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On the Motion of a Rigid Body with a Fixed Point in a Flow of Particles

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Abstract—The problem of motion of a rigid body with a fixed point in a free molecular flow of particles is considered. It is shown that the equations of motion of this body generalize the classical Euler—Poisson equations of motion of a heavy rigid body with a fixed point, and they are represented in the form of the classical Euler—Poisson equations in the case when the surface of the body in a flow of particles is a sphere. The existence of first integrals in the considered system is discussed.

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1. PROBLEM FORMULATION. COMPUTATION OF THE MOMENT ACTING ON A BODY WITH A FIXED POINT

We consider the problem of motion of a rigid body about a fixed point in a flow of particles. We assume that the flow of particles is a free molecular flow of constant density ρ whose particles are in translational motion with the constant absolute velocity v_0:

$$-\mathbf{v} = v_0\boldsymbol{\gamma},$$

where $\boldsymbol{\gamma}$ is the unit vector directed along the incoming flow. We ignore the thermal motion of molecules in the flow.

We consider the following mechanism of interaction between the molecules of the incoming flow and the surface of the body. A particle, having transferred almost all its energy to the body at collision, arrives at the temperature equilibrium with the location of impact (somewhat heated now). When heating is released, the particle moves towards the space with the thermal velocity equal to the thermal velocity of molecules of the body surface. Because this thermal velocity is considerably lower than the thermal velocity of external particles, this interaction can be simplistically described by the hypothesis of absolutely inelastic impact, when the particles lose its energy at collision with the body (and are not reflected).

We obtain the expressions for the force and moment acting on the body with a fixed point from the particle flow. We use the approach provided in the monograph by Beletskii [1]. Denote by O the fixed point of the rigid body. The distribution of velocities in the rigid body is determined by the Euler formula:

$$\mathbf{u}_M = [\boldsymbol{\omega} \times \boldsymbol{OM}],$$

where M is an arbitrary point of the rigid body and $\boldsymbol{\omega}$ is the absolute angular velocity of the rigid body. If we denote the angle between the vectors $\boldsymbol{\omega}$ and $\boldsymbol{OM}$ by α, then

$$|\mathbf{u}_M| = |\boldsymbol{\omega}|\,|\boldsymbol{OM}|\sin\alpha \leqslant |\boldsymbol{\omega}|\,|\boldsymbol{OM}|.$$

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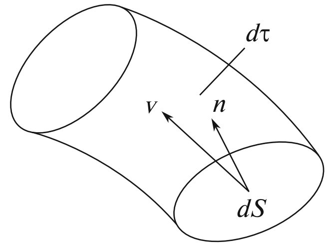

Fig. 1.

Assume that the value of the incoming flow velocity v_0 is considerably higher than the product of the characteristic value of the rigid body angular velocity and the characteristic distance from the fixed point to any point of the rigid body, that is,

$$\frac{|\boldsymbol{\omega}|\,|\boldsymbol{OM}|}{v_0} \ll 1. \tag{1}$$

Hence, we assume that in the absolute space the velocities of all points of the rigid body are zero. We determine the action of the flow on the body when the body is motionless and the flow has a constant velocity. We proceed to the coordinate system that translates together with the flow. In this coordinate system we follow the fixed point O of the rigid body (or its any other point due to assumption (1)). The absolute velocity $\mathbf{v}_O^{\mathrm{abs}}$ of the point O is zero, because O is the fixed point of the rigid body. The transfer velocity $\mathbf{v}_O^{\mathrm{trans}}$ of the point O is the absolute velocity of the point of the moving space (that is, the space that translates together with the chosen coordinate system) at which the point O is situated at the current time instance. This velocity is

$$\mathbf{v}_O^{\mathrm{trans}} = -\mathbf{v} = v_0\boldsymbol{\gamma}.$$

The relative velocity $\mathbf{v}_O^{\mathrm{rel}}$ of the point O is the velocity of the point O with respect to the flow. By the velocity addition formula we have

$$0 = \mathbf{v}_O^{\mathrm{abs}} = \mathbf{v}_O^{\mathrm{trans}} + \mathbf{v}_O^{\mathrm{rel}},$$

from which we find out that the point O (and, consequently, due to assumption (1) the entire body) moves relative to the flow with the velocity $\mathbf{v}_O^{\mathrm{rel}} = \mathbf{v} = -v_0\boldsymbol{\gamma}$.

We separate an elementary area dS on the body surface and compute an elementary momentum received by the area dS translating with respect to the flow with the velocity $\mathbf{v}$ for a time dt (Fig. 1). We assume the impact of particles to the body to be absolutely inelastic. In the course of such motion, the area covers the volume

$$d\tau = (\mathbf{v}\cdot\mathbf{n})\,dS\,dt,$$

where $\mathbf{n}$ is the unit normal vector to the area and $(\mathbf{v}\cdot\mathbf{n}) > 0$. Inside the volume $d\tau$ there is the mass $dm = \rho\,d\tau$, where ρ is the flow density. An elementary momentum received by the area and the force acting upon it have the form

$$d\mathbf{Q} = -\mathbf{v}\,dm = -\mathbf{v}\rho\,d\tau = -\rho\mathbf{v}\,(\mathbf{v}\cdot\mathbf{n})\,dS\,dt, \quad \mathbf{F} = \frac{d\mathbf{Q}}{dt} = -\rho\mathbf{v}\,(\mathbf{v}\cdot\mathbf{n})\,dS.$$

Consider a convex body bounded by a smooth closed surface and translating with the velocity $\mathbf{v} = -v_0\boldsymbol{\gamma}$ with respect to the flow. The force resultant of interaction between the body and the molecules is given by the formula

$$\mathbf{F} = -\int\limits_{S_*} \rho\mathbf{v}\,(\mathbf{v}\cdot\mathbf{n})\,dS, \tag{2}$$

where S_* denotes the part of the body surface passed over by the molecular flow: on its boundary $(\mathbf{v}\cdot\mathbf{n}) = 0$, because at the boundary the flow direction is tangent to S_* and in the internal points of

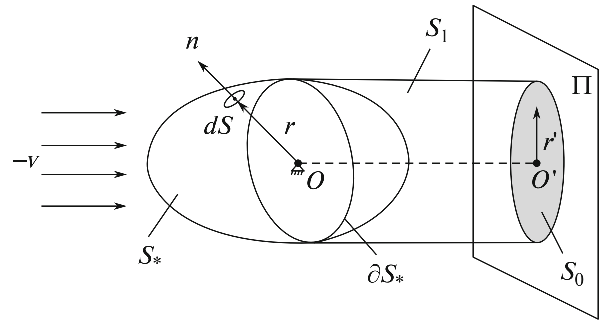

Fig. 2.

the surface S_* the outer normal $\mathbf{n}$ satisfies the inequality $(\mathbf{v}\cdot\mathbf{n}) > 0$. The boundary of this surface is denoted by ∂S_* (Fig. 2).

We assume that the direction of the velocity vector $\mathbf{v}$ is independent of the choice of the elementary area dS and, consequently, the integral in Eq. (2) can be rewritten as

$$\mathbf{F} = -\rho\mathbf{v}\int\limits_{S_*}(\mathbf{v}\cdot\mathbf{n})\,dS. \tag{3}$$

Now, let us compute the resultant moment of interaction forces between the molecules and the body relative to the fixed point O. This moment is calculated by the formula

$$\mathbf{M}_O = -\rho\int\limits_{S_*}[\mathbf{r}\times\mathbf{v}]\,(\mathbf{v}\cdot\mathbf{n})\,dS = \rho\left[\mathbf{v}\times\int\limits_{S_*}\mathbf{r}\,(\mathbf{v}\cdot\mathbf{n})\,dS\right], \tag{4}$$

where $\mathbf{r}$ is the position vector of a point of the body surface relative to the fixed point O.

To compute the integrals entering formulas (3) and (4), we introduce the new body T that we construct in the following manner. We place the plane Π perpendicular to the vector $\mathbf{v}$. It is convenient to place this plane at a certain distance to the point O behind (with respect to the vector $\mathbf{v}$) the body. The projection of the body onto the plane Π along the vector $\mathbf{v}$ (the orthogonal projection) is some planar figure S_0. In addition, we introduce a cylindrical surface S_1 with the generatrix $\mathbf{v}$ and the boundary ∂S_* as the directrix. On the one side, the surface S_1 is bounded by this directrix; on the other side, the surface is bounded by the line of intersection with the plane Π. The surface $\Sigma = S_*\bigcup S_1\bigcup S_0$ bounds the body T whose volume is denoted by τ (Fig. 2). According to the Gauss–Ostrogradsky theorem, the following relation is valid:

$$\int\limits_{\Sigma}(\mathbf{v}\cdot\mathbf{n})\,dS = \int\limits_{T}\operatorname{div}\mathbf{v}\,d\tau = 0,$$

because $\operatorname{div}\mathbf{v} = 0$. In addition to that, the relations hold:

$$(\mathbf{v}\cdot\mathbf{n})|_{S_1} = 0, \quad (\mathbf{v}\cdot\mathbf{n})|_{S_0} = -v_0\,(\boldsymbol{\gamma}\cdot\boldsymbol{\gamma}) = -v_0. \tag{5}$$

Hence,

$$\int\limits_{\Sigma}(\mathbf{v}\cdot\mathbf{n})\,dS = \int\limits_{S_*}(\mathbf{v}\cdot\mathbf{n})\,dS + \int\limits_{S_1}(\mathbf{v}\cdot\mathbf{n})\,dS + \int\limits_{S_0}(\mathbf{v}\cdot\mathbf{n})\,dS = 0,$$

and, consequently,

$$\int\limits_{S_*}(\mathbf{v}\cdot\mathbf{n})\,dS = -\int\limits_{S_0}(\mathbf{v}\cdot\mathbf{n})\,dS = v_0\int\limits_{S_0}dS = v_0 S,$$

where S is the area of the figure S_0. Thus,

$$\mathbf{F} = -\rho\mathbf{v}v_0 S = \rho v_0^2 S\boldsymbol{\gamma}. \tag{6}$$

We introduce the coordinate system $Oxyz$ with the origin in the fixed point O and the axes directed along the principal axes of inertia for the point O. Suppose that in this coordinate system $\mathbf{r} = (x,\, y,\, z)$, $\mathbf{v} = -v_0\,(\gamma_1,\, \gamma_2,\, \gamma_3)$. By the Gauss–Ostrogradsky theorem

$$\int\limits_{\Sigma} x\,(\mathbf{v}\cdot\mathbf{n})\,dS = \int\limits_{\Sigma} (x\mathbf{v}\cdot\mathbf{n})\,dS = \int\limits_{T} \operatorname{div}(x\mathbf{v})\,d\tau = -v_0\gamma_1\tau,$$

and, similarly,

$$\int\limits_{\Sigma} y\,(\mathbf{v}\cdot\mathbf{n})\,dS = -v_0\gamma_2\tau, \quad \int\limits_{\Sigma} z\,(\mathbf{v}\cdot\mathbf{n})\,dS = -v_0\gamma_3\tau.$$

Consequently,

$$\int\limits_{\Sigma} \mathbf{r}\,(\mathbf{v}\cdot\mathbf{n})\,dS = \tau\mathbf{v}.$$

On the other side, by formulas (5) we can write

$$\int\limits_{\Sigma} \mathbf{r}\,(\mathbf{v}\cdot\mathbf{n})\,dS = \int\limits_{S_*} \mathbf{r}\,(\mathbf{v}\cdot\mathbf{n})\,dS - v_0\int\limits_{S_0} \mathbf{r}\,dS.$$

On S_0 the vector $\mathbf{r}$ is the vector connecting the fixed point with various points of the figure S_0. Therefore, on S_0 we represent the vector $\mathbf{r}$ in the form

$$\mathbf{r} = -\frac{l\mathbf{v}}{|\mathbf{v}|} + \mathbf{r}' = l\boldsymbol{\gamma} + \mathbf{r}',$$

where l is the length of the normal from the fixed point onto the plane Π. For the vector $\mathbf{r}'$ the condition $(\mathbf{v}\cdot\mathbf{r}') = 0$ is met, because the vector $\mathbf{r}'$ lies in the plane Π (Fig. 2). Then,

$$v_0\int\limits_{S_0} \mathbf{r}\,dS = v_0 l\boldsymbol{\gamma}\int\limits_{S_0} dS + v_0\int\limits_{S_0} \mathbf{r}'dS = -lS\mathbf{v} + v_0\int\limits_{S_0} \mathbf{r}'dS = -lS\mathbf{v} + v_0\mathbf{P}_{O'}.$$

The integral

$$\mathbf{P}_{O'} = \int\limits_{S_0} \mathbf{r}'dS \tag{7}$$

is the first moment of the figure S_0 relative to the point O', the projection of the fixed point O onto the plane Π. Thus,

$$\tau\mathbf{v} = \int\limits_{S_*} \mathbf{r}\,(\mathbf{v}\cdot\mathbf{n})\,dS + lS\mathbf{v} - v_0\mathbf{P}_{O'}.$$

Hence,

$$\int\limits_{S_*} \mathbf{r}\,(\mathbf{v}\cdot\mathbf{n})\,dS = (\tau - lS)\,\mathbf{v} + v_0\mathbf{P}_{O'},$$

and, according to formula (4),

$$\mathbf{M}_O = \rho v_0\,[\mathbf{v}\times\mathbf{P}_{O'}] = -\rho v_0^2\,[\boldsymbol{\gamma}\times\mathbf{P}_{O'}]\,. \tag{8}$$

Now, we compute integral (7). In this integral the vector $\mathbf{r}'$ is the vector passed from the point O' to various points of the figure S_0. Suppose that the figure S_0 is an infinitely thin homogeneous plate with a density $\rho_1 = \mathrm{const}$ glued on the plane Π. Then,

$$\int\limits_{S_0} \mathbf{r}'dS = \frac{1}{\rho_1}\int\limits_{S_0} \rho_1\mathbf{r}'dS = \frac{\rho_1 S}{\rho_1}\,\boldsymbol{O'G} = S\cdot\boldsymbol{O'G}.$$

Here, $S = S(\boldsymbol{\gamma})$ is the area of the figure S_0 and $\boldsymbol{O'G}$ is the vector connecting the point O', projection of the fixed point O onto the plane Π, with the center of mass G of the plate bounded by the figure S_0. In the general case

$$S = S(\boldsymbol{\gamma}), \quad \boldsymbol{O'G} = \mathbf{c} = \mathbf{c}(\boldsymbol{\gamma}).$$

We also introduce the denotation $\rho v_0^2 = f$. As a result, formula (8) takes its final form

$$\mathbf{M}_O = -fS(\boldsymbol{\gamma})\left[\boldsymbol{\gamma} \times \mathbf{c}(\boldsymbol{\gamma})\right]. \tag{9}$$

Thus, we have obtained the expression for the moment acting on the rigid body with a fixed point occurring in a flow of particles. It is clear that this moment is independent of the flow direction passed over this body. Note that in the derivation of this formula we have used assumption (1). This means that formula (9) should be applied only in studying slow rotational movements of a body with a fixed point.

The equations of motion of a rigid body with a fixed point in a flow of particles have the form

$$\mathbb{J}\dot{\boldsymbol{\omega}} + [\boldsymbol{\omega} \times \mathbb{J}\boldsymbol{\omega}] = -fS(\boldsymbol{\gamma})\left[\boldsymbol{\gamma} \times \mathbf{c}(\boldsymbol{\gamma})\right], \quad \dot{\boldsymbol{\gamma}} + [\boldsymbol{\omega} \times \boldsymbol{\gamma}] = 0, \tag{10}$$

where $\mathbb{J} = \operatorname{diag}(A_1, A_2, A_3)$ is the tensor of inertia of the body relative to the fixed point O.

2. EXPLICIT EXPRESSION FOR THE MOMENT ACTING UPON THE BODY BOUNDED BY THE SPHERICAL AND ELLIPSOIDAL SURFACE

Consider some examples of computing the moment $\mathbf{M}_O$, determined by formula (9), for some bodies with simple geometry.

Example 1. Let us compute the moment $\mathbf{M}_O$, determined by formula (9), in the case when the body with a fixed point is bounded by the spherical surface of radius R and the fixed point is the center of this sphere. Then, the figure S_0 is a circle whose radius is equal to the radius of the sphere R. The area of this circle is constant and is equal to

$$S(\boldsymbol{\gamma}) = \pi R^2 = \text{const}.$$

It is clear that the center of mass of the homogeneous plate with a shape of the figure S_0 is located in the center of the circle. This means that the vector $\mathbf{c}(\boldsymbol{\gamma})$ connecting the point O', projection of the fixed point O onto the plane perpendicular to the flow, and the center mass of the plate vanishes in this case. Therefore, $\mathbf{M}_O = 0$.

Now, we compute the moment $\mathbf{M}_O$ in the case when the fixed point is chosen to be an arbitrary point O_1 inside the sphere. We introduce the coordinate system O_1xyz whose axes are directed along the principal axes of inertia of the body relative to the point O_1. Suppose that $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are unit basis vectors of this coordinate system. We denote the coordinates of the sphere center, point O, in the coordinate system O_1xyz by a_1, a_2, and a_3, that is,

$$\boldsymbol{O_1O} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3.$$

According to the well-known formula of the theoretical mechanics, we have

$$\mathbf{M}_{O_1} = \mathbf{M}_O - [\boldsymbol{OO_1} \times \mathbf{F}] = [\boldsymbol{O_1O} \times \mathbf{F}] = [\boldsymbol{O_1O} \times fS(\boldsymbol{\gamma})\boldsymbol{\gamma}] = f\pi R^2\left[\boldsymbol{O_1O} \times \boldsymbol{\gamma}\right].$$

Suppose that $\mathbf{M}_{O_1} = M_1\mathbf{e}_x + M_2\mathbf{e}_y + M_3\mathbf{e}_z$ in the coordinate system O_1xyz. Then,

$$M_1 = f\pi R^2(a_2\gamma_3 - a_3\gamma_2), \quad M_2 = f\pi R^2(a_3\gamma_1 - a_1\gamma_3), \quad M_3 = f\pi R^2(a_1\gamma_2 - a_2\gamma_1).$$

Equations (10) in the scalar form are written as

$$A_1\dot{\omega}_1 + (A_3 - A_2)\omega_2\omega_3 = f\pi R^2(a_2\gamma_3 - a_3\gamma_2), \quad A_2\dot{\omega}_2 + (A_1 - A_3)\omega_1\omega_3 = f\pi R^2(a_3\gamma_1 - a_1\gamma_3),$$
$$A_3\dot{\omega}_3 + (A_2 - A_1)\omega_1\omega_2 = f\pi R^2(a_1\gamma_2 - a_2\gamma_1);$$
$$\dot{\gamma}_1 = \omega_3\gamma_2 - \omega_2\gamma_3, \quad \dot{\gamma}_2 = \omega_1\gamma_3 - \omega_3\gamma_1, \quad \dot{\gamma}_3 = \omega_2\gamma_1 - \omega_1\gamma_2.$$

It is clear that in this case Eqs. (10) take the form of the classical Euler–Poisson equations describing the motion of a heavy rigid body about a fixed point. Consequently, we can consider the system of equations (10) to be a possible generalization of the classical Euler–Poisson equations.

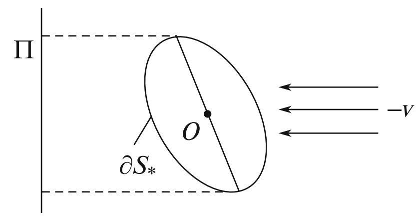

Fig. 3.

Example 2. We are going to compute the moment $\mathbf{M}_O$ acting upon a body with a fixed point when this body has the ellipsoidal shape and the fixed point coincides with the center of the ellipsoid. We direct the axes of the coordinate system $Oxyz$ with the origin in the fixed point O along the principal axes of inertia of the body relative to the point O. Suppose that the equation of ellipsoid in the coordinate system $Oxyz$ has the form

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} + \frac{z^2}{a_3^2} = 1. \tag{11}$$

This means that the principal axes of inertia $Oxyz$ are the principal axes of the ellipsoidal surface as well. We find the boundary ∂S_* (Figs. 2 and 3). The tangent plane to the ellipsoid at the point $(x,\, y,\, z)$ is given by the following equation for X, Y, and Z:

$$\frac{xX}{a_1^2} + \frac{yY}{a_2^2} + \frac{zZ}{a_3^2} = 1.$$

Suppose that a point with the coordinates $(x,\, y,\, z)$ belongs to the boundary ∂S_*. Then, the straight line

$$X = x + v_0\gamma_1 t, \quad Y = y + v_0\gamma_2 t, \quad Z = z + v_0\gamma_3 t$$

lies in the tangent plane to the body surface, that is, for any t it is true that

$$\frac{x}{a_1^2}\left(x + v_0\gamma_1 t\right) + \frac{y}{a_2^2}\left(y + v_0\gamma_2 t\right) + \frac{z}{a_3^2}\left(z + v_0\gamma_3 t\right) = 1.$$

Using (11), we therefore obtain

$$\frac{x\gamma_1}{a_1^2} + \frac{y\gamma_2}{a_2^2} + \frac{z\gamma_3}{a_3^2} = 0. \tag{12}$$

This equation together with the equation of ellipsoid (11) prescribes the boundary ∂S_*. It is well-known (see, for instance, [2]), any plane passing through the center of ellipsoid intersects the ellipsoid along an ellipse. Therefore, the section of the surface (11) by plane (12) is an ellipse. Let us find its area. It is equal to the product of the semiaxes of the ellipse multiplied by π. The squares of the semiaxes of the ellipse are extremums of the function $f = x^2 + y^2 + z^2$ under conditions (11) and (12). We use the method of Lagrange multipliers and consider the function

$$\mathcal{L} = x^2 + y^2 + z^2 + \lambda\left(\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} + \frac{z^2}{a_3^2} - 1\right) + \mu\left(\frac{x\gamma_1}{a_1^2} + \frac{y\gamma_2}{a_2^2} + \frac{z\gamma_3}{a_3^2}\right).$$

At the points of its extremum, the following equalities hold:

$$\frac{\partial\mathcal{L}}{\partial x} = \frac{\partial\mathcal{L}}{\partial y} = \frac{\partial\mathcal{L}}{\partial z} = 0.$$

We write these conditions in the explicit form:

$$2x + \frac{2\lambda x}{a_1^2} + \frac{\mu\gamma_1}{a_1^2} = 0, \quad x = -\frac{\mu\gamma_1}{2\left(a_1^2 + \lambda\right)}, \quad 2y + \frac{2\lambda y}{a_2^2} + \frac{\mu\gamma_2}{a_2^2} = 0, \quad y = -\frac{\mu\gamma_2}{2\left(a_2^2 + \lambda\right)},$$

$$2z + \frac{2\lambda z}{a_3^2} + \frac{\mu\gamma_3}{a_3^2} = 0, \quad z = -\frac{\mu\gamma_3}{2\left(a_3^2 + \lambda\right)}. \tag{13}$$

We substitute the found x, y, and z into Eq. (12):

$$\frac{\gamma_1^2}{a_1^2}\left(\lambda+a_2^2\right)\left(\lambda+a_3^2\right)+\frac{\gamma_2^2}{a_2^2}\left(\lambda+a_1^2\right)\left(\lambda+a_3^2\right)+\frac{\gamma_3^2}{a_3^2}\left(\lambda+a_1^2\right)\left(\lambda+a_2^2\right)=0.$$

Thus, λ satisfies the quadratic equation

$$k_0\lambda^2+k_1\lambda+k_2=0, \tag{14}$$

where

$$k_0=\frac{\gamma_1^2}{a_1^2}+\frac{\gamma_2^2}{a_2^2}+\frac{\gamma_3^2}{a_3^2},\quad k_2=a_1^2a_2^2a_3^2\left(\frac{\gamma_1^2}{a_1^4}+\frac{\gamma_2^2}{a_2^4}+\frac{\gamma_3^2}{a_3^4}\right).$$

It is not necessary to solve this equation. To clarify the meaning of λ, we multiply the equations of system (13) by x, y, and z, respectively, and sum,

$$0=2x^2+2y^2+2z^2+2\lambda\left(\frac{x^2}{a_1^2}+\frac{y^2}{a_2^2}+\frac{z^2}{a_3^2}\right)+\mu\left(\frac{x\gamma_1}{a_1^2}+\frac{y\gamma_2}{a_2^2}+\frac{z\gamma_3}{a_3^2}\right)=2\left(x^2+y^2+z^2+\lambda\right).$$

Therefore, we obtain $\lambda=-x^2-y^2-z^2$ at the extremum points. The area of the ellipse that is the section of the ellipsoid (11) by the plane (12) is $S_1=\pi\sqrt{\lambda_1\lambda_2}$, where λ_1 and λ_2 are roots of the quadratic equation (14). By Vieta's formulas

$$\lambda_1\lambda_2=\frac{k_2}{k_0}=a_1^2a_2^2a_3^2\frac{\left(\dfrac{\gamma_1^2}{a_1^4}+\dfrac{\gamma_2^2}{a_2^4}+\dfrac{\gamma_3^2}{a_3^4}\right)}{\left(\dfrac{\gamma_1^2}{a_1^2}+\dfrac{\gamma_2^2}{a_2^2}+\dfrac{\gamma_3^2}{a_3^2}\right)},$$

and, consequently,

$$S_1=\pi a_1a_2a_3\sqrt{\frac{\dfrac{\gamma_1^2}{a_1^4}+\dfrac{\gamma_2^2}{a_2^4}+\dfrac{\gamma_3^2}{a_3^4}}{\dfrac{\gamma_1^2}{a_1^2}+\dfrac{\gamma_2^2}{a_2^2}+\dfrac{\gamma_3^2}{a_3^2}}}.$$

The figure S_0 is the projection of the ellipse (11), (12) onto the plane Π; therefore, the area of this figure is

$$S\left(\boldsymbol{\gamma}\right)=S_1\frac{\left(\mathbf{N}\cdot\boldsymbol{\gamma}\right)}{|\mathbf{N}|},\quad \mathbf{N}=\left(\frac{\gamma_1}{a_1^2},\,\frac{\gamma_2}{a_2^2},\,\frac{\gamma_3}{a_3^2}\right).$$

Here, $\mathbf{N}$ is the normal vector to the plane (12), that is, to the plane in which the boundary ∂S_* lies. Thus, we conclude that

$$S\left(\boldsymbol{\gamma}\right)=\pi a_1a_2a_3\sqrt{\frac{\dfrac{\gamma_1^2}{a_1^4}+\dfrac{\gamma_2^2}{a_2^4}+\dfrac{\gamma_3^2}{a_3^4}}{\dfrac{\gamma_1^2}{a_1^2}+\dfrac{\gamma_2^2}{a_2^2}+\dfrac{\gamma_3^2}{a_3^2}}}\,\frac{\left(\dfrac{\gamma_1^2}{a_1^2}+\dfrac{\gamma_2^2}{a_2^2}+\dfrac{\gamma_3^2}{a_3^2}\right)}{\sqrt{\dfrac{\gamma_1^2}{a_1^4}+\dfrac{\gamma_2^2}{a_2^4}+\dfrac{\gamma_3^2}{a_3^4}}}=\pi a_1a_2a_3\sqrt{\frac{\gamma_1^2}{a_1^2}+\frac{\gamma_2^2}{a_2^2}+\frac{\gamma_3^2}{a_3^2}}.$$

According to (6), the components of the force resultant vector of interaction between the body and the molecules of the flow in the coordinate system $Oxyz$ have the form

$$F_1=f\pi a_1a_2a_3\gamma_1\sqrt{\frac{\gamma_1^2}{a_1^2}+\frac{\gamma_2^2}{a_2^2}+\frac{\gamma_3^2}{a_3^2}},\quad F_2=f\pi a_1a_2a_3\gamma_2\sqrt{\frac{\gamma_1^2}{a_1^2}+\frac{\gamma_2^2}{a_2^2}+\frac{\gamma_3^2}{a_3^2}},$$

$$F_3=f\pi a_1a_2a_3\gamma_3\sqrt{\frac{\gamma_1^2}{a_1^2}+\frac{\gamma_2^2}{a_2^2}+\frac{\gamma_3^2}{a_3^2}}.$$

The center of mass of the figure S_0 is at the point O', projection of the fixed point O onto the plane Π perpendicular to the flow. Hence, $\mathbf{c}\left(\boldsymbol{\gamma}\right) = 0$ and $\mathbf{M}_O = 0$.

Now, we compute the moment $\mathbf{M}$ relative to an arbitrary point O_1 belonging to the body. We introduce the coordinate system $O_1x_1x_2x_3$ whose axes are directed along the principal axes of inertia of the body relative to the point O_1. We still denote the components of the vector $\boldsymbol{\gamma}$ in the coordinate system $O_1x_1x_2x_3$ by γ_1, γ_2, and γ_3. We denote the coordinates of the point O in the coordinate system $O_1x_1x_2x_3$ in the following manner:

$$\boldsymbol{O_1O} = h_1\mathbf{e}_x + h_2\mathbf{e}_y + h_3\mathbf{e}_z.$$

According to the well-known formula of the theoretical mechanics, we have

$$\mathbf{M}_{O_1} = \mathbf{M}_O - [\boldsymbol{OO_1} \times \mathbf{F}] = [\boldsymbol{O_1O} \times \mathbf{F}] = [\boldsymbol{O_1O} \times fS\left(\boldsymbol{\gamma}\right)\boldsymbol{\gamma}] = fS\left(\boldsymbol{\gamma}\right)[\boldsymbol{O_1O} \times \boldsymbol{\gamma}].$$

Suppose that $\mathbf{M}_{O_1} = M_1\mathbf{e}_x + M_2\mathbf{e}_y + M_3\mathbf{e}_z$ in the coordinate system $O_1x_1x_2x_3$; then,

$$M_1 = f\pi a_1a_2a_3\sqrt{\frac{\gamma_1^2}{a_1^2} + \frac{\gamma_2^2}{a_2^2} + \frac{\gamma_3^2}{a_3^2}}\left(h_2\gamma_3 - h_3\gamma_2\right), \quad M_2 = f\pi a_1a_2a_3\sqrt{\frac{\gamma_1^2}{a_1^2} + \frac{\gamma_2^2}{a_2^2} + \frac{\gamma_3^2}{a_3^2}}\left(h_3\gamma_1 - h_1\gamma_3\right),$$

$$M_3 = f\pi a_1a_2a_3\sqrt{\frac{\gamma_1^2}{a_1^2} + \frac{\gamma_2^2}{a_2^2} + \frac{\gamma_3^2}{a_3^2}}\left(h_1\gamma_2 - h_2\gamma_1\right).$$

Thus, in the case of a flow of particles about a rigid body with a fixed point that is bounded by the ellipsoidal surface and whose principal axes coincide with the principal axes of inertia of the body with respect to the fixed point, the equations of motion of the body (10) are writte in the scalar form as

$$A_1\dot{\omega}_1 + \left(A_3 - A_2\right)\omega_2\omega_3 = f\pi a_1a_2a_3\sqrt{\frac{\gamma_1^2}{a_1^2} + \frac{\gamma_2^2}{a_2^2} + \frac{\gamma_3^2}{a_3^2}}\left(h_2\gamma_3 - h_3\gamma_2\right),$$
$$A_2\dot{\omega}_2 + \left(A_1 - A_3\right)\omega_1\omega_3 = f\pi a_1a_2a_3\sqrt{\frac{\gamma_1^2}{a_1^2} + \frac{\gamma_2^2}{a_2^2} + \frac{\gamma_3^2}{a_3^2}}\left(h_3\gamma_1 - h_1\gamma_3\right),$$
$$A_3\dot{\omega}_3 + \left(A_2 - A_1\right)\omega_1\omega_2 = f\pi a_1a_2a_3\sqrt{\frac{\gamma_1^2}{a_1^2} + \frac{\gamma_2^2}{a_2^2} + \frac{\gamma_3^2}{a_3^2}}\left(h_1\gamma_2 - h_2\gamma_1\right);$$

$$\dot{\gamma}_1 = \omega_3\gamma_2 - \omega_2\gamma_3, \quad \dot{\gamma}_2 = \omega_1\gamma_3 - \omega_3\gamma_1, \quad \dot{\gamma}_3 = \omega_2\gamma_1 - \omega_1\gamma_2.$$

Now, we present some conclusions about the structure of the moment $\mathbf{M}_O$ in the case of a flow of particles about a rigid body with a fixed point bounded by an axisymmetric surface.

3. FLOW ABOUT A BODY BOUNDED BY AN AXISYMMETRIC SURFACE

Suppose that the surface of a body passed over by a flow of particles is the surface of revolution. We are going to obtain the expression for the moment $\mathbf{M}_O$ in this case. In our reasoning we use some results of works [3, 4]. Suppose that a body with a fixed point is bounded by a surface of revolution whose equation in the principal axes of inertia with the origin in the fixed point is given by

$$F\left(x,\, y,\, z\right) = x^2 + y^2 - f\left(z\right) = 0. \tag{15}$$

Thus, the axis Oz is the axis of symmetry of this surface of revolution. We find the partial derivatives:

$$\frac{\partial F}{\partial x} = 2x, \quad \frac{\partial F}{\partial y} = 2y, \quad \frac{\partial F}{\partial z} = -\frac{df}{dz}.$$

We introduce the denotation

$$G\left(z\right) = \sqrt{x^2 + y^2 + \frac{1}{4}\left(\frac{df}{dz}\right)^2} = \sqrt{f\left(z\right) + \frac{1}{4}\left(\frac{df}{dz}\right)^2}.$$

Then, the normal vector to the surface takes the form

$$\mathbf{n} = \left(\frac{x}{G(z)}, \frac{y}{G(z)}, \frac{-\dfrac{1}{2}\dfrac{df}{dz}}{G(z)} \right).$$

The boundary of the region passed over by the flow of particles is determined by the equation of the surface (15) and by the equation $(\mathbf{n} \cdot \boldsymbol{\gamma}) = 0$, or, in the explicit form,

$$\gamma_1 x + \gamma_2 y = \frac{1}{2} \gamma_3 \frac{df}{dz}. \tag{16}$$

Now, we introduce the new coordinate system. Above, we have considered a system of principal axes of inertia that have coincided with the principal axes of the surface. We have denoted the unit basis vectors of this system by $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$. The new basis vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ are associated with the old ones $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, and $\boldsymbol{e}_z$ by the formulas

$$\mathbf{e}_1 = \frac{\gamma_2}{\sqrt{\gamma_1^2 + \gamma_2^2}} \mathbf{e}_x - \frac{\gamma_1}{\sqrt{\gamma_1^2 + \gamma_2^2}} \mathbf{e}_y, \quad \mathbf{e}_2 = \boldsymbol{\gamma} = \gamma_1 \mathbf{e}_x + \gamma_2 \mathbf{e}_y + \gamma_3 \mathbf{e}_z,$$
$$\mathbf{e}_3 = -\frac{\gamma_1 \gamma_3}{\sqrt{\gamma_1^2 + \gamma_2^2}} \mathbf{e}_x - \frac{\gamma_2 \gamma_3}{\sqrt{\gamma_1^2 + \gamma_2^2}} \mathbf{e}_y + \sqrt{\gamma_1^2 + \gamma_2^2}\, \mathbf{e}_z.$$

It is easy to see that the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are mutually orthogonal. This means that $\mathbf{e}_1$ and $\mathbf{e}_3$ are two mutually orthogonal unit vectors in the plane Π perpendicular to the flow. Suppose that $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$ is the position vector of some point of the surface relative to the old coordinate system. In new axes the same vector has the form $\mathbf{r} = x_1\mathbf{e}_1 + y_1\mathbf{e}_2 + z_1\mathbf{e}_3$. Then, the coordinates x, y, z and x_1, y_1, z_1 are linked by the relations

$$x = \frac{x_1 \gamma_2}{\sqrt{\gamma_1^2 + \gamma_2^2}} + \frac{\gamma_1}{\sqrt{\gamma_1^2 + \gamma_2^2}} \left(y_1 \sqrt{\gamma_1^2 + \gamma_2^2} - z_1 \gamma_3 \right),$$
$$y = -\frac{x_1 \gamma_1}{\sqrt{\gamma_1^2 + \gamma_2^2}} + \frac{\gamma_2}{\sqrt{\gamma_1^2 + \gamma_2^2}} \left(y_1 \sqrt{\gamma_1^2 + \gamma_2^2} - z_1 \gamma_3 \right), \quad z = y_1 \gamma_3 + z_1 \sqrt{\gamma_1^2 + \gamma_2^2}.$$

We substitute these relations into Eq. (15) of the surface and into Eq. (16) of the boundary and find out that the equation of the surface in the new coordinates becomes

$$x_1^2 + \left(y_1 \sqrt{\gamma_1^2 + \gamma_2^2} - z_1 \gamma_3 \right)^2 = f\left(y_1 \gamma_3 + z_1 \sqrt{\gamma_1^2 + \gamma_2^2} \right). \tag{17}$$

The equation of the boundary takes the form

$$\sqrt{\gamma_1^2 + \gamma_2^2} \left(y_1 \sqrt{\gamma_1^2 + \gamma_2^2} - z_1 \gamma_3 \right) = \frac{1}{2} \frac{df}{dz} \gamma_3. \tag{18}$$

We eliminate the coordinate y_1 from Eqs. (17) and (18) and obtain the constraint equation between x_1 and z_1, that is, the equation of the projection of the boundary onto the plane perpendicular to the flow. This projection bounds the region whose area enters the expression for the moment of forces acting upon the body with a fixed point. Let us clarify some properties of this area. From the general form of Eqs. (17) and (18), we can draw the following conclusions:

1. The value of this area depends only on the parameters of the surface itself and on the variable γ_3, the angle between the axis of symmetry of the body and the flow direction.

2. The curve bounding the projection is symmetrical with respect to the axis z_1, that is, the center of gravity of the projection necessarily lies on the axis z_1.

Thus, the area of the projection can be considered a function of the variable γ_3, that is, in this case $S(\boldsymbol{\gamma}) = S(\gamma_3)$. We take into account that the center of gravity of the projection lies on the axis z_1 and write the position vector $\mathbf{c}(\boldsymbol{\gamma})$ as follows:

$$\mathbf{c}(\boldsymbol{\gamma}) = c(\gamma_3)\, \mathbf{e}_3.$$

Now, we note that the vector $\mathbf{e}_3$ can be represented by

$$\mathbf{e}_3 = -\frac{\gamma_3}{\sqrt{\gamma_1^2+\gamma_2^2}}\,\boldsymbol{\gamma} + \frac{1}{\sqrt{\gamma_1^2+\gamma_2^2}}\,\mathbf{e}_z.$$

Consequently,

$$\mathbf{c}(\boldsymbol{\gamma}) = c(\gamma_3)\,\mathbf{e}_3 = -\frac{\gamma_3 c(\gamma_3)}{\sqrt{\gamma_1^2+\gamma_2^2}}\,\boldsymbol{\gamma} + \frac{c(\gamma_3)}{\sqrt{\gamma_1^2+\gamma_2^2}}\,\mathbf{e}_z,$$

and the expression for the moment of forces (9) can be rewritten as

$$\mathbf{M}_O = -fS(\gamma_3)\,\frac{c(\gamma_3)}{\sqrt{1-\gamma_3^2}}\,[\boldsymbol{\gamma}\times\mathbf{e}_z]\,.$$

Thus, we can think that, in the case of flow of particles about an axisymmetric body, we have

$$\mathbf{c}(\boldsymbol{\gamma}) = \frac{c(\gamma_3)}{\sqrt{1-\gamma_3^2}}\,\mathbf{e}_z$$

and $\mathbf{e}_z$ is the unit vector of the axis of the geometrical symmetry of the body. If the principal axes of inertial of the body do not coincide with the principal axes of the surface, then in the principal axes of inertia the unit vector of the axis of geometrical symmetry of the surface has the components $\mathbf{e}_z = \boldsymbol{\alpha} = (\alpha_1,\, \alpha_2,\, \alpha_3)$ and we should everywhere write $(\boldsymbol{\alpha}\cdot\boldsymbol{\gamma})$ instead of γ_3. Thus, the following statement is true.

Theorem 1. *In the case of flow of particles about a rigid body with a fixed point that is bounded by an axisymmetric surface, the unit vector of the axis of geometrical symmetry is*

$$\mathbf{e}_z = \boldsymbol{\alpha} = (\alpha_1,\, \alpha_2,\, \alpha_3)\,,$$

for the area $S(\boldsymbol{\gamma})$ of the figure S_0 and vector $\mathbf{c}(\boldsymbol{\gamma})$ the following formulas are valid:

$$S = S((\boldsymbol{\alpha}\cdot\boldsymbol{\gamma}))\,,\quad \mathbf{c}(\boldsymbol{\gamma}) = \frac{c((\boldsymbol{\alpha}\cdot\boldsymbol{\gamma}))}{\sqrt{1-(\boldsymbol{\alpha}\cdot\boldsymbol{\gamma})^2}}\,\boldsymbol{\alpha} = \chi((\boldsymbol{\alpha}\cdot\boldsymbol{\gamma}))\,\boldsymbol{\alpha}.$$

4. POTENTIAL PROPERTY OF MOMENT. EXISTENCE OF ENERGY-TYPE INTEGRAL. SIMPLEST CASES OF INTEGRABILITY

The equations of motion of a rigid body with a fixed point in a flow of particles have the form (10) and possess the integral invariant with a unit density and the first integrals

$$J_1 = (\mathbb{J}\boldsymbol{\omega}\cdot\boldsymbol{\gamma})\,,\quad J_2 = (\boldsymbol{\gamma}\cdot\boldsymbol{\gamma}) = 1.$$

Equations (10) are reversible, that is, hold at replacement of the variables and time $(\boldsymbol{\omega},\,\boldsymbol{\gamma},\,t)\to(-\boldsymbol{\omega},\,\boldsymbol{\gamma},\,-t)$. However, in the general case these equations are not a Hamilton system with some Poisson structure. The following statement is true:

Theorem 2. *If for any i, j, $i\neq j$ the relations*

$$c_i\,\frac{\partial S}{\partial\gamma_j} + \frac{\partial c_i}{\partial\gamma_j}\,S(\boldsymbol{\gamma}) = c_j\,\frac{\partial S}{\partial\gamma_i} + \frac{\partial c_j}{\partial\gamma_i}\,S(\boldsymbol{\gamma}) \tag{19}$$

are satisfied, then the equations of motion are Hamiltonian with the Poisson structure determined by the algebra $E(3)$ and admit an additional energy-type first integral.

Proof. Suppose that

$$S(\boldsymbol{\gamma})\,\mathbf{c}(\boldsymbol{\gamma}) = \frac{\partial U}{\partial\boldsymbol{\gamma}} \tag{20}$$

for a certain function $U(\boldsymbol{\gamma})$. Then, if the function U is sufficiently smooth, then for satisfaction of relations (20) it is necessary and sufficient that conditions (19) are met. In this case the equations of motion can be represented in the form

$$\dot{\mathbf{L}} = \{\mathbf{L}, H\}\,,\quad \dot{\boldsymbol{\gamma}} = \{\boldsymbol{\gamma}, H\}\,,$$

$$\{L_i, L_j\} = \varepsilon_{ijk} L_k, \quad \{L_i, \gamma_j\} = \varepsilon_{ijk}\gamma_k, \quad \{\gamma_i, \gamma_j\} = 0,$$

where the Hamilton function

$$H = \frac{1}{2}\left(\mathbb{J}^{-1}\boldsymbol{L}\cdot\mathbf{L}\right) - fU(\boldsymbol{\gamma}) \tag{21}$$

determines $J_0 = H$, which is the additional first integral of Eqs. (10), an analog of the energy integral.

It is clear that, when a body with a fixed point is bounded by a spherical surface, the area $S(\boldsymbol{\gamma})$ of the figure S_0 is constant and relations (19) are fulfilled. As we have already said, in this case the equations of motion of the body coincide with the equations of motion of a rigid body in a homogeneous force field.

However, relations (19) are fulfilled not often. For instance, consider the case when the body is bounded by an ellipsoidal surface (see Section 2) and the vector connecting the fixed point O_1 and the center of the ellipsoid O has the form $\boldsymbol{O_1O} = h\mathbf{e}_x = (h,\, 0,\, 0)$. In this case the expression $S(\boldsymbol{\gamma})\,\mathbf{c}(\boldsymbol{\gamma})$ is written as

$$S(\boldsymbol{\gamma})\,\mathbf{c}(\boldsymbol{\gamma}) = \pi h a_1 a_2 a_3\sqrt{\frac{\gamma_1^2}{a_1^2} + \frac{\gamma_2^2}{a_2^2} + \frac{\gamma_3^2}{a_3^2}}\,\mathbf{e}_x.$$

Relations (19) are met only if $a_2 = a_3$, that is, the ellipsoid bounding the rigid body is an ellipsoid of revolution and the fixed point lies on the axis of geometrical symmetry of the ellipsoid.

In the case of motion of a body bounded by the axisymmetric surface, when the axis of symmetry is determined by the vector $\boldsymbol{\alpha} = (\alpha_1,\, \alpha_2,\, \alpha_3)$ and includes the fixed point, Theorem 1 is valid. In this case, as we know, the conditions are met:

$$\frac{\partial U}{\partial \gamma} = S(\boldsymbol{\gamma})\,\mathbf{c}(\boldsymbol{\gamma}) = \frac{S((\boldsymbol{\alpha}\cdot\boldsymbol{\gamma}))\,c((\boldsymbol{\alpha}\cdot\boldsymbol{\gamma}))}{\sqrt{1-(\boldsymbol{\alpha}\cdot\boldsymbol{\gamma})^2}}\,\boldsymbol{\alpha}.$$

It is clear that in this case the potential $U(\boldsymbol{\gamma})$ can be represented as

$$U(\boldsymbol{\gamma}) = \int\limits_0^{(\boldsymbol{\alpha}\cdot\boldsymbol{\gamma})} \frac{S(u)\,c(u)}{\sqrt{1-u^2}}\,du = U((\boldsymbol{\alpha}\cdot\boldsymbol{\gamma}))\,.$$

Thus, when a body bounded by an axisymmetric surface is passed over by a flow of particles, the equations of motion (10) always admit the first integral of type (21).

Let us specify some cases when the equations of motion of a rigid body in a flow of particles (10) possess the additional integral.

Euler–Poinsot case. Suppose that the surface of the body is centrally symmetric and the center of symmetry coincides with the suspension point. Then, Eqs. (10) admit the integral $J_3 = (\mathbb{J}\boldsymbol{\omega}\cdot\mathbb{J}\boldsymbol{\omega})$. In this case the problem is completely integrable and coincides with the Euler–Poinsot problem.

Case of axial symmetry. Suppose that the body is dynamically symmetric, that is, for instance, the condition $A_1 = A_2$ is met. Also, suppose that the surface of the body is centrally symmetric and the center of symmetry lies on the axis Ox_3. Then, the equations of motion admit the first integral $J_3 = \omega_3 = \mathrm{const}$. This case is similar to the Lagrange case.

Analogs of Hess case. 1. Suppose that the surface of the body is centrally symmetric and the center of symmetry and the moments of inertia are such that the conditions are valid:

$$A_1 < A_2 < A_3, \quad \sqrt{\frac{1}{A_1} - \frac{1}{A_2}}\,c_3 \mp \sqrt{\frac{1}{A_2} - \frac{1}{A_3}}\,c_1 = 0, \quad c_2 = 0.$$

Then, the equations of motion admit the partial integral

$$J_3 = \sqrt{\frac{1}{A_1} - \frac{1}{A_2}}\,A_1\omega_1 \pm \sqrt{\frac{1}{A_2} - \frac{1}{A_3}}\,A_3\omega_3 = 0. \tag{22}$$

2. Suppose that the surface of the body is axisymmetric and the axis of symmetry is determined by the vector $\boldsymbol{\alpha} = (\alpha_1,\, \alpha_2, \alpha_3)$ and includes the fixed point. Then, if the moment of inertia and the components of the vector $\boldsymbol{\alpha}$ satisfy the conditions

$$A_1 < A_2 < A_3, \quad \sqrt{\frac{1}{A_1} - \frac{1}{A_2}}\,\alpha_3 \mp \sqrt{\frac{1}{A_2} - \frac{1}{A_3}}\,\alpha_1 = 0, \quad \alpha_2 = 0,$$

then the equations of motion admit the partial integral (22).

Thus, in the considered mechanical system we have succeeded to detect particularly interesting dynamic properties.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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