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Constructing the Asymptotics of Solutions to Differential Sturm–Liouville Equations in Classes of Oscillating Coefficients

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Abstract—The article is focused on the development of a method allowing one to construct asymptotics for solutions to ODEs of arbitrary order with oscillating coefficients on the semiaxis. The idea of the method is presented on the example of studying the asymptotics of the Sturm–Liouville equation.

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1. INTRODUCTION

A substantial number of works are devoted to studying asymptotic properties of solutions to singular Sturm–Liouville equations and differential equations of arbitrary order, see [1–3] and the references cited there. However, in these works, the circumstance was essentially used that the coefficients of the equation have a regular growth at infinity.

In works [4–11] the asymptotic properties of solutions to ordinary differential equations were investigated for equations with coefficients from a broader classes, in particular, those not satisfying the Titchmarsh–Levitan conditions.

In work [11] we proposed a method for studying the asymptotic behavior of solutions to the Sturm–Liouville equation

$$y'' + (1 + q(x))y = 0, \quad x_0 < x < \infty$$

in the case when $q(x)$ is a rapidly oscillating function from the class σ (see [11]) that has the property $\int_{x_0}^{\infty} q(x)dx < \infty$. This method allows constructing asymptotic formulas for solutions both in the case when $q(x)$ affects the dominate term of the asymptotic and in the opposite case. Note that the above mentioned method does not allow studying the case when $q(x)$ oscillates, but not falls into the class described in [11]. An example of such function is a function given by $\sin x/x^{\alpha}$, $\alpha > 0$. The current work is aimed at developing this approach to construct asymptotics with perturbations of type $p(x)/x^{\alpha}$, $\alpha > 0$, where $p(x)$ is an almost periodic function.

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2. CONSTRUCTION OF ASYMPTOTIC FORMULAS

Consider the equation

$$y'' + \left(\mu^2 + \frac{p(x)}{x^\alpha}\right) y = 0, \quad x_0 < x < \infty, \quad \mu \in \mathbb{C}, \tag{1}$$

where $0 < \alpha$ and $p(x)$ is an almost periodic function of the form

$$p(x) = \sum_{k=1}^{m} s_k e^{ip_k x}, \quad s_k \in \mathbb{C}, p_k \in \mathbb{R}. \tag{2}$$

The main result of this paper is the following

Theorem. *Let a function $p(x)$ have form* (2) *and*

(1) *for any set of numbers $\{c_1, \dots, c_m\}$, where $c_j \in \{0\} \cup \mathbb{N}$, the condition is met:*

$$\sum_{k=1}^{m} c_k p_k \neq 0; \tag{3}$$

(2) *for any p_k, $k = \overline{1,m}$, it is true that*

$$2\mu \neq \pm p_k. \tag{4}$$

Then for the fundamental system of solutions to Eq. (1) *the following asymptotic relations hold as $x \to +\infty$:*

$$\begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} = \begin{pmatrix} ie^{-i\mu x} & -ie^{i\mu x} \\ e^{-i\mu x} & e^{i\mu x} \end{pmatrix} \begin{pmatrix} 1+o(1) & o(1) \\ o(1) & 1+o(1) \end{pmatrix}. \tag{5}$$

Proof. The scheme of the proof is as follows. We reduce Eq. (1) to an equivalent system of equations. We introduce a vector function $\mathbf{z}(x,\lambda) = (z_1, z_2)$: $z_1 = y$, $z_2 = y'$.

Then Eq. (1) is rewritten as

$$\mathbf{z}' = \begin{pmatrix} 0 & 1 \\ -\mu^2 - p(x)/x^\alpha & 0 \end{pmatrix} \mathbf{z}. \tag{6}$$

Replacement

$$\mathbf{z} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \mathbf{u} \tag{7}$$

transits Eq. (6) to the system

$$\mathbf{u}' = i\mu L_0 \mathbf{u} + \frac{1}{x^\alpha} D(x)\mathbf{u},$$

where

$$L_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(x) = \frac{ip(x)}{2\mu} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}.$$

We make one more replacement:

$$\mathbf{u} = B(x)\mathbf{v}, \quad B(x) = B_0(x) + \frac{1}{x^\alpha} B_1(x) + \ldots + B_k(x)\frac{1}{x^{k\alpha}}, \tag{8}$$

where the parameter k is such that $\alpha \cdot (k+1) > 1$. Below, we show that by the conditions of the theorem the matrices B_i are bounded as $x \to \infty$. Replacement (8) leads to the system

$$B'(x)\mathbf{v} + B(x)\mathbf{v}' = i\mu L_0 B(x)\mathbf{v} + \frac{1}{x^\alpha} D(x)B(x)\mathbf{v}. \tag{9}$$

Assuming $x^{-\alpha}$ a small parameter as $x \to \infty$, we seek the matrices $B_i(x)$ from the following system of matrix equations:

$$\begin{cases} B_0' = i\mu L_0 B_0, \\ B_1' = i\mu L_0 B_1 + DB_0, \\ \dots \\ B_k' = i\mu L_0 B_k + DB_{k-1}. \end{cases} \tag{10}$$

From the first equation of system (10) we have

$$B_0 = e^{i\mu L_0 x} = \begin{pmatrix} e^{-i\mu x} & 0 \\ 0 & e^{i\mu x} \end{pmatrix}.$$

We present the formulas for solutions to the following equations:

$$\begin{cases} B_1 = B_0 - B_0 \cdot D_1, \quad D_1' = D_0 = B_0^{-1} D B_0, \\ B_2 = B_0 - B_0 \cdot D_1 + B_0 \cdot D_2, \quad D_2' = D_0 D_1, \\ \dots. \\ B_k = B_0 - B_0 \cdot D_1 + B_0 \cdot D_2 - \ldots + (-1)^k B_0 \cdot D_k, \quad D_k' = D_{k-2} D_{k-1}, \end{cases}$$

from which we finally obtain a representation for $B(x)$

$$B(x) = B_0 \cdot \left(E - \frac{1}{x^{\alpha}} D_1 + \frac{1}{x^{2\alpha}} D_2 + \ldots + \frac{(-1)^k}{x^{k\alpha}} D_k \right),$$

which also implies nondegeneracy of the matrix $B(x)$.

At integration of system (9) we need to perform operations of matrix multiplication and finding the antiderivative. It is easy to prove that due to conditions (3), (4) the mean value of any product $D \cdot B_j,\ j = \overline{0,k}$, is zero, which results in the boundedness of $B_j(x)$.

We denote

$$K(x) = -B'(x) + B_0'(x) + \frac{1}{x^{\alpha}} B_1'(x) + \ldots + B_k'(x) \frac{1}{x^{k\alpha}} + \frac{1}{x^{(k+1)\alpha}} D(x) B_k(x),$$

and due to the inequality $\alpha \cdot (k+1) > 1$ the matrix $K(x)$ is summable on (x_0, ∞). Taking into account (10), we can write system (9) as

$$\mathbf{v}' = B^{-1}(x) K(x) \mathbf{v}.$$

We proceed here to an equivalent system of integral equations and apply the method of successive approximations, thus obtaining the expressions for the dominate term of asymptotic of the fundamental system of solutions to the latter system:

$$\begin{pmatrix} v_1^1(x) & v_1^2(x) \\ v_2^1(x) & v_2^2(x) \end{pmatrix} = \begin{pmatrix} 1 + o(1) & o(1) \\ o(1) & 1 + o(1) \end{pmatrix}, \quad x \to \infty.$$

Performing the inverse replacements (7) and (8), we finally obtain the required asymptotic formulas (5) for the fundamental system of solutions to Eq. (1). The theorem is proved.

Note that condition (3) of the theorem may be excessive. In view of this, we provide a more accurate formulation of the conditions ensuring the boundedness of the matrices $B_j(x)$.

We introduce a matrix column $\mathfrak{B} = (B_0, B_1, \ldots B_k)^T$ and a matrix $\mathfrak{A} = i\mu I \otimes L_0 + J \otimes D$, where I is the identity matrix of order $k+1$, J is the lower-shear matrix of order $k+1$. Then system (10) is equivalent to the system

$$\mathfrak{B}' = \mathfrak{A}\mathfrak{B}. \tag{11}$$

Note that the Floquet theory is applicable to system (11). Let ρ_j, $j = \overline{0,k}$, be multiplicators of system (11). It is clear that the condition

$$|\rho_j| \leqslant 1, \quad j = \overline{0,k}, \tag{12}$$

provides existence of a bounded solution to system (11) and, therefore, to system (10) equivalent to it. It is important to note that ρ_j can be computed for any $\mu \in \mathbb{C}$; therefore, condition (12) can be checked.

Remark 1. From the proved theorem it follows that the perturbation $p(x)/x^\alpha$ does not affect the dominate part of asymptotic of solutions to Eq. (1) if conditions (3) and (4) are met.

Remark 2. Formulas (8) allow refining the asymptotic formulas up to order $1/x^{k\alpha}$ inclusively.

Remark 3. The proof of the theorem implies that, in the case of a periodic function $p(x)$, despite the fact that condition (3) of the theorem is not met, the given algorithm of constructing the asymptotic of the fundamental system of solutions for the original equation can be executed under the condition $\alpha > 1/2$. Let us illustrate the above said by the following example.

3. EXAMPLE

Consider an equation

$$y'' + \left(\mu^2 + \frac{\sin x}{x^\alpha}\right) y = 0, \quad x_0 < x < \infty, \quad 1/2 < \alpha, \quad 2\mu \neq \pm 1.$$

In this case $p(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$, $p_1 = 1$, $p_2 = -1$, and, for instance, for $c_1 = c_2 = 1$ we have $c_1 p_1 + c_2 p_2 = 0$, that is, condition (3) of the theorem is violated. In (8) we put $k = 1$, then the matrix $B(x)$ has the form

$$B(x) = B_0(x) + \frac{1}{x^\alpha} B_1(x), \quad B_1 = B_0 - B_0 D_1,$$

where

$$D_1'(x) = D_0(x) = B_0^{-1}(x) D(x) B_0(x).$$

For this example we compute the matrices D_0 and D_1. We have

$$D_0(x) = \frac{i \sin x}{2\mu} \begin{pmatrix} -1 & e^{2i\mu x} \\ -e^{-2i\mu x} & 1 \end{pmatrix}.$$

To compute the matrix D_1, we need to integrate the elements of the matrix D_0. The condition $2\mu \neq \pm 1$ provides a nonzero imaginary part for all elements of the matrix D_0, that is, absence of a resonance. We have

$$D_1(x) = \frac{i}{4\mu} \begin{pmatrix} 2\cos x & -\frac{1}{1-2\mu} e^{-ix(1-2\mu)} - \frac{1}{1+2\mu} e^{ix(1+2\mu)} \\ \frac{1}{1+2\mu} e^{-ix(1+2\mu)} + \frac{1}{1-2\mu} e^{ix(1-2\mu)} & -2\cos x \end{pmatrix}.$$

Note that, to make the matrix $B(x)$ contain three and more terms, we need to compute the elements of the matrix $D_0 \cdot D_1$ and then integrate this product. Because condition (3) of the theorem is violated for this example, some elements of the matrix $D_0 \cdot D_1$ contain zero imaginary part, which results in unboundedness of the matrices $B_k,\ k > 1$.

Thus, for this example we succeed in finding and explicitly computing the bounded matrices $B_0(x)$, $B_1(x)$ and obtaining the first two terms of the asymptotic of the fundamental system of solutions to Eq. (1) as $x \to \infty$:

$$\begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \left(B_0(x) + \frac{1}{x^\alpha} B_1(x) \right) \begin{pmatrix} 1 + o(1) & o(1) \\ o(1) & 1 + o(1) \end{pmatrix}.$$

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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