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Self-Induced Vibrations in a String–Bow System

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Abstract—The vibrations of a thin stretched string is studied in the case when a bow slides on it with a constant velocity orthogonal to the string. The interaction between the bow and the string is governed by a smooth nonlinear law of friction with a falling segment of the characteristic. The motion of this mechanical system is described by an infinite coupled system of nonlinear ordinary differential equations. Some averaged equations of motion are derived in terms of the action–angle variables. The stationary points corresponding to self-vibration modes are found. The stability of these modes is analyzed.

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1. A MODEL OF INTERACTION BETWEEN A STRING AND A BOW. EQUATIONS OF MOTION.

Wave and self-vibration processes are discussed in [1–7] for systems with finite and infinite degrees of freedom.

In this paper a string is considered as a thin elastic rod under longitudinal and bending deformation. Let an undeformed stretched rod be situated along the Ox axis. By $u\,(s,t)$, $0 \le s \le l$, we denote the displacements of its points along the Oy axis. The motion of the rod proceeds on the plane Oxy. The ends of the rod are hinged on the Ox axis at the points whose coordinates are $s=0$ and $s=l$. The kinetic and potential energies of a deformed rod can be expressed by the formulas [8]

$$T=\frac{\rho}{2}\int\limits_0^l \dot{u}^2\,ds,\qquad \Pi=\frac{1}{2}\int\limits_0^l\left(N_1u'^2+N_2u''^2\right)ds,\qquad \dot{u}=\frac{\partial u}{\partial t},\qquad u'=\frac{\partial u}{\partial s},\qquad u''=\frac{\partial^2 u}{\partial s^2},\tag{1}$$

where ρ is the linear density of the rod's material, N_1 is the tension of the rod, and N_2 is the bending stiffness of the rod. It is assumed that the rod's material possesses the dissipation properties described by the Rayleigh dissipation functional [1]

$$D=\frac{\chi}{2}\int\limits_0^l\left(N_1\dot{u}'^2+N_2\dot{u}''^2\right)ds,\qquad \dot{u}'=\frac{\partial^2 u}{\partial t\partial s},\qquad \dot{u}''=\frac{\partial^3 u}{\partial t\partial s^2},$$

where χ is the coefficient specifying the dissipation properties of the rod's material.

Let us assume that, at the point with the s_0 coordinate, a force ρF directed along the Oy axis is exerted on the rod. This force used to model the interaction between the string and the bow is represented as [6]

$$\rho F\,(V)=\rho f\left(V-g_1V^3+g_2V^5\right),\qquad V=v-\dot{u}\,(s_0,t),\tag{2}$$

where v is the velocity of the bow moving translationally along the Oy axis and f is a constant coefficient. In our further discussion, we assume that v is constant on a certain time interval. The coefficients g_1 and g_2 used in (2) are constant and can be expressed in terms of other constants V_1 and V_2 as follows:

$$g_1=\frac{V_1^2+V_2^2}{3V_1^2V_2^2},\qquad g_2=\frac{1}{5V_1^2V_2^2},\qquad V_2^2<5V_1^2.\tag{3}$$

When $V \ge 0$, the function $F\,(V)$ has a positive maximum for $V=V_1$ and a positive minimum for $V=V_2$. The derivative $F'\,(V)$ is positive for $v\in[0,V_1)\cup(V_2,\infty)$ and is negative for $v\in(V_1,V_2)$. The friction law (2) can be characterized as a nonlinear model of viscous friction. This model has no stagnation zones and approximates the dry friction law when the friction of rest is greater than the sliding friction. An advantage of our model consists in the fact that it uses a smooth function without the absolute values of relative velocities of two bodies at the points of contact. This fact significantly simplifies the studies connected with the motion of systems with dry friction.

The Hamilton–Ostrogradskii variational principle

$$\int_0^T \Big[\delta T - \delta\Pi + \rho F \delta u\left(s_0, t\right) + \int_0^l \nabla_{\dot u} D \delta u\left(s, t\right) ds\Big]\, dt = 0,$$

$$\forall\ \delta u\left(s, t\right),\ \ \delta u\left(0, t\right) = \delta u\left(l, t\right) = 0$$

is used to derive the following equations of motion and the dynamic boundary conditions:

$$\frac{\partial^2 u}{\partial t^2} - \left(1 + \chi \frac{\partial}{\partial t}\right)\left(\frac{N_1}{\rho}\frac{\partial^2 u}{\partial s^2} - \frac{N_2}{\rho}\frac{\partial^4 u}{\partial s^4}\right) = F\left(v - \dot u\left(s, t\right)\right)\delta\left(s - s_0\right),$$

$$u''\left(0, t\right) = u''\left(l, t\right) = 0.$$

The configuration space of our mechanical system is the Hilbert space

$$H_2 = \big\{u\left(s, t\right) : u''\left(s, t\right) \in L_2\left([0, l]\right), \quad u\left(0, t\right) = u\left(l, t\right) = 0\big\}.$$

The velocity space is defined as the following subspace of square-integrable functions:

$$H_0 = \big\{\dot u\left(s, t\right) : \dot u\left(s, t\right) \in L_2\left([0, l]\right), u\left(0, t\right) = u\left(l, t\right) = 0\big\}.$$

In these spaces we choose the orthogonal basis $\{\psi_k\left(s\right)\}_1^\infty$, where $\psi_k\left(s\right) = \sqrt{2/l}\,\sin \pi k s l^{-1}$. This basis satisfies the conditions

$$\int_0^{2\pi} \psi_i\left(s\right)\psi_j\left(s\right) ds = \delta_{ij},$$

where δ_{ij} is the Kronecker symbol. We represent the function $u\left(s, t\right)$ as

$$u\left(s, t\right) = \sum_{k=1}^{\infty} q_k\left(t\right)\psi_k\left(s\right). \tag{4}$$

The convergence of the series

$$\sum_{k=1}^{\infty} \dot q_k^2\left(t\right) < \infty, \qquad \sum_{k=1}^{\infty} k^4 q_k^2\left(t\right) < \infty$$

follows from the existence condition (1) for the functionals of the kinetic and potential energies.

The Lagrange generalized coordinates $\mathbf{q} = \left(q_1, q_2, \ldots\right)$ and the representation expressed by (4) are used to obtain the following expressions for the kinetic and potential energies and the dissipation function:

$$T_1 = \frac{1}{2}\sum_{k=1}^{\infty}\dot q_k^2, \quad \Pi_1 = \frac{1}{2}\sum_{k=1}^{\infty}\nu_k^2 q_k^2, \quad D = \frac{\chi}{2}\sum_{k=1}^{\infty}\nu_k^2\dot q_k^2, \quad \nu_k^2 = \frac{N_1\pi^2k^2}{\rho l^2} + \frac{N_2\pi^4k^2}{\rho l^4}. \tag{5}$$

The multiplier ρ is omitted in (5). The generalized forces $Q_k = F\left(V\right)\psi_k\left(s_0\right)$, $k = 1, 2, \ldots$, can be determined from the following formula for the elementary work of the force expressed by (2) on virtual displacements:

$$\delta A = F\left(V\right)\sum_{k=1}^{\infty}\delta q_k\psi_k\left(s_0\right), \quad V = v - \sum_{n=1}^{\infty}\dot q_n\psi_n\left(s_0\right), \quad F\left(V\right) = f\left(V - g_1V^3 + g_2V^5\right).$$

The equations of motion are written in the form of the second-kind Lagrange equations:

$$\ddot q_k + \chi\nu_k^2\dot q_k + \nu_k^2 q_k = f\psi_k\left(s_0\right)\left(V - g_1V^3 + g_2V^5\right), \quad V = v - \sum_{n=1}^{\infty}\dot q_n\psi_n\left(s_0\right), \quad k = 1, 2, \ldots. \tag{6}$$

Equations (6) can be considered as an infinite system of nonlinear second-order ordinary differential equations. The mechanical system corresponding to (6) can be treated as an infinite system of harmonic oscillators related to each other by the nonlinear forces of viscous friction.

Let q_{k0} be defined by the relations $\nu_k^2 q_{k0} = f\psi_k(s_0)\left(v - g_1 v^3 + g_2 v^5\right)$. Denoting $q_k - q_{k0}$ by q_k, we represent Eqs. (6) as

$$\begin{aligned} &\ddot{q}_k + \chi\nu_k^2\dot{q}_k + \nu_k^2 q_k = f\psi_k(s_0)\left[V - v - g_1\left(V^3 - v^3\right) + g_2\left(V^5 - v^5\right)\right], \\ &V = v - \sum_{n=1}^{\infty}\dot{q}_n\psi_n(s_0), \qquad k = 1, 2, \ldots. \end{aligned} \tag{7}$$

In order to study the dynamics of the system expressed by (7), we use the action–angle variables

$$(p_k, q_k) \Leftrightarrow (I_k, \varphi_k), \qquad p_k = \dot{q}_k = \sqrt{2I_k\nu_k}\,\cos\varphi_k, \qquad q_k = \sqrt{2I_k/\nu_k}\,\sin\varphi_k.$$

Then, the Hamiltonian $H = \sum_{k=1}^{\infty}\left(p_k^2 + \nu_k^2 q_k^2\right)/2$ takes the form $K = \sum_{k=1}^{\infty}\nu_k I_k < \infty$. Now we rewrite the equations of motion in the following form of the canonical Hamiltonian equations with the generalized forces [8]:

$$\begin{aligned} &\dot{I}_k = -2\chi\nu_k^2 I_k\cos^2\varphi_k + \left[F(V) - F(v)\right]\psi_k(s_0)\left(2I_k/\nu_k\right)^{1/2}\cos\varphi_k, \\ &\dot{\varphi}_k = \nu_k + \chi\nu_k^2\cos\varphi_k\sin\varphi_k - \left[F(V) - F(v)\right]\psi_k(s_0)\left(2I_k\nu_k\right)^{-1/2}\sin\varphi_k. \end{aligned} \tag{8}$$

Here

$$V = v - \sum_{n=1}^{\infty}\psi_n(s_0)\sqrt{2I_n\nu_n}\,\cos\varphi_n, \qquad k = 1, 2, \ldots.$$

If $\varphi_k(s_0) = 0$ for a particular value of k, then the corresponding equation of (7) can be separated from the other equations. This equation describes the damped eigenvibrations with respect to the variable q_k.

2. SELF-INDUCED VIBRATIONS IN THE STRING–BOW SYSTEM

System (8) is well suitable for the application of the averaging method over the fast variables $(\varphi_1, \varphi_2, \ldots)$. Since the total energy is bounded, the series $\sum_{k=1}^{\infty}\nu_k I_k$ is convergent and $I_k < bk^{-3-\alpha}$, where the constant b is bounded and $\alpha > 0$. It is assumed that the velocity of the bow is bounded with respect to the string: $V < \infty$. The series specifying the velocity V in (8) is convergent and $I_k < ck^{-4-\beta}$, where the positive constant c is bounded and $\beta > 0$. Hence, the right-hand sides of (8) are small if the coefficients χ and f are small, where χ characterizes the internal friction in the rod and f characterizes the friction during the interaction between the bow and the rod.

Let us assume that the rod's eigenfrequencies ν_k, $k = 1, 2, \ldots$, are independent. This means that the linear combination $\sum_{k=1}^{N} m_k\nu_k$, where m_k are integer, becomes equal to zero for any N only for $m_k = 0$, $k = 1, \ldots, N$. Applying the averaging operation over the angle variables $(\varphi_1, \varphi_2, \ldots)$ to Eqs. (8), we get

$$\begin{aligned} \dot{Z}_k = -\chi\nu_k^2 Z_k + \frac{f}{16\nu_k}Z_k\Bigg[&-8\left(1 - 3g_1v^2 + 5g_2v^4\right) + 6\left(g_1 - 10g_2v^2\right)\Big(Z_k + 2\sum_{n\neq k}^{\infty}Z_n\Big) - 5g_2\Big(Z_k^2 \\ &+ 6Z_k\sum_{n\neq k}^{\infty}Z_n + 3\sum_{n\neq k}^{\infty}Z_n^2 + 12\sum_{n\neq m\neq k}^{\infty}Z_nZ_m\Big)\Bigg], \quad \dot{\theta}_k = \nu_k, \quad Z_k = 2J_k\nu_k\psi_k^2(s_0), \quad k = 1, 2, \ldots. \end{aligned} \tag{9}$$

The averaging operation consists in the calculation of the mean value for the expansions of the corresponding functions in the infinite-dimensional Fourier series

$$\left\langle G\left(I_1, I_2, \ldots, \varphi_1, \varphi_2, \ldots\right)\right\rangle = \lim_{n\to\infty}\frac{1}{(2\pi)^n}\int_0^{2\pi}\ldots\int_0^{2\pi}G\,d\varphi_1\ldots d\varphi_n. \tag{10}$$

After calculating (10), we replace the variables $(I_1, I_2, \ldots)$ by the variables $(J_1, J_2, \ldots)$ and the derivatives $(\dot{\varphi}_1, \dot{\varphi}_2, \ldots)$ in (9) by the variables $\left(\dot{\theta}_1, \dot{\theta}_2, \ldots\right)$. From (9) it follows that the derivatives of the angle variables $\dot{\theta}_k$ are coincident with the eigenfrequencies ν_k of the unperturbed system of equations describing the harmonic oscillations of an infinite system of independent oscillators.

The series of (9) are convergent, since

$$\sum_{n\neq k}^{\infty} Z_n < b_1 \sum_{n\neq k}^{\infty} \frac{1}{n^{1+\beta}}, \qquad \sum_{n\neq m\neq k}^{\infty} Z_n Z_m < b_1^2 \sum_{n\neq m\neq k}^{\infty} \frac{1}{n^{1+\beta} m^{1+\beta}},$$

where b_1 and β are positive constants.

Based on (3), we represent the first group of equations of (9) in the form

$$\nu_k \dot{Z}_k = -CZ_k \left[A_k - 2E\Big(Z_k + 2\sum_{n\neq k}^{\infty} Z_n\Big) + Z_k^2 + 6Z_k \sum_{n\neq k}^{\infty} Z_n + 3\sum_{n\neq k}^{\infty} Z_n^2 + 12 \sum_{n\neq m\neq k}^{\infty} Z_n Z_m\right], \quad k = 1, 2, \ldots, \quad (11)$$

where

$$C = f/(16V_1^2V_2^2) > 0, \quad A_k = 16\chi\nu_k^3 V_1^2 V_2^2 f^{-1} + 8\left(v^2 - V_1^2\right)\left(v^2 - V_1^2\right), \quad E = V_1^2 + V_2^2 - 6v^2.$$

Now we find the stationary solutions to Eqs. (11). Obviously, these equations have the trivial solutions $Z_k = 0$, $k = 1, 2, \ldots$. The stability of these solutions is studied using the system of equations in variations. In the right-hand sides of (11), we keep the terms linear in Z_k. As a result, we come to the following infinite system of first-order ordinary differential equations:

$$\nu_k \dot{Z}_k = -CA_k Z_k, \quad k = 1, 2, \ldots. \quad (12)$$

The stability of the trivial solution depends on the sign of the coefficient A_k: the solution $Z_k = 0$ is stable if $A_k > 0$; otherwise, this solution is unstable. The coefficients A_k are positive if $v \in (0, V_1) \cup (V_2, \infty)$, since they are the sums of the two positive summands. The coefficients A_k are dependent on v; their minima with respect to v are

$$\min_{0<v<\infty} A_k = 16\chi\nu_k^3 V_1^2 V_2^2 f^{-1} - 2(V_1^2 - V_1^2)^2.$$

Since the eigenfrequencies ν_k increase as k^2, the minimum of A_k becomes positive for k greater than or equal to n. The function $A_k(v)$ is greater than zero for $k > n$; the zero equilibrium positions $Z_k = 0$, $k > n$, are stable. If $n = 0$, then the zero equilibrium position is stable for all normal modes of vibrations, which is the case for a sufficiently large value of χ.

Let us consider the existence of nonzero vibrations when $Z_k \neq 0$ and $Z_n = 0$ for all $n \neq k$. In this case the right-hand side of (11) becomes equal to zero:

$$A_k - 2EZ_k + Z_k^2 = 0. \quad (13)$$

The quadratic equation (13) has real roots $Z_{jk} = E \pm \sqrt{D_k}$, $j = 1, 2$, if its determinant is not negative: $D_k = E^2 - A_k \geq 0$. If $A_k < 0$, then there exists a negative root $Z_{1k} < 0$ and a positive root $Z_{2k} > 0$. The trivial solution becomes unstable; as a result, there appears a nonzero stationary solution Z_{2k} corresponding to the self-vibration mode with the frequency ν_k. The stability of this solution is studied on the basis of the variational equations

$$\nu_k \dot{\xi}_k = -2CZ_{2k}\sqrt{D_k}\,\dot{\xi}_k, \quad \nu_m \dot{Z}_m = -CA_m Z_m, \quad m = 1, 2, \ldots, \quad m \neq k, \quad \xi = Z_k - Z_{2k}.$$

Analyzing these equations, we come to the conclusion that this self-vibration mode is stable. For other modes, however, the trivial solutions may be unstable, depending on the sign of A_m. Since $A_1 < A_2 < \ldots < A_k < \ldots$, the self-vibration mode is stable with respect to all variables if this mode corresponds to the mode with the lowest frequency when $A_1 < 0$ and $A_m > 0$, $m = 2, 3, \ldots$.

The roots of (13) are positive if

$$D_k > 0, \quad A_k > 0, \quad v^2 < \left(V_1^2 + V_2^2\right)/6. \quad (14)$$

Taking into account the inequality in (3), we obtain the estimate $v^2 < V_1^2$ from the last inequality of (14). Based on this estimate, we conclude that $A_k > 0$ and $D_k < 0$, since

$$(V_1^2 + V_2^2 - 6v^2)^2 - 8\left(v^2 - V_1^2\right)\left(v^2 - V_2^2\right) = 28v^4 - 4\left(V_1^2 + V_2^2\right)v^2 - 7(V_1^2 + V_2^2)^2 < 0$$

for $v^2 < \left(V_1^2 + V_2^2\right)/6$. The existence of two stationary modes is impossible.

Now we study the existence of stable two-frequency self-vibration modes when $Z_k > 0$ and $Z_m > 0$. Since the other vibration modes should be absent ($Z_j = 0$, $j \neq k$, $j \neq m$) and should be stable ($A_j > 0$), the stable two-frequency self-vibration modes are possible for $k = 1$ and $m = 2$, which corresponds to the case $A_1 < A_2 < 0$. Hence, the system of equations expressed by (11) can be rewritten as

$$\begin{aligned} \nu_1 \dot{Z}_1 &= -CZ_1 \left[A_1 - 2E\left(Z_1 + 2Z_2\right) + Z_1^2 + 6Z_1Z_2 + 3Z_2^2\right], \\ \nu_2 \dot{Z}_2 &= -CZ_2 \left[A_2 - 2E\left(Z_2 + 2Z_1\right) + Z_2^2 + 6Z_1Z_2 + 3Z_1^2\right]. \end{aligned} \tag{15}$$

The nonzero stationary solutions to system (15) satisfy the equations

$$\begin{aligned} A_1 - 2E\left(Z_1 + 2Z_2\right) + Z_1^2 + 6Z_1Z_2 + 3Z_2^2 = 0, \\ A_2 - 2E\left(Z_2 + 2Z_1\right) + Z_2^2 + 6Z_1Z_2 + 3Z_1^2 = 0. \end{aligned} \tag{16}$$

Hence,

$$A_2 - A_1 - 2E\left(Z_1 - Z_2\right) + 2\left(Z_1^2 - Z_2^2\right) = 0. \tag{17}$$

The difference $A_2 - A_1$ is proportional to the coefficient χ specifying the energy dissipation in the string during its vibrations. Assuming that χ is small, we put $A_2 - A_1 \cong 0$. Equation (17) is valid if $Z_1 \cong Z_2$ or $Z_1 + Z_2 \cong E$. In the first case, from (16) we get

$$Z_1 \cong Z_2 \cong Z_0, \qquad Z_0 = \left(\sqrt{9E^2 - 10A} + 3E\right)/10 > 0, \qquad A \cong A_1 \cong A_2 < 0. \tag{18}$$

In the second case, the system expressed by (16) has no solution, since the inequality $\left(Z_1 - Z_2\right)^2 \cong A - E^2 < 0$ follows from it.

The solution expressed by (18) describes the self-vibrations for the frequencies ν_1 and ν_2 and is unstable. Let us consider the variational equations

$$\dot{\xi}_1 = -2C_1Z_0\left[\left(4Z_0 - E\right)\xi_1 + 2\left(3Z_0 - E\right)\xi_2\right],$$

$$\dot{\xi}_2 = -2C_2Z_0\left[\left(4Z_0 - E\right)\xi_2 + 2\left(3Z_0 - E\right)\xi_1\right]$$

and represent them as

$$\begin{aligned} \nu_1\xi_1' + \left(2\sqrt{9E^2 - A} + E\right)\xi_1 + \left(3\sqrt{9E^2 - A} - E\right)\xi_2 = 0, \\ \nu_2\xi_2' + \left(2\sqrt{9E^2 - A} + E\right)\xi_2 + \left(3\sqrt{9E^2 - A} - E\right)\xi_1 = 0, \end{aligned} \tag{19}$$

where $\xi_k = Z_k - Z_0$ and the prime indicates the differentiation with respect to $\tau = fZ_0t/(40V_1^2V_2^2)$. The characteristic equation of (19) is of the form

$$\nu_1\nu_2\lambda^2 + \left(\nu_1 + \nu_2\right)\left(2\sqrt{9E^2 - A} + E\right)\lambda - 5\sqrt{9E^2 - A}\left(\sqrt{9E^2 - A} - 2E\right) = 0.$$

This characteristic equation has a positive root and a negative root, since, according to Vieta's theorem, the product of the roots is negative, whereas their sum is positive. This means that the singular point of (15) is a saddle. The two-frequency self-vibrations of the string are unstable. Depending on perturbations, the self-vibrations become stable for the frequency ν_1 or for the frequency ν_2. If our mechanical system tends to the self-vibrations at the lowest frequency, then, during the transition process, there are the vibrations at the fundamental tone and the damping vibrations at the frequency ν_2. On the plane $(Z_1,\ Z_2)$, the domain $Z_1 \geq 0$, $Z_2 \geq 0$ is divided into two parts by two separatrices: one of them connects the coordinate origin with the singular point $(Z_0,\ Z_0)$, whereas the second one comes to this singular point from infinity. Each of these parts is the attraction domain of the corresponding one-frequency self-vibration. The two other separatrices connect the singular point $(Z_0,\ Z_0)$ with the singular points $(Z_{10},\ 0)$ and $(0,\ Z_{20})$.

Let us show that the self-vibration mode $Z_1 = \ldots = Z_N = Z_{0N} > 0$, $Z_j = 0$, $j = N+1, \ldots$, is unstable for $\chi = 0$ if $N \geq 2$. Here

$$Z_{0N} = \frac{\left(2N-1\right)E + \sqrt{(2N-1)^2E^2 - A\left(12N^2 - N - 8\right)}}{12N^2 - N - 8},$$

$$A = 8\left(v^2 - V_1^2\right)\left(v^2 - V_2^2\right) < 0.$$

The case $N = 2$ was considered above. For system (11), the variational equations are of the form

$$\begin{gathered}\nu_k \xi_k' + \alpha_N \Big(\xi_k + 2\sum_{n\neq k}^{N} \xi_n\Big) + \xi_k + \big(12N^2 - 24N + 54\big)\sum_{n\neq m}^{N} \xi_n = 0, \qquad k = 1, 2, \ldots, N,\\ (\cdot)' = \frac{1}{2CZ_{0N}^2}\frac{d(\cdot)}{dt}, \qquad \alpha_N = -E/Z_{0n} > 0.\end{gathered} \tag{20}$$

The characteristic equation of (20) can be represented as [9]

$$f_N(\lambda) = \begin{vmatrix} a_1 & u & \ldots & u \\ u & a_2 & \ldots & u \\ \cdot & \cdot & \ldots & \cdot \\ u & u & \ldots & a_N \end{vmatrix} = 0 \ \Rightarrow\ \prod_{k=1}^{N}(a_k - u) + u\sum_{i=1}^{N}\prod_{k\neq i}^{N}(a_k - u) = 0.$$

Here $a_k = \nu_k\lambda + (\alpha_N + 1)$, $u = 2\alpha_N + 12N^2 - 24N + 54$, and $f_N(\lambda) = b_N\lambda^N + \ldots + b_1\lambda + b_0$, where $b_N = \nu_1 + \ldots + \nu_N > 0$. Let us find the coefficient $b_0 = f_N(0) = (a-u)^{N-1}\big[a + (N-1)u\big]$, where $a = \alpha_N + 1 > 0$ and $a - u < 0$. If N is even, then $b_0 < 0$ and there exists a positive root of the characteristic equation. Hence, the self-vibration mode $Z_1 = \ldots = Z_N = Z_{0N} > 0$ is unstable. If N is odd, then $b_1 < 0$. Indeed,

$$b_1 = \frac{\partial f(0)}{\partial\lambda} = \sum_{k=1}^{N}\nu_k(a-u)^{N-2}(a + Nu) < 0.$$

The necessary condition of stability is not valid, since one of the coefficients of the characteristic equation is negative; hence, the corresponding self-vibration mode is unstable. If $\chi \neq 0$ is small, then the stationary mode $Z_1 = \ldots = Z_N = Z_{0N} > 0$ exists for bounded values of N and is unstable, since the roots of the characteristic equation are slightly perturbed and a number of roots remain with negative real parts.

Now we summarize our analysis of the existence of self-vibration modes and their stability when the internal dissipative forces are taken into account. The self-vibration modes exist when the velocity of the bow belongs to the interval (V_1, V_2), where the derivative $F'(V)$ is negative. The friction coefficient f should be sufficiently large in order that the inequality $A_1 > 0$ be valid in (12). The one-frequency stable vibrations exist as long as the coefficients A_m remain positive. From a certain number M, we have $A_k < 0$; hence, the self-vibration modes are absent at the corresponding frequencies. All multifrequency self-vibration modes are unstable. The domain $D_* = \{Z_1 \geq 0, \ldots, Z_{M-1} \geq 0\}$ is divided into the attraction domains of the one-frequency stable self-vibration modes. The transition process toward the corresponding one-frequency mode is accompanied by the overtone vibrations with decaying amplitudes.

If, before its vibrations, the string is in the state of rest, then the initial conditions take the form

$$Z_k(0) = \nu_k^{-2} f^2 \psi_k^4(s_0)\,(v - g_1 v^3 + g_2 v^5)^2, \qquad k = 1, 2, \ldots.$$

The motion trajectory described by (11) comes to the attraction domain specified by the point of contact between the bow and the string (more specifically, by the value of $\psi_k(s_0)$) and by the value of v. The attraction domain of the first harmonic is larger than those of higher harmonics. Varying the parameters s_0 and v, we can change the initial conditions and the overtone amplitudes during the transition process in the attraction domain of the first harmonic. Note that $\psi_1(s_0) > 0$ if $s_0 \in (0, l)$. The vibrations of the first fundamental tone are always excited at the beginning of the transition process. This is not the case for higher overtones, i.e., it may occur that a certain overtone is absent if $\psi_k(s_0) = 0$.

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