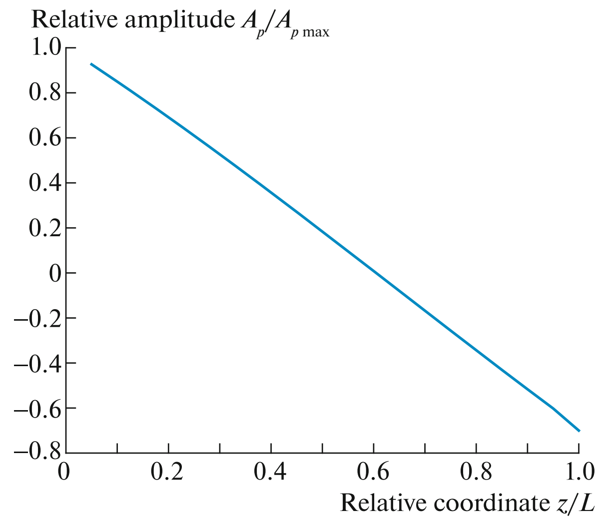

Fig. 8. The relative pump amplitude $\frac{A_p}{A_{pmax}}$ as a function of the relative coordinate z/L in the three-cavity interaction scheme under the same conditions as in Fig. 7.

Now, $A(z)$ and $A_p(z)$ can be calculated by Eqs. (10) and (11).

Similarly to Eq. (17), the threshold value of the pump amplitude

$$A_p^{0th} = (1 - r_p)\frac{1}{\beta}\left(\alpha + \frac{1 - r^2}{8Lr^2}\right). \tag{27}$$

The calculation result is shown in Figs. 7 and 8. The approximation is valid if the pumping power exceeds the threshold power $P = \left(\frac{A_{p0}}{A_{p0}^{th}}\right)^2 \le 8$. The error is less than 5% in this case (Fig. 9).

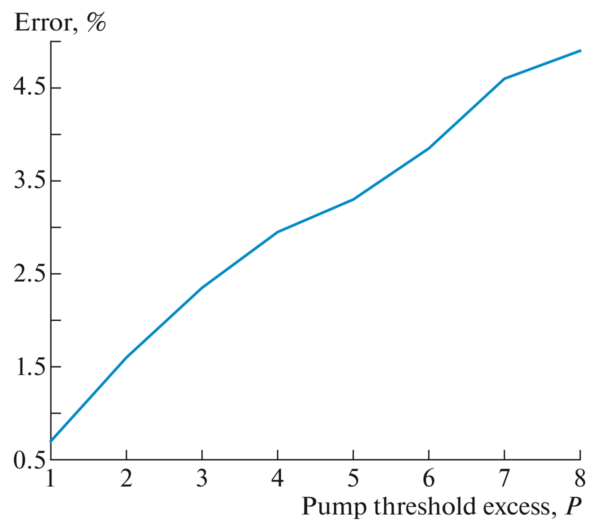

Fig. 9. The error of the theoretical approximation as a function of the parameter $P = \left(\frac{A_p^0}{A_p^{0th}}\right)^2$ in the three-cavity case.

CONCLUSIONS

In this work, we considered an analytical solution of the problem of parametric generation of light in a cavity. It significantly exceeds previously known solutions in accuracy, until the conditions for significant depletion of not only the pump, but also the signal. The possibility of this approximation follows from the numerical simulation results, which have shown that the pumping decreases linearly even under significantly nonlinear behavior of the signal along the nonlinear medium. Exactly this property has been used in our description, where the pumping is considered to be linear in z. As a result, our analytical solution is completely adequate, even at high values of the nonlinear interaction coefficient. This was confirmed by corresponding computations of the numerical solution of the initial set of equations without any simplifications.

The results of this work were reported at a conference [19] and at the 9th workshop devoted to the memory of D.N. Klyshko.

An alternative method of increasing the accuracy of the description of fields was suggested in [20], where both pump and generated waves were described in quadratic or cubic approximations.

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