ISSN 0027-1330, Moscow University Mechanics Bulletin, 2018, Vol. 73, No. 5, pp. 101–116. © Allerton Press, Inc., 2018.
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The General Mathematical Theory of Plasticity and the Il'yushin Postulates of Macroscopic Definability and Isotropy

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Received November 3, 2017; in final form, June 23, 2018

Abstract—The physical laws characterizing the relation between stresses and strains are considered and analyzed in the general modern theory of elastoplastic deformations and in its postulates of macroscopic definability and isotropy for initially isotropic continuous media. The fundamentals of this theory in continuum mechanics were developed by A.A. Il'yushin in the mid-twentieth century. His theory of small elastoplastic deformations under simple loading became a generalization of Hencky's deformation theory of flow, whereas his theory of elastoplastic processes which are close to simple loading became a generalization of the Saint-Venant–Mises flow theory to the case of hardening media. In these theories, the concepts of simple and complex loading processes and the concept of directing form change tensors are introduced; the Bridgman law of volume elastic change and the universal Roche–Eichinger laws of a single hardening curve under simple loading are adopted; and the Odquist hardening for plastic deformations is generalized to the case of elastoplastic hardening media for the processes of almost simple loading without consideration of a specific history of deformations for the trajectories with small and mean curvatures. In this paper we discuss the possibility of using the isotropy postulate to estimate the effect of forming parameters in the stress-strain state appeared due to the strain-induced anisotropy during the change of the internal structures of materials. We also discuss the possibility of representing the second-rank symmetric stress and strain tensors in the form of vectors in the linear coordinate six-dimensional Euclidean space. An identity principle is proposed for tensors and vectors.

DOI: 10.3103/S0027133018050011

1. THE STRESS–STRAIN STATE AND ITS INVARIANTS

A new direction in the mathematical theory of plasticity under the complex stress–strain state and loading was proposed by A.A. Il'yushin in [1–11]. His new theory became known as the theory of elastoplastic deformation processes and was based on the theory of plasticity and on experimental studies of materials [12–14]. The application of linear algebra and Euclidean tensor spaces allows one to adequately describe the processes of deformation and loading of materials at each pont of physical space [15–17]. This new approach in the theory of plasticity was widely discussed by the specialists in the plasticity and flow theories [18–21]. In [10] another important discussion is described among the specialists in mechanics and physics as well as in metallurgy. The subject matter of this discussion was the effect of structural changes in materials on their mechanical characteristics. The above discussions were important for the development of the plasticity theory and for the study of stress–strain states in the processes of elastoplastic deformation of materials. At a later time, the new theory was intensively developed in [22–54]. In this paper we consider some new concepts necessary for the further development of the theory of elastoplastic deformation processes.

Let us consider the body's point defined by the radius vector $\overline{x} = x_i \hat{e}_i$ $(i = 1, 2, 3)$, where x_i are the coordinates of this point and $\{\hat{e}_i\}$ is the orthonormal coordinate basis of physical space. At this point, the stress–strain state is characterized by the stress tensor (σ_{ij}) and by the strain tensor (ε_{ij}), where σ_{ij} and ε_{ij} are their components $(i, j = 1, 2, 3)$. In the process of loading by external forces, the stresses and strains

$$\sigma_{ij} = \sigma_{ij}(\overline{x}, t), \quad \varepsilon_{ij} = \varepsilon_{ij}(\overline{x}, t)$$

are continuous functions of time t [1–6, 25–27]. In the plasticity theory, the tensors (σ_{ij}) and (ε_{ij}) are decomposed into the spherical tensors and the deviators [4–6, 11]:

$$(\sigma_{ij}) = S(\sigma^*_{ij}) = \sigma_0(\delta_{ij}) + \sigma(S^*_{ij}), \quad (\varepsilon_{ij}) = \varepsilon(\varepsilon^*_{ij}) = \varepsilon_0(\delta_{ij}) + Э(Э^*_{ij}).$$

Here δ_{ij} is the Kronecker symbol. The deviator components are of the form

$$S_{ij} = \sigma S^*_{ij} = \sigma_{ij} - \delta_{ij}\sigma_0, \quad Э_{ij} = ЭЭ^*_{ij} = \varepsilon_{ij} - \delta_{ij}\varepsilon_0.$$

The moduli of the tensors (σ_{ij}) and (ε_{ij}) and the moduli of the spherical tensors and deviators can be written as

$$\sigma_0 = \frac{1}{3}\sigma_{ij}\delta_{ij}, \quad \varepsilon_0 = \frac{1}{3}\varepsilon_{ij}\delta_{ij}; \quad \sigma = \sqrt{S_{ij}S_{ij}}, \quad Э = \sqrt{Э_{ij}Э_{ij}},$$

$$S = \sqrt{\sigma_{ij}\sigma_{ij}} = \sqrt{3\sigma_0^2 + \sigma^2}, \quad \varepsilon = \sqrt{\varepsilon_{ij}\varepsilon_{ij}} = \sqrt{3\varepsilon_0^2 + Э^2}.$$

The components of the direction tensors (σ^*_{ij}) and (ε^*_{ij}) are related as

$$\sigma^*_{ij}\sigma^*_{ij} = 1, \quad \varepsilon^*_{ij}\varepsilon^*_{ij} = 1 \quad (i, j = 1, 2, 3).$$

Each of these tensors is specified by five independent quantities defining the vector properties of materials. The components of the tensor deviators are related as

$$S^*_{ii} = 0, \quad S^*_{ij}S^*_{ij} = 1; \quad Э^*_{ii} = 0, \quad Э^*_{ij}Э^*_{ij} = 1.$$

Each of these deviators is specified by four independent quantities defining their vector properties. In the case of simple loading, we have

$$(\sigma^*_{ij}) = (\varepsilon^*_{ij}), \quad (S^*_{ij}) = (Э^*_{ij}).$$

In the case of complex loading, we have $(\sigma^*_{ij}) \neq (\varepsilon^*_{ij})$ and $(S^*_{ij}) \neq (Э^*_{ij})$ [1, 25]. The moduli of the tensors $S = |\overline{S}|$ and $\varepsilon = |\overline{\varepsilon}|$ characterize the scalar properties of materials. The stress tensor (σ_{ij}) is characterized by the following three vectors on the three mutually orthogonal coordinate areas at a given point on the general position area with the unit normal $\hat{n} = n_i\hat{e}_i$, where $n_i = \cos(\hat{n}, \hat{e}_i)$:

$$\overline{S}_i = \sigma_{ji}\hat{e}_j, \quad \overline{S}_j = \sigma_{ij}\hat{e}_i \quad (i, j = 1, 2, 3). \tag{1}$$

On the general position area, the stress vector $\overline{S}_n$ is specified by the Cauchy formula [24]

$$\begin{cases} \overline{S}_n = \overline{S}_i n_i = X_i\hat{e}_i = (\sigma_{ij}n_j)\hat{e}_i, \\ X_n = \sigma_{ij}n_j \quad (i, j = 1, 2, 3), \end{cases} \tag{2}$$

where X_n are the projections of the vector onto the coordinate axes x_i. In the case of the fixed vector $\overline{S}_n$ and after rotation of these coordinate axes, their new position is characterized by the following transformation:

$$x'_i = l_{ij}x_j, \quad \hat{e}'_i = l_{ij}\hat{e}_j \quad \hat{e}_i = l_{ji}\hat{e}'_j \quad (i, j = 1, 2, 3).$$

Here (l_{ij}) is the direction cosine matrix. After rotation of the coordinate axes, the fixed vector takes the form

$$\overline{S}'_i = l_{ij}\overline{S}_j, \quad \overline{S}_n = X_i\hat{e}_i = X'_j\hat{e}'_j.$$

Taking into account (1) and rewriting indices, we get

$$\overline{S}_q = l_{qj}\overline{S}_j = (\sigma_{ij}l_{pi}l_{qj})\hat{e}'_p \quad (p, q, i, j = 1, 2, 3). \tag{3}$$

From (3) we come to the following formula for the transformation of the stress tensor components in the new coordinate axes:

$$\sigma'_{pq} = \sigma_{ij}l_{pi}l_{qj} = \sigma_0\delta_{pq} + S_{ij}l_{pi}l_{qj}.$$

In a similar way, for the strain tensor we obtain

$$\varepsilon'_{pq} = \varepsilon_{ij}l_{pi}l_{qj} = \varepsilon_0\delta_{pq} + Э_{ij}l_{pi}l_{qj}.$$

During the rotation of the coordinate axes, there exists their position such that the shear stresses σ_{ij} $(i \neq j)$ are equal to zero and the normal stresses take the extreme values σ_k $(k = 1, 2, 3)$ known as the principal or eigen stresses. In this case we have

$$\overline{S}_n = \sigma_k(\delta_{ij}n_j)\hat{e}_i \quad (i, j = 1, 2, 3). \tag{4}$$

Comparing (4) and (2), we come to the following system of equations:

$$(\sigma_{ij} - \delta_{ij}\sigma_k)n_j = 0. \tag{5}$$

The additional relation is $n_j n_j = n_1^2 + n_2^2 + n_3^2 = 1$. From this relation it follows that this system has nonzero solutions and, hence, its determinant is equal to zero:

$$D = -|\sigma_{ij} - \delta_{ij}\sigma_k| = \sigma_k^3 - I_1\sigma_k^2 + I_2\sigma_k - I_3 = 0. \tag{6}$$

Here the coefficients are invariant and are of the form

$$\begin{cases} I_1 = \sigma_{ii} = \sigma_1 + \sigma_2 + \sigma_3 = 3\sigma_0, \quad S^2 = 3\sigma_0^2 + \sigma^2; \\ I_2 = \dfrac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}) = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 = \dfrac{1}{2}(9\sigma_0^2 - S^2); \\ I_3 = |\sigma_{ij}| = \sigma_1\sigma_2\sigma_3. \end{cases} \tag{7}$$

In the theory of plasticity, it is convenient to consider the following invariant instead of the invariant I_2:

$$2I_2' = \sigma_{ij}\sigma_{ij} = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = S^2.$$

The invariants of the stress deviator can be determined from an equation similar to (6) and are of the from

$$\begin{cases} J_1 = S_{11} + S_{22} + S_{33} = S_1 + S_2 + S_3 = 0, \\ 2J_2 = S_{ij}S_{ij} = S_1^2 + S_2^2 + S_3^2 = \sigma^2, \\ J_3 = |S_{ij}| = I_3 - \sigma_0^3 + \dfrac{1}{2}\,\sigma_0\sigma^2 = \dfrac{\sigma^3\cos 3\varphi}{3\sqrt{6}}, \end{cases} \tag{8}$$

where $S_k = \sigma_k - \sigma_0$ are the principal stresses of the stress deviator. Similar formulas are valid for the invariants of the strain tensors and the strain deviators. The principal normal and shear stresses are of the form

$$\begin{cases} S_1 = \sigma_1 - \sigma_0 = \sqrt{\dfrac{2}{3}}\,\sigma\cos\varphi, \\ S_2 = \sigma_2 - \sigma_0 = \sqrt{\dfrac{2}{3}}\,\sigma\cos\left(\dfrac{2\pi}{3} - \varphi\right), \\ S_3 = \sigma_3 - \sigma_0 = \sqrt{\dfrac{2}{3}}\,\sigma\cos\left(\dfrac{2\pi}{3} + \varphi\right), \end{cases} \qquad \begin{cases} T_{12} = \dfrac{\sigma_1 - \sigma_2}{2} = \dfrac{\sigma}{\sqrt{2}}\sin\left(\dfrac{2\pi}{3} + \varphi\right), \\ T_{23} = \dfrac{\sigma_2 - \sigma_3}{2} = \dfrac{\sigma}{\sqrt{2}}\sin\varphi, \\ T_{13} = \dfrac{\sigma_1 - \sigma_3}{2} = \dfrac{\sigma}{\sqrt{2}}\sin\left(\dfrac{2\pi}{3} - \varphi\right). \end{cases}$$

If the coordinate axes are rotated, the stress tensor remains unchanged as a physical quantity, although its components are changed. This covariance of tensor components under coordinate transformations can be considered as a definition of the stress tensor if its three invariants remain the same. A similar situation takes place for the strain tensor. In [13] the concept of the octahedral stresses $\sigma_{\rm oct} = \sigma_0$ and

$$\tau_{\rm oct} = \frac{2}{3}\sqrt{T_{12}^2 + T_{23}^2 + T_{13}^2} = \frac{\sigma}{\sqrt{3}}$$

are introduced. Here $\tau_{\rm oct}$ is proportional to the modulus of the stress deviator σ. These stresses are the same on all areas of a mechanical particle of octahedron shape. On each area, the direction of the shear stress is specified by the forming angle φ measured from the projection of the first principal axis onto this area using the formula

$$\cos 3\varphi = \frac{3\sqrt{6}|S_{ij}|}{\sigma^3}.$$

In the case of strains, the forming angle ψ is determined by the formula

$$\cos 3\psi = \frac{3\sqrt{6}|Э_{ij}|}{Э^3}.$$

On an octahedron area, the stress vector $\overline{S}_n^0$ makes an angle α_0 with the unit normal to this area. Then, the projections of this vector onto the unit normal $\hat{n}$ and onto the unit tangent vector $\hat{t} \perp \hat{n}$ are as follows:

$$\sigma_0 = S_n^0\cos\alpha_0 = \frac{S}{\sqrt{3}}\cos\alpha_0, \quad \tau_{\rm oct} = \frac{\sigma}{\sqrt{3}} = S_n^0\sin\alpha_0 = \frac{S}{\sqrt{3}}\sin\alpha_0. \tag{9}$$

Here

$$S_n^0 = \sqrt{\sigma_0^2 + \tau_{\text{oct}}^2} = \frac{S}{\sqrt{3}}, \quad S = \sqrt{3\sigma_0^2 + \sigma^2}.$$

From (9) we obtain the following parameter characterizing the deviation of $\overline{S}_n$ from $\hat{n}$:

$$\chi = \cot\alpha_0 = \frac{\sqrt{3}\,\sigma_0}{\sigma} = \frac{3\sigma_0}{\tau_{\text{oct}}}.$$

The stress tensor invariants (7) can be represented as

$$\begin{cases} I_1 = 3\sigma_0 = \sqrt{3}S\cos\alpha_0, \quad S^2 = 3\sigma_0^2 + \sigma^2, \\ 2I_2 = 9\sigma_0^2 - S^2 = S^2(3\cos^2\alpha_0 - 1), \\ I_3 = \dfrac{S^3}{3\sqrt{6}}\left[\cos 3\varphi \sin^3\alpha_0 + \sqrt{2}\cos^3\alpha_0 - \dfrac{3}{\sqrt{2}}\sin^2\alpha_0\right]. \end{cases} \tag{10}$$

According to (7), (8), and (10), thus, the stress tensor can be specified by the six components σ_{ij}, or by the three principal stresses σ_k, or by the three invariants I_k and the three principal directions (the three Eulerian angles) as well as by the three invariants σ_0, σ, and J_3 or S, α_0, and φ and by the three principal directions (the three Eulerian angles). The tensor modulus S characterizes the scalar properties of a material, whereas the three Eulerian angles and the two aspect angles α_0 and φ of the stress–strain state characterize the vector properties of this material. A similar situation takes place for the strain tensor. In the case of uniform tension and compression, we have $\alpha_0 = 0$, $\tau_{\text{oct}} = 0$, and $\sigma_0 = S/\sqrt{3}$. Then,

$$I_1 = 3\sigma_0, \quad I_2 = 3\sigma_0^2, \quad I_3 = \frac{S^3}{3\sqrt{3}} = \sigma_0^3.$$

In the case of uniform forming, we have $\alpha_0 = \pi/2$, $3\sigma_0 = \sigma_1 + \sigma_2 + \sigma_3 = 0$, and $\tau_{\text{oct}} = S/\sqrt{3} = \sigma/\sqrt{3}$. Then,

$$I_1 = J_1 = 0, \quad 2I_2 = -S^2 = -2J_2 = -\sigma^2, \quad I_3 = J_3 = \frac{\sigma^3\cos 3\varphi}{3\sqrt{6}}.$$

Now we consider the tensor transformation when the coordinate axes and the coordinate basis $\{\hat{e}_i\}$ are fixed. In this case, $x_i = \text{const}$ and the stress tensor and the vector $\overline{S}_n = X_i\hat{e}_i$ are transformed as

$$\overline{S}'_n = (\alpha_{ij})\overline{S}_n = X'_i\hat{e}'_i, \quad \alpha_{ij} = l_{ij} \quad (i,j = 1,2,3)$$

if the lengths of vectors remain the same:

$$X'_iX'_i = X_jX_j, \quad \overline{X}'_i = \alpha_{ij}X_j = \alpha_{ik}X_k, \quad X_j = \delta_{jk}X_k \quad (i,j,k = 1,2,3).$$

From here we obtain

$$\alpha_{ij}\alpha_{ik} = \delta_{jk}, \quad |\alpha_{ij}\alpha_{ik}| = 1, \quad |\alpha_{ij}| = \pm 1,$$

where the plus sign corresponds to the vector rotation, whereas the minus sign corresponds to the vector reflection. The first and third invariants remain indefinite. Their changes may be possible, since in a particle the physical processes are different under transformation and correspond to modified stress–strain states. Although the transformation matrices are coincident for the tensor and for the coordinate axes ($\alpha_{ij} = l_{ij}$), the tensor transformations correspond to a new physical state.

In an initially isotropic body, the appearance of plastic deformations is usually associated with the Tresca–Saint-Venant criterion [30, 49]

$$T_{\max} = T_{mn} = \frac{\sigma_m - \sigma_n}{2} = \begin{cases} k = \sigma_T/2, \\ k_0 = \sigma_T/\sqrt{3} \end{cases} \quad (m < n;\ m,n = 1,2,3) \tag{11}$$

or with the Mises–Nadai criterion [13, 30, 47, 49]

$$\sigma = \sqrt{3}\,\tau_{\text{oct}} = \frac{2}{\sqrt{3}}\sqrt{T_{12}^2 + T_{23}^2 + T_{13}^2} = \sigma^T. \tag{12}$$

Here k is the shear yield stress in the case of spatially complex shear in the Tresca sense, k_0 is the shear yield stress in the case of simple shear, and σ^T given in (12) is

$$\sigma^T = \sqrt{\frac{2}{3}}\,\sigma_T = \sqrt{6}k_* = \sqrt{\frac{2}{3}}\,(2k), \tag{13}$$

where σ_T is the tensile yield stress and k_* is the spatial pure shear yield stress [30, 47, 49]. In [14] it is stated that, in the Tresca experiments, the distribution of shear stresses is not uniform and is not accurate enough. In [13] a geometrically and physically clear explanation is given to the Mises criterion (12); according to this criterion, the transition of a material from the elastic state to the plastic state is observed when, on all areas, the value of τ_{oct} simultaneously becomes equal to the limit value $\tau_{\text{oct}} = k_{\text{oct}} = \sqrt{2}\,k_*$, since $\tau_{\text{oct}} < T_{\max} = 1.5\,k_*$.

The concepts of full and partial plasticity were introduced by Haar and Kármán in [12]. The Tresca plasticity criterion (11) can be represented in the following more general form:

$$(T_{12}^2 - k^2)(T_{23}^2 - k^2)(T_{13}^2 - k^2) = 0.$$

Using (8), from here we get

$$\sin^2 3\varphi\sigma^6 - 18k^2\left[\sigma^2 - \left(\frac{8}{3}\,k^2\right)\right]^2 = 0. \tag{14}$$

The Mises–Nadai criterion follows from (14) for $\sigma = \sigma^T$ and $\sin^2 3\varphi = 0$. In the vector space of principal stresses (directions), geometrically this criterion specifies a circle of radius $\sigma = \sigma^T$ on the deviatoric plane. The angles $\varphi = 0°, 60°, 120°, 180°, 240°, 300°$ specify the six singular points of full plasticity ($\sigma_{ij} = \tau = k_*$) on this circle. The corresponding radii form the six sectors of angle 60°. The other points of the circle correspond to the partial plasticity [23, 30, 47, 49]. In the mentioned sectors, the maximum shear stresses $T_{\max}$ are different. However, their behavior is the same in each sector:

$$T_{\max} = \frac{\sigma}{\sqrt{2}}\sin\omega = \sqrt{\frac{3}{2}}\,\tau_{\text{oct}}\sin\omega. \tag{15}$$

The values of ω are given in the table for each sector.

Sector	$T_{\max}$	ω	φ
I, IV	T_{13}	$2\pi/3 - \varphi$	$0° \le \varphi \le 60°;\ 180° \le \varphi \le 240°$
II, V	T_{23}	φ	$120° \le \varphi \le 180°;\ 300° \le \varphi \le 360°$
III, VI	T_{12}	$2\pi/3 + \varphi$	$60° \le \varphi \le 120°;\ 240° \le \varphi \le 300°$

From this table it follows that $T_{\max}$ is variable in each sector and the Tresca criterion is approximate. From (15) and from the table, it follows that, at the extreme points of sector arcs and at their midpoints, we have

$$\frac{\tau_{\text{oct}}}{T_{\max}} = \frac{2\sqrt{2}}{3} = 0.945, \quad \frac{\tau_{\text{oct}}}{T_{\max}} = \sqrt{\frac{2}{3}} = 0.816.$$

In other words, the following inequality is valid: $\tau_{\text{oct}} < T_{\max}$. Hence, the transition from the elastic state into the plastic state is performed according to the Mises–Nadai criterion if the material is initially isotropic and stable:

$$\tau_{\text{oct}} = k_{\text{oct}} = \sqrt{2}\,k_* = \frac{\sqrt{2}}{3}\,\sigma_T.$$

The unstable (metastable) materials (alloys) can satisfy the Mises–Nadai criterion with a sufficient accuracy, but can significantly deviate from this criterion. For example, this is the case for the magnesium alloys [10] when their unstable structure causes the large deviations of the modulus σ and the single hardening curve law is violated. In this case, for a conditional single hardening curve under simple loading, we should take the hardening curve obtained by a torsion test performed to estimate the error in the fulfilment of the main hypothesis of continuum mechanics.

2. BASIC POSTULATES OF CONTINUUM MECHANICS IN TENSOR SPACES

The postulate of macroscopic definability proposed by Il'yushin in [1, 2] is used to derive the stress–strain constitutive relations in the theory of elastoplastic deformation processes. According to this postulate, the stress process uniquely specifies the medium's state at each mechanical particle of a body under complex stress state and loading at each time instant t. From the macroscopic definability postulate, it follows that the appearing stresses σ_{ij} or σ_0, S_{ij} are dependent on ε_{ij} or ε_0, $Э_{ij}$ as well as on the temperature T and the pressure p and on the non-thermophysical parameters β. If a medium is initially isotropic under any stress–strain state during a complex loading process ($S^*_{ij} \neq Э^*_{ij}$), hence, we have

$$\sigma_0 = \blacksquare_0\{\varepsilon_0, Э_{ij}, T, \beta\}_t, \quad S_{ij} = \blacksquare_{ij}\{\varepsilon_0, Э_{ij}, T, \beta\}_t.$$

These relations are the functionals of the process and are invariant with respect to the orthogonal transformations of rotation and reflection for the coordinate axes x_i in the physical three-dimensional space. In linear algebra, the sets of any elements are said to be linear or affine spaces [15–17]. As such elements, we can consider the second-rank stress and strain tensors. In this case, the set of tensors is said to be the tensor space and the elements are said to be the generalized vectors. For the initially isotropic media, Il'yushin proposed the generalized tensor relations

$$\sigma_0 = 3K\varepsilon_0, \quad S_{ij} = \sum_{n=1}^{5} A_n \frac{d^n Э_{ij}}{ds^n} \quad (i, j = 1, 2, 3), \tag{16}$$

where K is the Bridgman bulk elastic modulus, $s(t)$ is the length of the strain tensor trajectory as a parameter of process tracking, $\{d^n Э_{ij}/ds^n\}_t$ is the linearly independent tensor basis of the six-dimensional tensor space T_6, and A_n are the coefficients dependent on the invariants of the deviators [5]. The above relation between stresses and strains was derived on the basis of universal tensor approach useful to develop the general mathematical theory of plasticity and to prove the invariance of its general physical laws irrespective of the coordinate system x_i and its coordinate basis $\{\hat{e}_i\}$ [23–31].

The relations expressed by (16) can be considered as the mathematical formulation of the macroscopic definability postulate for the initially isotropic media. In the case of small strains, these relations completely describe the properties of such media in the physical space for a particular process with respect to the orthogonal transformations of rotation and reflection for the coordinate axes $x'_i = l_{ij}x_j$ and for the corresponding transformations of the tensor components:

$$\sigma'_{pq} = \sigma_{ij} l_{pi} l_{qj}, \quad \varepsilon'_{pq} = \varepsilon_{ij} l_{pi} l_{qj}.$$

This assertion is only valid when the medium under study satisfies the hypothesis of material continuum. In this case, all the three tensor invariants remain unchanged.

When transforming the vector images of stress and strain tensors, we obtain a set of other physical processes at a given mechanical particle of a body. In this case the general form of the constitutive relations (16) remains the same; however, the coefficients A_n become dependent on the changes of the first and third invariants of stress states under uniform tension and compression and under uniform forming caused by changes in the structural-mechanical properties during forming processes in the stress–strain state.

If the changes in the structural-mechanical properties are ignored, then the constitutive relations (16) can be considered as the isotropy postulate for the initially isotropic media: under small elastoplastic deformations, these relations describe the physical properties of the medium with respect to the transformations of rotation and reflection of the stress and strain tensors if the basic continuum mechanics hypothesis does not take into account the changes in the medium's structure under complex loading [30, 31].

3. THE MACROSCOPIC DEFINABILITY AND ISOTROPY POSTULATES IN THE LINEAR VECTOR SPACE

In [1–3, 7–10], the linear Euclidean six-dimensional space E_6 is introduced instead of the linear tensor space. In E_6, the stress tensor σ_{ij} and the strain tensor ε_{ij} correspond to the ordered collections of their components called the coordinates of multidimensional vectors. As a result, the tensor space is considered as the coordinate-vector linear six-dimensional space. The tensor space becomes Euclidean if the rule of vector summation and the rule of the scalar product for two vectors are introduced. Since the product rule for the two second-rank tensors $(\sigma_{ij})\cdot(\varepsilon_{ij}) = (\sigma_{ik}\sigma_{kj})$ is not permutable, the scalar product of six-dimensional vectors is considered as their convolution, or the double scalar product [23, 26, 29]. In this case we have

$$\begin{cases} (\sigma_{ij})\cdot\cdot(\varepsilon_{ij}) = \sigma_{ik}\varepsilon_{ki} \quad (i,j,k=1,2,3), \\ (\sigma_{ij}\cdot\cdot(\sigma_{ij}) = \sigma_{ij}\sigma_{ij} = S^2, \quad (\varepsilon_{ij})\cdot\cdot(\varepsilon_{ij}) = \varepsilon_{ij}\varepsilon_{ij} = \varepsilon^2. \end{cases}$$

Hence, each of the tensors (σ_{ij}) and (ε_{ij}) can be put in correspondence with the generalized vectors of the six-dimensional linear Euclidean space [23, 24]:

$$\overline{S} = Y_n\hat{e}_n, \quad \overline{\varepsilon} = X_n\hat{e}_n \quad (n=1,2,\dots,6). \tag{17}$$

Here $\{\hat{e}_n\}$ is the coordinate orthonormal vector basis of this space and

$$\begin{cases} X_1=\varepsilon_{11}, \quad X_2=\varepsilon_{22}, \quad X_3=\varepsilon_{33}, \quad X_4=\sqrt{2}\,\varepsilon_{12}, \quad X_5=\sqrt{2}\,\varepsilon_{23}, \quad X_6=\sqrt{2}\,\varepsilon_{13}, \\ Y_1=\sigma_{11}, \quad Y_2=\sigma_{22}, \quad Y_3=\sigma_{33}, \quad Y_4=\sqrt{2}\,\sigma_{12}, \quad Y_5=\sqrt{2}\,\sigma_{23}, \quad Y_6=\sqrt{2}\,\sigma_{13} \end{cases} \tag{18}$$

are the vector coordinates corresponding to the tensor components. In the decomposition of the tensors into the spherical tensors and the deviators, the corresponding vectors (17) with the coordinates expressed by (18) can be represented in the new basis $\{\hat{i}_k\}$ as [1–3, 23]

$$\overline{S} = S_k\hat{i}_k, \quad \overline{\varepsilon} = Э_k\hat{i}_k \quad (k=0,1,2,\dots,5), \tag{19}$$

where the vector coordinates S_k and $Э_k$ are related to the tensor components as

$$\begin{cases} S_0=\sqrt{3}\,\sigma_0, \quad S_1=\sqrt{\dfrac{3}{2}}\,S_{11}, \quad S_2=\dfrac{S_{22}-S_{33}}{\sqrt{2}}, \quad S_3=\sqrt{2}\,S_{12}, \quad S_4=\sqrt{2}\,S_{23}, \quad S_5=\sqrt{2}\,S_{13}, \\ Э_0=\sqrt{3}\,\varepsilon_0, \quad Э_1=\sqrt{\dfrac{3}{2}}\,Э_{11}, \quad Э_2=\dfrac{Э_{22}-Э_{33}}{\sqrt{2}}, \quad Э_3=\sqrt{2}\,Э_{12}, \quad Э_4=\sqrt{2}\,Э_{23}, \quad Э_5=\sqrt{2}\,Э_{13} \end{cases} \tag{20}$$

and the unit vectors of the orthonormal basis $\{\hat{e}_k\}$ are related to the unit vectors of the Il'yushin orthonormal basis $\{\hat{i}_k\}$ as [23]

$$\begin{cases} \hat{i}_0=\dfrac{1}{\sqrt{3}}(\hat{e}_1+\hat{e}_2+\hat{e}_3), \quad \hat{i}_1=\sqrt{\dfrac{2}{3}}\left[\hat{e}_1-\dfrac{1}{2}(\hat{e}_2+\hat{e}_3)\right], \quad \hat{i}_2=\dfrac{\hat{e}_2-\hat{e}_3}{\sqrt{2}}, \\ \hat{i}_3=\hat{e}_4, \quad \hat{i}_4=\hat{e}_5, \quad \hat{i}_5=\hat{e}_6. \end{cases}$$

The transformations expressed by (17) and (19) satisfy the identity conditions for the above tensors and vectors [3, 23, 30]:

$$\begin{cases} S^2=\sigma_{ij}\sigma_{ij}=Y_nY_n=3\sigma_0^2+S_{ij}S_{ij}=S_kS_k, \\ \varepsilon^2=\varepsilon_{ij}\varepsilon_{ij}=X_nX_n=3\varepsilon_0^2+Э_{ij}Э_{ij}=Э_kЭ_k. \end{cases} \tag{21}$$

From the identity conditions (21), it follows that the tensor moduli and the vector lengths are equal, whereas from the one-to-one transformations (18) and (20) it follows that the three tensor invariants remain the same also in the six-dimensional vector space E_6 for the fixed vectors. Using (20), hence, we can transform the tensor form of the basic postulate of continuum mechanics expressed by (16) to the vector form

$$\sigma_0=3K\varepsilon_0, \quad \overline{\sigma}=\sum_{n=1}^{5}A_n\frac{d^n\overline{Э}}{ds^n} \quad \text{or} \quad S_k=\sum_{n=1}^{5}A_n\frac{d^nЭ_k}{ds^n} \quad (k=1,2,\dots,5), \tag{22}$$

where $\{d^n\overline{Э}/ds^n\}$ is the local linearly independent skew-angular frame at each point of the strain trajectory $\overline{Э}=\overline{Э}(s)$ whose arc length is $s(t)$. In E_6 with the basis $\{\hat{i}_k\}$, the strain trajectory $\overline{Э}(s)$ with the stress vector $\overline{\sigma}$, the temperature $T(s)$, the pressure $p(s)$, and other non-thermophysical parameters is said to be the image of the strain process [1–3]. Instead of the local skew-angular frame $\{d^k\overline{Э}/ds^k\}$, at each point of the strain trajectory we can construct the orthonormal frame $\{\hat{p}_k\}$ whose unit vectors satisfy the recurrence formulas [1–3, 23]

$$\frac{d\hat{p}_k}{ds}=-\varkappa_{k-1}\hat{p}_{k-1}+\varkappa_k\hat{p}_{k+1},$$

where $k = 0, 1, \ldots, 5$ and $\varkappa_0 = \varkappa_6 = 0$ in E_6 or $k = 1, 2, \ldots, 5$ and $\varkappa_0 = \varkappa_5 = 0$ in E_5, and

$$\hat{p}_1 = \frac{d\overline{Э}}{ds}, \quad \hat{p}_2 = \frac{1}{\varkappa_1}\frac{d^2\overline{Э}}{ds^2}, \quad \hat{p}_3 = \frac{1}{\varkappa_2}\left[\varkappa_1\frac{d\overline{Э}}{ds} + \frac{d}{ds}\left(\frac{1}{\varkappa_1}\frac{d^2\overline{Э}}{ds^2}\right)\right], \quad \ldots.$$

The vector $\hat{p}_1$ is a tangent vector at each point of the analytical strain trajectory. The stress vector and its first derivative can be represented in the following form in the natural frame of E_5 [23–26]:

$$\overline{S}_0 = 3K\overline{Э}_0, \quad \overline{\sigma} = P_k\hat{p}_k, \quad \frac{d\overline{\sigma}}{ds} = P_k^*\hat{p}_k \quad (k = 1, 2, \ldots, 5). \tag{23}$$

Here the unit stress vector is of the following form in E_5:

$$\hat{\sigma} = \frac{\overline{\sigma}}{\sigma} = \cos\beta_k\hat{p}_k \quad (k = 1, 2, \ldots, 5). \tag{24}$$

The functionals of the process are

$$P_k = \sigma\cos\beta_k, \quad P_k^* = \frac{d\sigma}{ds}\cos\beta_k + \sigma\left[\frac{d\cos\beta_k}{ds} + \varkappa_{k-1}\cos\beta_{k-1} - \varkappa\cos\beta_{k+1}\right], \tag{25}$$

where β_k are the angular coordinates of the vectors $\overline{\sigma}$ and $\hat{\sigma}$ in the natural Frenét frame.

The constitutive relations (23) can be rewritten as

$$\begin{cases} \overline{S}_0 = 3K\overline{Э}_0, \quad \dfrac{d\overline{\sigma}}{ds} = M_k\hat{p}_k + \left(\dfrac{d\sigma}{ds} - M_k\cos\beta_k\right)\hat{\sigma}, \\ \overline{\sigma} = \sigma\hat{\sigma} = \sigma(\cos\beta_k\hat{p}_k) \quad (k = 1, 2, \ldots, 5), \end{cases} \tag{26}$$

where the strain process functionals

$$M_k = \sigma\left[P_k^0 - P_2^0\frac{\cos\beta_k}{\cos\beta_2}\right] \quad (M_2 = 0) \tag{27}$$

are dependent on the internal geometry parameters $(s, \varkappa_m)$, the temperature T, the strain rate $\dot{s}$, and the other non-thermophysical parameters β as well on the strain tensor invariants ε_0, $Э$, and $J_3^\varepsilon = |Э_{ij}|$ for each fixed image of the strain process irrespective of the coordinate system and its transformations of rotation and reflection. The angular coordinates β_k of the vector $\hat{\sigma}$ can be expressed in terms of the spherical coordinates ϑ_m $(m = 1, 2, 3, 4)$ as [23, 25]

$$\begin{cases} \cos\beta_1 = \cos\vartheta_1, \quad \cos\beta_2 = \sin\vartheta_1\cos\vartheta_2, \quad \cos\beta_3 = \sin\vartheta_1\sin\vartheta_2\cos\vartheta_3, \\ \cos\beta_4 = \sin\vartheta_1\sin\vartheta_2\sin\vartheta_3\cos\vartheta_4, \quad \cos\beta_5 = \sin\vartheta_1\sin\vartheta_2\sin\vartheta_3\sin\vartheta_4. \end{cases} \tag{28}$$

Hence, the quantities expressed by (23)–(27) can be represented in the form [23–26]

$$\frac{d\overline{\sigma}}{ds} = M_k\hat{p}_k + M\hat{\sigma} \quad (k = 1, 2, \ldots, 5), \tag{29}$$

$$M = \frac{d\sigma}{ds} - M_1\cos\vartheta_1 - M_0\sin\vartheta_1\sin\vartheta_2, \tag{30}$$

$$\begin{cases} \dfrac{d\vartheta_1}{ds} + \varkappa_1\cos\vartheta_2 = \dfrac{1}{\sigma}[-M_1\sin\vartheta_1 + M_0\cos\vartheta_1\sin\vartheta_2], \\ \sin\vartheta_1\left(\dfrac{d\vartheta_2}{ds} + \varkappa_2\cos\vartheta_3\right) = \dfrac{1}{\sigma}[-M_2\sin\vartheta_2 + M_0\cos\vartheta_2] + \varkappa_1\cos\vartheta_1\sin\vartheta_2, \\ \sin\vartheta_1\sin\vartheta_2\left(\dfrac{d\vartheta_3}{ds} + \varkappa_3\cos\vartheta_3\right) = \dfrac{1}{\sigma}[-M_3\sin\vartheta_3 + M_*\cos\vartheta_3] + \varkappa_2\sin\vartheta_1\cos\vartheta_2\sin\vartheta_3, \\ \sin\vartheta_1\sin\vartheta_2\sin\vartheta_3\left(\dfrac{d\vartheta_4}{ds} + \varkappa_4\right) = \dfrac{1}{\sigma}[-M_4\sin\vartheta_4 + M_5\cos\vartheta_4] + \varkappa_3\sin\vartheta_1\sin\vartheta_2\cos\vartheta_3\sin\vartheta_4, \end{cases} \tag{31}$$

where

$$M_2 = 0, \quad M_0 = M_3 \cos\vartheta_3 + M_* \sin\vartheta_3, \quad M_* = M_4 \cos\vartheta_4 + M_5 \sin\vartheta_4. \tag{32}$$

In the three-dimensional case, we have $\vartheta_3 = \vartheta_4 = 0$ and $M_4 = M_5 = 0$. Therefore, the basic relations take the form

$$\begin{cases} \dfrac{d\overline{\sigma}}{ds} = M_1\hat{p}_1 + \left(\dfrac{d\sigma}{ds} - M_1\cos\vartheta_1 - M_3\sin\vartheta_1\sin\vartheta_2\right)\hat{\sigma} + M_3\hat{p}_3, \\ \hat{\sigma} = \cos\vartheta_1\hat{p}_1 + \sin\vartheta_1(\cos\vartheta_2\hat{p}_2 + \sin\vartheta_2\hat{p}_3), \\ \dfrac{d\vartheta_1}{ds} + \varkappa_1\cos\vartheta_2 = \dfrac{1}{\sigma}[-M_1\sin\vartheta_1 + M_3\cos\vartheta_1\sin\vartheta_2], \\ \sin\vartheta_1\left(\dfrac{d\vartheta_2}{ds} + \varkappa_2\right) = \varkappa_1\cos\vartheta_1\sin\vartheta_2 + \dfrac{M_3}{\sigma}\cos\vartheta_2. \end{cases} \tag{33}$$

If $\vartheta_2 = 0$, the torsion $\varkappa_2$ is small, and $M_3 = \sigma\varkappa_2\sin\vartheta_1$, then we come to the following equations for the trajectories of small torsion given in (33):

$$\begin{cases} \dfrac{d\overline{\sigma}}{ds} = M_1\hat{p}_1 + \left(\dfrac{d\sigma}{ds} - M_1\cos\vartheta_1\right)\hat{\sigma} + \sigma\varkappa_2\sin\vartheta_1\hat{p}_3, \\ \hat{\sigma} = \cos\vartheta_1\hat{p}_1 + \sin\vartheta_1\hat{p}_2, \quad \dfrac{d\vartheta_1}{ds} + \varkappa_1 = -\dfrac{M_1}{\sigma}\sin\vartheta_1. \end{cases} \tag{34}$$

In (34), for the plane trajectories we have $\varkappa_2 = 0$ and $\vartheta_2 = 0$. After rotating the coordinate basis of E_6, we get

$$x'_i = \beta_{ij}x_j, \quad \hat{e}'_i = \beta_{ij}\hat{e}_j \quad (i, j = 1, 2, \ldots, 6),$$

where (β_{ij}) is the transformation matrix. If the vectors and the images of the processes are fixed, then we have

$$\overline{Э} = Э_i\hat{e}_i = Э'_j\hat{e}'_j, \quad \hat{e}_j = \beta_{ji}\hat{e}_i, \quad Э_i = \beta_{ji}Э'_j.$$

Hence,

$$Э'_i = \beta_{ij}Э_j \quad (i, j = 1, 2, \ldots, 6). \tag{35}$$

In other words, the vector coordinates $Э_i$ are changed, whereas the vector itself and all its invariants remain the same, since this vector is fixed in E_6. In this case the constitutive relations expressed by (22) and (23) as well as by (29)–(32) can be considered as a mathematical formulation of the macroscopic definability postulate for the initially isotropic media: in the case of small strains, these constitutive relations completely specify the medium's properties in the physical space with respect to the orthogonal transformations of rotation and reflection of the coordinate basis $\{\hat{e}_i\}$ in E_6 if the tensor invariants in the physical space remain unchanged in the space E_6.

If the coordinate basis $\{\hat{e}_i\}$ is fixed, then in E_6 the transformation of the process image leads us to the relations

$$\overline{Э}' = (\alpha_{ij})\overline{Э}, \quad Э'_i = \alpha_{ij}Э_j \quad (i, j = 1, 2, \ldots, 6), \tag{36}$$

where (α_{ij}) is the orthogonal transformation matrix. If the transformation of rotation or reflection leaves the length of the vector $\overline{Э}$ unchanged, then we have

$$Э^2 = Э'_iЭ'_i = Э_jЭ_j, \quad Э'_i = \alpha_{ij}Э_j = \alpha_{ik}Э_k, \quad Э_j = \delta_{jk}Э_k.$$

Hence, we come to the following system of equations:

$$\alpha_{ij}\alpha_{ik} = \delta_{jk} \quad (i, j, k = 1, 2, \ldots, 6). \tag{37}$$

Here

$$|\alpha_{ij}\alpha_{ik}| = 1, \quad |\alpha_{ij}| = \pm 1, \tag{38}$$

where the plus sign corresponds to rotations, whereas the minus sign corresponds to reflections.

It is not clear yet whether the other invariants remain unchanged. Comparing (35) and (36), we conclude that the transformation matrices (β_{ij}) and (α_{ij}) are mathematically the same: $\alpha_{ij} = \beta_{ij}$. Using the orthogonal transformation (α_{ij}) of the process image, we can obtain a set of strain transformations with the same internal geometry $(s, \varkappa_m)$, but with different stress–strain states and, hence, for different physical processes with different structural-mechanical properties of the material. For some classes of structural materials, the effect of the type of the stress–strain state is weak as shown by many experiments [22–40]. In [1–11] Il'yushin proposed to ignore this effect on the relation between stresses and strains and to replace the set of transformations with the same internal geometry by a single original trajectory. In this case any possible history of loading and deformation is specified only by the internal geometry of the strain trajectory and by its orientation in E_6. This remarkable idea is very successful in the development of the general mathematical theory of plasticity and its applications in practice [41]. The isotropy postulate is formulated in [1, 2] as an approximation to the basic postulate of macroscopic definability: the strain process image is invariant with respect to its transformations of rotation and reflection in a fixed coordinate basis; in other words, the constitutive relations (22) and (23) as well as the constitutive relations (29)–(32) are invariant with respect to these transformations. The isotropy postulate proposed by Il'yushin significantly simplifies the experimental studies to test its reliability and to construct the functionals of elastoplastic processes for its refinement [2].

According to the identity condition (21), the three invariants of the tensors and vectors also remain invariant in E_6. However, the transformation of the process image leaves unchanged only the second invariants, whereas the first and third invariants may be changed. In this case, the isotropy postulate is violated. This postulate is often said to be the particular postulate, whereas the macroscopic definability postulate is said to be the general isotropy postulate, which is not appropriate. The isotropy postulate was widely discussed in 1961–1962 from the standpoint of the general mathematical theory of plasticity and from the standpoint of the theory of elastoplastic processes of deformation under complex loading [18–21]. The second theory is more general than the theory of flow. This can be confirmed by considering the following constitutive relations for the plane trajectories [31]:

$$\begin{cases} \dfrac{d\overline{\sigma}}{ds} = M_1 \dfrac{d\overline{Э}}{ds} + \left(\dfrac{d\sigma}{ds} - M_1 \cos\vartheta_1 \right) \hat{\sigma}, \\ \dfrac{d\vartheta_1}{ds} + \varkappa_1 = -\dfrac{M_1}{\sigma} \sin\vartheta_1, \quad \sigma = \blacksquare(s, \varkappa_1). \end{cases} \tag{39}$$

From here it follows that [23, 27]

$$\begin{cases} d\overline{Э}^* = d\overline{Э} - \dfrac{1}{M_1} d\overline{\sigma} = d\lambda\overline{\sigma}, \\ d\lambda = \dfrac{ds}{\sigma} \left(\cos\vartheta_1 - \dfrac{1}{M_1} \dfrac{d\sigma}{ds} \right), \end{cases} \tag{40}$$

where $\sigma = \blacksquare(s, \varkappa_1)$ is the Odquist–Il'yushin universal curve illustrating the history of deformation. In the theory of flow, the total strains are decomposed into the elastic parts ε_{ij}^e and the plastic parts ε_{ij}^p:

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p, \quad Э_{ij} = Э_{ij}^e + Э_{ij}^p.$$

This is possible only under simple loading or under simple unloading. Under complex loading, according to Haar and Kármán, a material can be in an incomplete plastic state and can be elastically deformed in one direction and can be plastically deformed in another direction [12]. Hence, the basic hypothesis of the flow theory is very restricted. A dependence between stresses and total strains can be theoretically studied and is reasonable theoretically and experimentally.

If we assume that $M_1 = 2G$ in (39) and (40), where G is the elastic shear modulus, then we come to the following more precise constitutive relations of the flow theory:

$$\overline{Э}^p = d\overline{Э} - \frac{1}{2G} d\overline{\sigma} = d\lambda\overline{\sigma}, \quad d\lambda = \frac{ds}{\sigma} \left(\cos\vartheta_1 - \frac{1}{2G} \frac{d\sigma}{ds} \right). \tag{41}$$

These relations take into account the scalar and vector properties of materials and generalize the constitutive relations of the Prandtl–Reuss–Hill flow theory to the case of hardening media [14, 23]. The linearity hypothesis is adopted in the flow theory; this hypothesis is equivalent to the assumption that the relations expressed by (41) are not dependent on the rotation angle ϑ_1 and on the curvature $\varkappa_1$ and is also said to be the hypothesis on free plastic flow. Assuming that $\vartheta_1 = 0$, from (41) we come to the relations of the classical Prandtl–Reuss–Hill flow theory [14]. That is why the application of the flow theory in the stability theory beyond the elastic limit leads to unacceptable results [23–25]. Another critical remark on the applicability of the flow theory is formulated in [8]: during the process of plastic deformation, there appears the strain-induced isotropy changing significantly the elastoplastic properties of a material. The basic remarks stated by the defenders of the flow theory can be reduced to the following. In [18] it is mentioned that the isotropy postulate proposed in the vector form (22) in the five-dimensional deviatoric space should take into account the effect of the third invariants of the stress–strain state, since the coefficients characterizing the medium's state are not defined for other different strain trajectories with the same internal geometry under the orthonormal transformation of the coordinate axes. This question is clarified in [20] as follows. Many experiments were performed to study the initially isotropic materials under normal and high temperatures when the deformation processes were short or long. These experiments show that the effect of the third invariant of the strain (stress) deviator on the mechanical properties is little at small strain. This fact is in agreement with the theory of elastoplastic deformation processes. In the widespread formulations of the isotropy postulate, hence, we assume that the coefficients A_n in (16) and (22) are not dependent on the third tensor invariants. This means that the five-dimensional stress and strain spaces are isotropic; in other words, the stress–strain relations are invariant with respect to the rotation transformations of the coordinates in the body and with respect to the rotation and reflection transformations in the five-dimensional spaces. Hence, only the arc length s and the four curvature parameters $\varkappa_n$ are the internal characteristics of complex loading processes. Thus, the mathematical measure is introduced to quantitatively characterize the degree of complexity for complex loading processes. This remarkable idea proposed by Il'yushin points the future directions in the development of the general mathematical theory of plasticity.

In [19] it is stated that some metastable materials do not have a single hardening curve and the isotropy postulate eliminates all the plasticity criteria with the exception of the Mises plasticity criterion. According to [7], the appearance of plastic strains is characterized by the Mises criterion accurately enough; this criterion can be replaced by the similar Tresca criterion. As noted above, we have $\tau_{\text{oct}} < T_{\max}$; hence, the transition from the elastic state to the plastic one happens when the limit value $\tau_{\text{oct}} = \sqrt{2}\,k_*$ is simultaneously achieved on all octahedral areas of a mechanical particle according to the Mises–Nadai criterion. It is well known that the rigid bodies can be amorphous and crystalline. The amorphous bodies are isotropic, whereas the crystalline bodies (metals and their alloys) are quasi-isotropic.

In the elastic state, all initially isotropic bodies retain the isotropy up to the instant of transition to the elastoplastic quasi-isotropic state when $\sigma \geq \sigma^T$. The mechanism of elastoplastic deformation processes may be different for different materials. The plastic deformation proceeds as a result of the intergranular sliding, the translational shear, and twinning [13]. As a consequence of these mechanisms, the structure of materials and their structural-mechanical properties are changed together with changes in the stress–strain state. A stable pure material with a cubic atomic lattice (such as copper, aluminum, and their alloys with equilibrium structure) possesses the mechanism of translational shear during elastoplastic deformation and has a single hardening curve under simple loading. Magnesium has a hexagonal atomic lattice with a limited number of sliding systems; for magnesium, the effect of a simple loading type on the single hardening curve is large [10]. For the materials with metastable structures (such as magnesium alloys, high-strength steel, etc.), the modulus of the stress deviator is different during the transition from the elastic state to the plastic one for different types of simple loading; hence, a single hardening curve may also be absent. If this deviation is little, then the isotropy postulate is valid accurately enough. Otherwise, some difficulties appear in refining the functionals of the strain process with consideration of strain-induced anisotropy under complex loading. In [21] it is stated that the tensor form of the isotropy postulate (16) is invariant in the physical space and the coefficients A_n should depend on the invariants of the physical space. The vector form of the isotropy postulate (22) is invariant in E_5 and the coefficients A_n should be the invariants of E_5. In [1–10] the following hypothesis is adopted: the coefficients of (16) dependent on the three invariants in the physical space remain invariant in E_5. In section 4 of this paper, we formulated the identity principle for tensors and vectors to clarify this question. The deviatoric space $E_5 \in E_6$ is important in the theory of processes and is separated from E_6 as an independent space [1–10]. In E_5, the strain process image is a set of forming trajectories $\overline{Э}(s)$ dependent on the stress vectors $\overline{\sigma}$, the pressure $p(s)$, the temperature $T(s)$, and on the other non-thermophysical parameters β [1, 2]. In this space, the strain processes are divided into the normal forming processes and the shear forming processes. If

$$3\sigma_0 = \sigma_1 + \sigma_2 + \sigma_3 = 0, \quad 3\varepsilon_0 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$$

and the principal stresses and strains are not equal to zero and have different signs, then the tensor invariants take the form

$$\begin{cases} I_1^\sigma = J_1^\sigma = 0, \quad 2I_2^\sigma = -2J_2^\sigma = -\sigma^2, \quad I_3^\sigma = J_3^\sigma = \dfrac{\sigma^3 \cos 3\varphi}{3\sqrt{6}}, \\ I_1^\varepsilon = J_1^\varepsilon = 0, \quad 2I_2^\varepsilon = -2J_2^\varepsilon = -Э^2, \quad I_3^\varepsilon = J_3^\varepsilon = \dfrac{Э^3 \cos 3\psi}{3\sqrt{6}}. \end{cases}$$

If the process image is fixed and an orthogonal transformation of the coordinate basis $\{\hat{i}_k\}$ is used, then the invariants of tensors and vectors remain the same at each point of the strain trajectory. If an orthogonal transformation of the process image and its vectors is used and the coordinate basis $\{\hat{i}_k\}$ is the same, then the vectors $\overline{\sigma}$ and $\overline{Э}$ retain their lengths σ and $Э$, whereas the angles φ and ψ are changed; hence, the third invariants are also changed. In this case the isotropy postulate is violated, since the form of the stress–strain state and the structural-mechanical properties are changed. If the loading process is simple or quasisimple when $(S_{ij}^*) = (Э_{ij}^*)$, then we come to the relations

$$\overline{\sigma} = \frac{\sigma}{Э}\,\overline{Э}, \quad S_{ij} = \frac{\sigma}{Э}\,Э_{ij}, \quad \sigma = \blacksquare(Э).$$

Since $\varphi = \psi$, we have

$$\cos 3\varphi = \frac{3\sqrt{6}|S_{ij}|}{\sigma^3} = \frac{3\sqrt{6}|Э_{ij}|}{Э^3} = \cos 3\psi.$$

Thus, the isotropy postulate is always valid in the case of the simple and quasisimple loading in the uniform shear forming process. A generalization of the isotropy postulate in E_5 by introducing the internal pressure p or the relative volume strain $\theta = 3\varepsilon_0$ in the concept of the process image at each point of the strain trajectory causes some difficulties, since, in the deviatoric subspace E_5, only the forming strain is described by definition. As a consequence, some test programs with $\sigma_0 \neq 0$ and $\varepsilon_0 \neq 0$ are used in experimental studies to check the isotropy postulate, which leads to the discovery of effects of the third invariants and to the violation of the isotropy postulate. The consideration of the isotropy postulate in E_6 instead of E_5 eliminates these difficulties for the processes of simple and quasisimple loading.

4. THE IDENTITY PRINCIPLE FOR THE TENSORS AND VECTORS OF STRESSES AND STRAINS IN E_6

The six-dimensional Euclidean space E_6 is defined by the set of the strain and stress vectors [1, 2, 23]:

$$\overline{\varepsilon} = X_i\hat{e}_i = Э_k\hat{i}_k, \quad \overline{S} = Y_i\hat{e}_i = S_k\hat{i}_k \quad (i = 1, 2, \ldots, 6;\ k = 0, 1, \ldots, 5).$$

Here $\{\hat{e}_i\}$ and $\{\hat{i}_k\}$ are the orthonormal coordinate bases in E_6.

As shown in Section 3 of this paper, the vector coordinates are changed according to the following relations if the vector $\overline{Э}$ is fixed and the coordinate basis $\hat{e}'_i = \beta_{ij}\hat{e}_j$, $\hat{e}_i = \beta_{ji}\hat{e}'_j$ is orthogonally transformed:

$$X_i = \beta_{ij}X_j. \tag{42}$$

Here all the invariants of a six-dimensional vector remain the same in E_6, since this vector remains the same as a whole. On the other hand, if the coordinate basis is fixed and the orthogonal transformation of the vector $\overline{\varepsilon}$ is used, then we have

$$X'_i = \alpha_{ij}X_j. \tag{43}$$

If the length of this vector remains unchanged, then the conditions expressed by (37) and (38) should be valid. From (42) and (43) it follows that $\alpha_{ij} = \beta_{ij}$. In other words, the transformations of rotation and reflection are mathematically the same, but are physically different if the coordinate axes, or the bases for fixed vectors, or the process images for fixed coordinate bases are transformed. In the case of (42), all the invariants of the transformation remain the same, whereas, in the case of (43), only the moduli of vectors remain the same and the other invariants remain indefinite. In the first case, the vector $\overline{\varepsilon}$ being transformed to a new position $\overline{\varepsilon}'$ may take a position such that

$$\overline{\varepsilon}' = \lambda\overline{\varepsilon}, \quad X'_i = \alpha_{ij}X_j = \lambda X_i. \tag{44}$$

The vector $\overline{\varepsilon}$ satisfying (44) is said to be the eigenvector with the eigenvalue λ [15, 16]. Since

$$X_i = \delta_{ij}X_j, \quad X_j = \varepsilon X_j^*, \quad X_j^*X_j^* = 1, \tag{45}$$

from (44) and (45) we come to the following system of homogeneous linear algebraic equations:

$$(\alpha_{ij} - \delta_{ij}\lambda)X_j^* = 0, \tag{46}$$

According to (45), its determinant is of the form

$$D = -|\alpha_{ij} - \delta_{ij}\lambda| = \lambda^6 - M_1\lambda^5 + M_2\lambda^4 - M_3\lambda^3 + M_4\lambda^2 - M_5\lambda + M_6 = 0, \tag{47}$$

whose coefficients M_n $(n = 1, 2, \ldots, 6)$ are the sums of the principal minors of order n for the transformation matrix $A = (\alpha_{ij})$:

$$M_1 = \alpha_{ii}, \quad M_2 = \frac{1}{2}(\alpha_{ii}\alpha_{jj} - \alpha_{ij}\alpha_{ij}), \quad \ldots, \quad M_6 = |\alpha_{ij}|.$$

The equation expressed by (47) is said to be the characteristic equation of system (46) and its coefficients are the invariants of the space E_6 [15]. The roots of Eq. (47) are the eigenvalues of the matrix $A = (\alpha_{ij})$. According to Vieta's theorem, the coefficients M_n can be expressed in terms of λ_n using the formulas

$$M_1 = \sum_{i=1}^{6} \lambda_i, \quad M_2 = \sum_{i,j=1}^{6} \lambda_i\lambda_j \ \ (i < j), \quad M_3 = \sum_{i,j,k=1}^{6} \lambda_i\lambda_j\lambda_k \ \ (i < j < k),$$
$$M_4 = \textstyle\sum_{i,j,k,r=1}^{6} \lambda_i\lambda_j\lambda_k\lambda_r \ (i < j < k < r), \quad \ldots, \quad M_6 = \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6.$$

Equation (47) has no more than six roots. If all λ_n are different and real, then the transformation matrix $A = (\alpha_{ij})$ is diagonalizable. Substituting λ_n into (46), we can find the eigenvectors $\overline{\varepsilon}_n$ and their moduli ε_n. Multiplying M_n by ε^n, where $\varepsilon = |\overline{\varepsilon}|$ is the modulus of the strain vector, we obtain the following six invariants of the strain vector in E_6:

$$I_n^\varepsilon = \varepsilon^n M_n \quad (n = 1, 2, \ldots, 6).$$

Similarly, for the stress invariants we have $I_n^\sigma = S^n M_n$.

If Eq. (47) has multiple roots, then the number of linearly independent eigenvectors in E_6 is less than six [16]. Let Eq. (47) have the multiple roots λ_n. It is important that the physical processes are different when the vectors $\overline{\varepsilon}(t)$ and $\overline{S}(t)$ are transformed. It is well known that, in the physical space, the transformation of the coordinate axes x_i is defined by the three parameters called the Euler angles being a particular case of the rotation transformation in E_6 for some values of α_{ij} [2]. If the transformations of rotation and reflection for the vectors $\overline{\varepsilon}' = (\alpha_{ij})\overline{\varepsilon}$ and $\overline{S}' = (\alpha_{ij})\overline{S}$ are not coincident with the three-parameter rotation, then we obtain the class of trajectories that do not correspond to the physical processes for the strain and stress tensors. This fact allows us to assume that in E_6 the following three multiple roots should be equal to zero: $\lambda_4 = \lambda_5 = \lambda_6 = 0$. In this case, Eq. (47) takes the form

$$\lambda^3(\lambda^3 - M_1\lambda^2 + M_2\lambda - M_3) = 0,$$

where

$$M_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad M_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \quad M_3 = \lambda_1\lambda_2\lambda_3.$$

In E_6 the invariants of the stress–strain state are

$$\begin{cases} I_1^\varepsilon = \varepsilon M_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 3\varepsilon_0, \quad I_2^\varepsilon = \varepsilon^2 M_2 = \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1, \\ I_3^\varepsilon = \varepsilon^3 M_3 = \varepsilon_1\varepsilon_2\varepsilon_3, \quad I_4^\varepsilon = I_5^\varepsilon = I_6^\varepsilon = 0, \quad 2(I_2^\varepsilon)' = \varepsilon^2 = \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2, \end{cases}$$

$$\begin{cases} I_1^\sigma = SM_1 = \sigma_1 + \sigma_2 + \sigma_3 = 3\sigma_0, \quad I_2^\sigma = S^2 M_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1, \\ I_3^\sigma = S^3 M_3 = \sigma_1\sigma_2\sigma_3, \quad I_4^\sigma = I_5^\sigma = I_6^\sigma = 0, \quad 2(I_2^\sigma)' = S^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2, \end{cases}$$

where ε_k and σ_k $(k = 1, 2, 3)$ are the principal strains and stresses. This result is in agreement with the fact that, in the particular case of the physical space, we obtain the three-dimensional space of principal stress and strain directions [23]. The obtained result is an additional requirement formulated in [21] and predicted in [1, 2, 10, 20]. We call this requirement as a generalized identity principle for the second-rank tensors in E_6: the vectors of E_6 and the stress and strain tensors in the physical three-dimensional space are identical if their moduli and their three eigenvectors are coincident; note that in E_6 these eigenvectors form a local three-dimensional invariant subspace equivalent to the three-dimensional subspace of principal directions in the physical space. According to the identity principle, the three invariants of each of the stress and strain tensors are valid in the physical space and are also invariant in E_6 [10].

The above result is related to the macroscopic definability postulate and to its constitutive relations (22), (23) and (29)–(32) in E_6 when the process image and its vectors are fixed. If the coordinate basis is fixed and the process image and its vectors (43) are orthogonally transformed, then the situation is physically changed when the isotropy postulate is formulated as a consequence of the macroscopic definability postulate. In this case, only the length of a vector remains invariant. The first and third invariants remain mathematically indefinite in the framework of the basic hypothesis on the material continuum and may become not invariant because of changes in structural-mechanical properties of the medium at the mesolevel. In the theory of elastoplastic deformation processes, hence, experimental studies become important for each class of materials to analyze the effect of the invariants of the stress–strain state on the fulfilment of the isotropy postulate. It is experimentally shown that, for the stable metals and their alloys, this effect is weak [20]. The Il'yushin isotropy postulate formulated in [2, 3, 7, 8, 10] originates a new direction in the plasticity theory and allows one to model the complex loading processes in continuum mechanics.

5. CONCLUSION

The above discussion allows us to state that, for the initially isotropic continuous media, the isotropy postulate completely corresponds to the fundamental hypothesis of continuum mechanics. In its original state, a material is isotropic (quasi-isotropic) and homogeneous near each point of small volume. The material possesses the elastoviscoplastic properties. During a deformation process, there appears the strain-induced anisotropy caused by changes in the structural-mechanical properties at the mesolevel when the stress–strain state and the elastic constants are changed. Under simple loading, the parameters of the strain-induced anisotropy are described for the stable materials on the basis of the single hardening curve law. Under complex loading, such a curve is absent and the functionals of elastoplastic processes (except for the complex loading parameters of internal geometry of the strain trajectory) are dependent on the forming parameters of the stress–strain state corresponding to the hidden structural-mechanical parameters at the mesolevel. Under complex loading or in the case of elastoplastic deformation, thus, the fundamental hypothesis should be refined.

On the other hand, the isotropy postulate contains the following fundamental idea: the history of complex loading processes is mainly defined by the parameters of internal geometry of strain trajectories (the arc length s and the curvature parameters $\varkappa_m$) rather than by the effect of the forming parameters of the stress–strain state caused by changing the structural-mechanical state at the mesolevel; this change has a secondary effect. Nevertheless, such changes become noticeable for some materials with the physical nonlinearity in an elastic domain (cast iron, concrete, graphite, etc.). The curvature parameters of complex trajectories can be considered as a mathematical measure of complexity in loading; the delay principle leads to the concept of the delay trace considered as a characteristic size in the classification of strain trajectories. The isotropy postulate defines the vector properties of materials and geometrically visualize the processes of complex loading. Under complex loading, the scalar properties define the functionals of processes. There is no universal functional describing the hardening process during the elastoplastic deformation. Therefore, the general mathematical theory of plasticity should be a theory of plasticity of anisotropic bodies in the case of complex loading [10]. This way of development takes time. However, this fact does not prevent from refining the isotropy postulate by constructing the approximate functionals for the complex elastoplastic processes of deformation. That is why the Il'yushin isotropy postulate provides a basis for the experimental and theoretical studies of various histories of complex deformation of elastoplastic media and shows the most efficient direction in the development of the plasticity theory for engineering studies compared to the other versions of plasticity theories that do not contain such a variety of complex loading histories. At present, a number of new results are obtained in this direction [22–31, 45–54].

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Translated by O. Arushanyan