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Spectrum of Schrödinger Operator in Covering of Elliptic Ring

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Abstract—The stationary Schrödinger equation is studied in a domain bounded by two confocal ellipses and in its coverings. The order of dependence of the Laplace operator eigenvalues on sufficiently small distance between the foci is obtained. Coefficients of the power series expansion of said eigenvalues are calculated up to and including the square of half the focal distance.

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1. INTRODUCTION

It is well-known that a billiard in a domain bounded by confocal quadrics is integrable. Recent works by Fomenko and Vedyushkina (see [1–4], as well as other publications of these authors) have again attracted attention of researchers to this topic. In particular, singularities of a billiard in a ring bounded by confocal ellipses were studied in work [1]. In the current work we consider the corresponding quantum system, namely, we study the spectrum of the Schrödinger operator in this domain and in its coverings. We obtain asymptotics of eigenvalues when the focal distance tends to zero.

Note that the problem of finding the eigenvalues and eigenfunctions of the Laplace operator in the disk under the condition that the function is zero at the disk boundary is classical (see [5, Section 200, p. 262; 6]). The corresponding equation is split in polar coordinates, the dependence on the angle is described by the (co)sine, and the dependence on the polar radius is described by the Bessel function of the first kind. In particular, the eigenvalues are proportional to the squares of zeros of Bessel functions.

A similar problem for the ellipse is split in elliptic coordinates and is reduced to two Mathieu equations, angular and radial ones (we bring the necessary information in Section 3).

The same problem in the circular ring bounded by concentric circles is split in polar coordinates and, in a certain sense, is similar to the problem in the disk. The difference is in that in the radial equation another boundary conditions are imposed. Therefore, in the solution we obtain a linear combination of Bessel functions of the first and second kinds (this result is also classical, see [5, Section 207, p. 276]).

The problem in the finite-sheeted covering of the circular ring is an easy generalization of the previous one and is solved by the same methods; for completeness we present derivation of eigenfunctions and eigenvalues. For the covering of multiplicity $p = 1$ the results coincide with the classical ones (for the circular ring).

Our main result concerns the finite-sheeted covering of elliptic ring, that is, the domain bounded by two ellipses with identical foci (see Theorems 2 and 3). Namely, we obtain the asymptotics of eigenvalues in dependence on the focal distance up to the second order inclusively (this is equivalent to the decomposition in terms of powers of eccentricity of the inner or outer ellipse). For coinciding foci (for zero eccentricity) the results coincide with the formulas for the covering of circular ring (see Theorem 1).

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2. MODEL PROBLEM

Before proceeding to the main result, we consider a minor generalization of the classical problem. Let Ω be a domain p-sheeted covering the ring bounded by two concentric circles with radii $0 < r_0 < r_1$. The case $p = 1$ is referred to the classical theory of oscillations (see [5]). We assume that both circles have centers at the origin of coordinates. In the domain Ω it is convenient to consider an analog of polar coordinates, the distance r to the origin of coordinates and the angle φ defined by $\mod 2\pi p$. In Ω we consider the stationary Schrödinger equation

$$\hat{H}\psi = \left(\frac{-\hbar^2}{2M}\nabla^2 + V(r)\right)\psi = E\psi,$$

where the potential $V(r)$ is zero inside the domain Ω and turns into infinity outside it. Such problem is equivalent to searching eigenfunctions and eigenvalues of the Laplace operator in the domain Ω for the functions vanishing at the boundary of Ω. We put $\varkappa^2 = \dfrac{2ME}{\hbar^2}$. Further, J_ν and Y_ν are Bessel functions of the first and second kinds, respectively.

Theorem 1 (for $p = 1$ see [6, p. 165.]). *In the domain Ω (p-sheeted covering of the circular ring) the eigenfunctions $\psi_{k,m}(r, \varphi)$ and the eigenvalues $E_{k,m}$ of the operator $\hat{H}$ have the form*

$$\psi_{k,m}(r,\varphi) = \left[Y_\nu(\alpha_{k,\nu}) J_\nu\left(\frac{\alpha_{k,\nu} r}{r_0}\right) - Y_\nu\left(\frac{\alpha_{k,\nu} r}{r_0}\right) J_\nu(\alpha_{k,\nu})\right] \cos(\nu\varphi + \varphi_0), \quad E_{k,m} = \frac{\varkappa_{k,m}^2 \hbar^2}{2M},$$

where $\nu = \dfrac{m}{p}$, $\lambda = \dfrac{r_1}{r_0}$, $\varkappa_{k,m}^2 = \dfrac{\alpha_{k,\nu}^2}{r_0^2}$, $k, m \in \mathbb{N}$, *$\alpha_{k,\nu}$ is the kth zero of the function* $f(x) = Y_\nu(x) J_\nu(\lambda x) - Y_\nu(\lambda x) J_\nu(x)$.

Proof. We write the sought function in the form $\psi(r, \varphi) = R(r)\Phi(\varphi)$; then the equation $(\nabla^2 + \varkappa^2)\psi = 0$ becomes $\dfrac{R(r)\Phi''(\varphi)}{r^2} + \dfrac{R'(r)\Phi(\varphi)}{r} + \Phi(\varphi)R''(r) + \varkappa^2 R(r)\Phi(\varphi) = 0$. We multiply both sides of the equation by $\dfrac{r^2}{R(r)\Phi(\varphi)}$:

$$\frac{\Phi''(\varphi)}{\Phi(\varphi)} + \frac{rR'(r)}{R(r)} + \frac{r^2 R''(r)}{R(r)} + \varkappa^2 r^2 = 0.$$

We introduce the dividing parameter ν and obtain two equations (below, we do not specify the variables explicitly, assuming $\Phi = \Phi(\varphi), R = R(r)$):

$$\begin{cases} \dfrac{\Phi''}{\Phi} = -\nu^2, \\ \dfrac{rR'}{R} + \dfrac{r^2 R''}{R} + \varkappa^2 r^2 = \nu^2. \end{cases}$$

The solution to the angular equation is the function $\Phi(\varphi) = \cos(\nu\varphi + \varphi_0)$ for some real value φ_0. The periodicity condition $\Phi(0) = \Phi(2\pi p)$ implies that $\nu = \dfrac{m}{p}$, where m is an arbitrary nonnegative integer number.

The solution to the radial equation is sought in the form of linear combination of Bessel functions of the first and second kinds [7, Section 9, p. 358]:

$$R(r) = AJ_\nu(\varkappa r) + BY_\nu(\varkappa r).$$

From the boundary condition $R(r_0) = 0$ we establish the values of the constants: $A = Y_\nu(\varkappa r_0), B = -J_\nu(\varkappa r_0)$ (or those proportional to them).

Now, we consider the function $f(x) = Y_\nu(x)J_\nu(\lambda x) - Y_\nu(\lambda x)J_\nu(x)$, where $\lambda = \dfrac{r_1}{r_0}$. Then the boundary condition $R(r_1) = Y_\nu(\varkappa r_0)J_\nu(\varkappa r_1) - J_\nu(\varkappa r_0)Y_\nu(\varkappa r_1) = 0$ can be written as $f(\varkappa r_0) = 0$. We denote the kth positive zero of this function by $\alpha_{k,\nu}$ (see also the below Lemma 1). Then $\varkappa r_0 = \alpha_{k,\nu}$ for some value of k, which implies that $\varkappa$ can take only the values $\varkappa^2_{k,m}$ given in the formulation of the theorem. □

Lemma 1 [7, Section 9.5, p. 374; 8]. *The asymptotically mth positive zero $\alpha_{m,\nu}$ of the function $f(x) = Y_\nu(x)J_\nu(\lambda x) - Y_\nu(\lambda x)J_\nu(x)$ as $m \to \infty$ behaves as*

$$\alpha_{m,\nu} = \sigma + \frac{\chi}{\sigma} + \frac{\omega - \chi^2}{\sigma^3} + \frac{\eta - 4\chi\omega + 2\chi^3}{\sigma^5} + \dots,$$

where $\mu = 4\nu^2$, $\sigma = \dfrac{\pi m}{\lambda - 1}$, $\chi = \dfrac{\mu - 1}{8\lambda}$, $\omega = \dfrac{(\mu-1)(\mu-25)(\lambda^3-1)}{6(4\lambda)^3(\lambda-1)}$, $\eta = \dfrac{(\mu-1)(\mu^2-114\mu+1073)(\lambda^5-1)}{5(4\lambda)^5(\lambda-1)}$. □

3. PRELIMINARY INFORMATION

3.1. Separation of Variables in Equation in Ellipse

We consider the area bounded by an ellipse with the lengths of the major and minor semiaxes equal to w and h, respectively. We denote the half of the ellipse focal distance by $\delta = \sqrt{w^2 - h^2}$. We introduce the elliptic coordinates ρ, φ, $\rho \geq 0, 0 \leq \varphi \leq 2\pi$, where

$$(x, y) = (\delta \cosh \rho \cos \varphi, \delta \sinh \rho \sin \varphi).$$

They are regular outside the line segment connecting the foci $(\pm\delta, 0)$. The considered domain is given by the inequality $0 \leqslant \rho \leqslant \operatorname{arccosh}(\dfrac{w}{\delta})$. Under a fixed $w = r_0$ and $\delta \to 0$ the domain "tends" to a circle of radius r_0.

In this coordinate system the Laplace operator is as follows:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\dfrac{\partial^2}{\partial \rho^2} + \dfrac{\partial^2}{\partial \varphi^2}}{\delta^2(\cosh^2\rho - \cos^2\varphi)} = \frac{\dfrac{\partial^2}{\partial \rho^2} + \dfrac{\partial^2}{\partial \varphi^2}}{\dfrac{\delta^2}{2}(\cosh 2\rho - \cos 2\varphi)}.$$

The stationary Schrödinger equation is rewritten as

$$\nabla^2\psi + \varkappa^2\psi = 0, \text{ where } \varkappa^2 = \frac{2ME}{\hbar^2},$$

under the condition that ψ vanishes at the domain boundary. Dividing the variables $\psi(\rho, \varphi) = R(\rho)\Phi(\varphi)$, we reduce the equation to the form

$$\Phi\frac{\partial^2}{\partial\rho^2}R + R\frac{\partial^2}{\partial\varphi^2}\Phi + \frac{(\varkappa\delta)^2}{2}(\cosh 2\rho - \cos 2\varphi)R\Phi = 0.$$

In parentheses we add and subtract the separating parameter $\dfrac{2a}{(\varkappa\delta)^2}$ and obtain the *Mathieu equations*, in which $q = \dfrac{(\varkappa\delta)^2}{4}$:

$$\begin{cases} \dfrac{\partial^2}{\partial\varphi^2}\Phi + (a - 2q\cos 2\varphi)\Phi = 0 \quad \text{(the angular Mathieu equation)}, \\ \\ \dfrac{\partial^2}{\partial\rho^2}R - (a - 2q\cosh 2\rho)R = 0 \quad \text{(the radial Mathieu equation)}. \end{cases} \tag{1}$$

Table 1. Periodic Mathieu functions of even order

a	Periodic solution of angular Mathieu equation*	Period	Evenness of function
$a_{2n}(q)$	$ce_{2n}(z,q)$	period π	even
$a_{2n+1}(q)$	$ce_{2n+1}(z,q)$	antiperiod** π	even
$b_{2n+1}(q)$	$se_{2n+1}(z,q)$	antiperiod π	odd
$b_{2n+2}(q)$	$se_{2n+2}(z,q)$	period π	odd

* In Table 1 we provide just the eigenvalues of the period π or 2π.
** Antiperiod π: $f(x+\pi) = -f(x)$.

Table 2. Periodic Mathieu functions of fractional order ν

a	Periodic solution of angular Mathieu equation	Period	Evenness of function
$\lambda_\nu(q)$	$ce_\nu(z,q)$	period πp	even
$\lambda_\nu(q)$	$se_\nu(z,q)$	antiperiod πp	odd

3.2. Some Properties of Mathieu Functions

Consider the angular Mathieu equation $\dfrac{d^2}{dz^2}\Phi(z) + (a - 2q\cos 2z)\Phi(z) = 0$. Because the coefficients of the angular Mathieu equation are periodic in z, by the Floquet theorem [7] there exists a solution in form $F_\nu(z) = e^{i\nu z}P(z)$, where ν depends on the parameters a and q, while the function $P(z)$ has the same period π as the coefficients of the equation. The constant ν is called the characteristic exponent. For $\nu \notin \mathbb{Z}$ the functions $F_\nu(z)$ and $F_\nu(-z)$ are independent solutions to the differential equation. For $\nu \in \mathbb{Z}$ the functions $F_\nu(z)$ and $F_\nu(-z)$ are proportional and have the period π or 2π (see [7]).

According to the Sturm theory, for $q \neq 0$ existence of at most one periodic solution with the period π or 2π is possible. Depending on the evenness and the period of this solution, the parameter[1] a is referred to one of the two types:

$$a = \left[\begin{array}{ll} a_\nu(q), & \nu \in \{0\} \cup \mathbb{N}; \\ b_{-\nu}(q), & -\nu \in \mathbb{N}, \end{array}\right.$$

more precisely, for $n \in \{0\} \cup \mathbb{N}$ (see Table 1).

The third type is also distinguished: $a = \lambda_\nu(q), \nu \notin \mathbb{Z}$, which corresponds to the Mathieu functions of fractional order $ce_\nu(z,q), se_\nu(z,q)$. In the general case for $\nu \notin \mathbb{Q}$ both functions are nonperiodic, but for $\nu \in \mathbb{Q} \setminus \mathbb{Z}, \nu = \dfrac{n}{p}$, both have a period not larger than $2\pi p$. Table 1 for $\nu = \dfrac{n}{p}$ can be continued (see Table 2).

The meaning of the parameter ν becomes understandable when we substitute $q = 0$ into the angular Mathieu equation (1). In this case the angular function appears to be the same as in the case of disk; consequently, $\lambda_\nu(0) = \nu^2, ce_\nu(z,0) = \cos(\nu z), se_\nu(z,0) = \sin(\nu z)$.

[1] We can consider the Mathieu equation as a eigenvalue problem for the operator $D(y) = \dfrac{d^2y}{dx^2} - 2q\cos(2x)y$ (or operator $D(y) = \dfrac{d^2y}{dx^2} - 2q\cosh(2x)y$). Therefore, in the literature a is often called the eigenvalues.

The Fourier series for the angular Mathieu functions converge uniformly and absolutely on all compact sets in the complex plane. In the below given formulas it is assumed that $n \in \{0\} \cup \mathbb{N}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}$:

$$ce_{2n}(z,q) = \sum_{m=0}^{\infty} A_{2m}^{2n}(q) \cos 2mz, \quad ce_{2n+1}(z,q) = \sum_{m=0}^{\infty} A_{2m+1}^{2n+1}(q) \cos{(2m+1)z},$$

$$se_{2n+1}(z,q) = \sum_{m=0}^{\infty} B_{2m+1}^{2n+1}(q) \sin{(2m+1)z}, \quad se_{2n+2}(z,q) = \sum_{m=0}^{\infty} B_{2m+2}^{2n+2}(q) \sin{(2m+2)z},$$

$$ce_{\nu}(z,q) = \sum_{m=-\infty}^{\infty} c_{2m}^{\nu}(q) \cos{(\nu+2m)z}, \quad se_{\nu}(z,q) = \sum_{m=-\infty}^{\infty} c_{2m}^{\nu}(q) \sin{(\nu+2m)z}.$$

The coefficients A_k^l, B_k^l, c_k^ν satisfy certain recurrent relations (see [7]).

Now, we proceed to the radial Mathieu functions. The radial Mathieu functions of the first and second orders are defined as $Ce_n(z,q) = ce_n(\pm iz, q)$, $Se_n(z,q) = \mp ise_n(\pm iz,q)$ (see, e.g., [9]). For them and for the radial function of fractional order $M_\nu^{(1)}(z,q)$, there exist expansions in terms of Bessel functions of the first kind (here, the equality is understood up to a constant factor independent of z, which is irrelevant for the current work):

$$Ce_{2n}(z,q) \propto \sum_{m=0}^{\infty} (-1)^m A_{2m}^{2n}(q) J_{2m}(x), \quad Ce_{2n+1}(z,q) \propto \sum_{m=0}^{\infty} (-1)^{m+1} A_{2m+1}^{2n+1}(q) J_{2m+1}(x),$$

$$Se_{2n}(z,q) \propto \tanh z \sum_{m=1}^{\infty} (-1)^m 2m B_{2m}^{2n}(q) J_{2m}(x),$$

$$Se_{2n+1}(z,q) \propto \tanh z \sum_{m=1}^{\infty} (-1)^m (2m+1) B_{2m+1}^{2n+1}(q) J_{2m+1}(x),$$

$$M_\nu^{(1)}(z,q) \propto \sum_{m=-\infty}^{\infty} (-1)^m c_{2m}^{\nu}(q) J_{\nu+2m}(x), \quad \text{everywhere, for brevity,} \quad x = 2\sqrt{q}\cosh z. \tag{2}$$

Here, the coefficients A_k^l, B_k^l, c_k^ν are the same as in the Fourier expansion of the functions $ce_l(z,q)$, $se_l(z,q)$, $ce_\nu(z,q)$ (see [9, Chapter VIII, pp. 158–169]). By replacing the Bessel functions of first kind $J_m(x)$ with the Bessel functions of the second kind $Y_m(x)$ in the above formulas, we can obtain independent solutions to the corresponding equations. For instance, for the radial functions of the first kind of integer order $Ce_n(z,q)$, we have the second solution $Fey_n(z,q)$, while for the functions $Se_n(z,q)$ such second solution is denoted by $Gey_n(z,q)$ (see [9, Chapter VIII, Sections 8.11–13, pp. 158–162]). The same reasoning applied to $M_\nu^{(1)}(z,q)$ leads to the independent solution $M_\nu^{(2)}(z,q)$ for the case of fractional order.

4. MAIN RESULT

We consider the domain ("elliptic ring") bounded by two ellipses with long semiaxes $0 < r_0 < r_1$ and with common foci at the points $(\pm\delta, 0)$. In the elliptic coordinates (ρ, φ) this domain is given by the inequalities $\rho_0 = \operatorname{arccosh}\left(\frac{r_0}{\delta}\right) \leqslant \rho \leqslant \operatorname{arccosh}\left(\frac{r_1}{\delta}\right) = \rho_1$, $0 \leqslant \varphi \leqslant 2\pi$. For the p-sheeted covering Ω_δ of the elliptic ring, the inequality on the angular coordinate is different: $0 \leqslant \varphi \leqslant 2\pi p$. For convenience we introduce $\varkappa^2 = \dfrac{2ME}{\hbar^2}$.

We want to obtain the solutions to the stationary Schrödinger equation in the p-sheeted covering Ω_δ and the asymptotics of the corresponding energy levels at focus distance 2δ tending to zero.

Theorem 2. *In the domain Ω_δ (p-sheeted covering of the elliptic ring) the eigenfunctions $\psi_{k,m}(\rho,\varphi)$ and the eigenvalues $E_{k,m}$ of the operator $\hat{H}$ have the form*

$$\psi_{k,m}(\rho,\varphi)=\left[\begin{array}{l}\Big[Ce_\nu(\rho_0,q)Fey_\nu(\rho,q)-Ce_\nu(\rho,q)Fey_\nu(\rho_0,q)\Big]ce_\nu(\varphi,q)\Big|_{q=\beta_{k,\nu}},\\ E_{k,m}=\dfrac{\varkappa_{k,m}^2\hbar^2}{2M},\quad \nu\in\{0\}\cup\mathbb{N};\\ \\ \Big[Se_{-\nu}(\rho_0,q)Gey_{-\nu}(\rho,q)-Se_{-\nu}(\rho,q)Gey_{-\nu}(\rho_0,q)\Big]se_{-\nu}(\varphi,q)\Big|_{q=\beta_{k,\nu}},\\ E_{k,m}=\dfrac{\varkappa_{k,m}^2\hbar^2}{2M},\quad -\nu\in\mathbb{N};\\ \\ \Big[M_\nu^{(1)}(\rho_0,q)M_\nu^{(2)}(\rho,q)-M_\nu^{(1)}(\rho,q)M_\nu^{(2)}(\rho_0,q)\Big](A_1ce_\nu(\varphi,q)+A_2se_\nu(\varphi,q))\Big|_{q=\beta_{k,\nu}},\\ E_{k,m}=\dfrac{\varkappa_{k,m}^2\hbar^2}{2M}\quad \textit{otherwise},\end{array}\right.$$

where $\nu=\dfrac{m}{p}$, $\varkappa_{k,m}^2=\dfrac{4\beta_{k,\nu}}{\delta^2}$, $k,m\in\mathbb{N}$, $\beta_{k,\nu}$ *is the kth zero of the function*

$$f(q)=\left[\begin{array}{ll}Ce_\nu(\rho_0,q)Fey_\nu(\rho_1,q)-Ce_\nu(\rho_1,q)Fey_\nu(\rho_0,q), & \nu\in\{0\}\cup\mathbb{N};\\ Se_{-\nu}(\rho_0,q)Gey_{-\nu}(\rho_1,q)-Se_{-\nu}(\rho_1,q)Gey_{-\nu}(\rho_0,q), & -\nu\in\mathbb{N};\\ M_\nu^{(1)}(\rho_0,q)M_\nu^{(2)}(\rho_1,q)-M_\nu^{(1)}(\rho_1,q)M_\nu^{(2)}(\rho_0,q) & \textit{otherwise}.\end{array}\right. \tag{3}$$

Proof. We will seek the solution to the equation $\dfrac{-\hbar^2}{2M}\nabla^2\psi=E\psi$ in the form $\psi(\rho,\varphi)=R(\rho)\Phi(\varphi)$, where ρ and φ are elliptic coordinates. Then R and Φ are the solutions to the Mathieu equations

$$\begin{cases}\dfrac{\partial^2}{\partial\varphi^2}\Phi+(a-2q\cos 2\varphi)\Phi=0,\\ \\ \dfrac{\partial^2}{\partial\rho^2}R-(a-2q\cosh 2\rho)R=0,\end{cases} \tag{4}$$

where $q=\dfrac{(\varkappa\delta)^2}{4}$ and a is the separating variable. Firstly, we consider the angular Mathieu equation and determine under which a the condition $\Phi(0)=\Phi(2\pi p)$ holds.

By the Floquet theorem, for some ν there exists a solution $\Phi_\nu(\varphi)$ to the Mathieu equation such that $\Phi_\nu(\varphi+2\pi p)=e^{2i\pi p\nu}\Phi_\nu(\varphi)$. In the case of p-sheeted covering we need to impose the periodicity condition $\Phi_\nu(0)=\Phi_\nu(2\pi p)$. Consequently, $e^{2i\pi p\nu}=1$. Hence, $p\nu=m\in\mathbb{Z}$ and, therefore, $\nu=\dfrac{m}{p}$, where $m\in\mathbb{Z}$. We put $\Phi(\varphi)=\Phi_\nu(\varphi)$.

By $R_1(\rho,q),R_2(\rho,q)$ we denote two independent solutions to the radial Mathieu equation (4) depending on the parameter q. The solution to Eq.(4) is their linear combination $R(\rho,q)=AR_1(\rho,q)+BR_2(\rho,q)$. From the condition $R(\rho_0,q)=0$ we establish the values of the constants: $A=R_2(\rho_0,q)$, $B=-R_1(\rho_0,q)$ (or the values proportional to them). Now, we consider the function $f(q)=R_2(\rho_0,q)\times R_1(\rho_1,q)-R_1(\rho_0,q)R_2(\rho_1,q)$; depending on the value ν this is one of functions (3). Then the condition $R(\rho_1,q)=0$ can be written as $f(q)=0$. We denote the kth positive zero of this function by $\beta_{k,\nu}$, then $q=\beta_{k,\nu}$ for some value of k, which implies that $\varkappa^2=\dfrac{4q}{\delta^2}$ can take on only the values $\varkappa_{k,m}^2$ given in the formulation of the theorem.

In conclusion of the proof, we just need to present the explicit form of the functions $\Phi(\varphi), R(\rho)$. Depending on the value $\nu = \dfrac{m}{p}$, the separating parameter a in the system of differential equations (4) is referred to one of the three types:

$$a = \left[\begin{array}{ll} a_\nu(q), & \nu \in \{0\} \cup \mathbb{N}; \\ b_{-\nu}(q), & -\nu \in \mathbb{N}; \\ \lambda_\nu(q) & \text{otherwise.} \end{array}\right.$$

The periodic angular solutions in the first two cases are the functions described in Table 1; they are the functions $\Phi(\varphi)$ depending on the value of ν.

The radial functions are obtained in as linear combinations of integer-order radial Mathieu functions. As $R_1(\rho, q)$ we take the radial Mathieu functions of the first kind (2): $Ce_\nu(\rho, q)$ for $\nu \in \{0\} \cup \mathbb{N}$ and $Se_\nu(\rho, q)$ for $-\nu \in \mathbb{N}$. The independent solutions $R_2(\rho, q)$ for these two cases are $Fey_\nu(\rho, q)$ and $Gey_\nu(\rho, q)$, respectively.

In the case $\nu = \dfrac{m_1}{m_2} \in \mathbb{Q} \setminus \mathbb{Z}$ both angular functions $ce_\nu(\varphi, q), se_\nu(\varphi, q)$ are periodic and have a period not larger than $2\pi m_2$ (see [7]); therefore, also their linear combination suits as $\Phi(\varphi)$. The solution to the radial Mathieu equation is represented as a linear combination of the functions $Ce_\nu(\rho, q)$ and $Se_\nu(\rho, q)$. However, in the next theorem it is more convenient to use a linear combination of the functions $M_\nu^{(1)}(\rho, q), M_\nu^{(2)}(\rho, q)$ (see [10, Section 28.23; 11, Chapter 2, Section 2.4, p. 165]), also forming a fundamental system. □

We introduce the function

$$W_{a,b}(u) = Y_a(u)J_b(\lambda u) - Y_a(\lambda u)J_b(u), \quad \lambda = \frac{r_1}{r_0}. \tag{5}$$

We denote: $\nu = \dfrac{m}{p}, m \in \mathbb{Z}$.

Theorem 3. *The value $\varkappa^2_{k,m}(\delta), k \in \mathbb{N}$, depends on the half of the focal distance δ with an accuracy up to $o(\delta^2)$ as follows:*

$$\varkappa^2_{k,m}(\delta) = \left[\begin{array}{l} \dfrac{\alpha^2_{k,\nu}}{r_0^2} + \delta^2 \dfrac{\alpha^3_{k,\nu}}{8\nu r_0^4} \left.\dfrac{\frac{\nu-2}{\nu-1}(W_{\nu-2,\nu}(u)+W_{\nu,\nu-2}(u)) - \frac{\nu+2}{\nu+1}(W_{\nu+2,\nu}(u)+W_{\nu,\nu+2}(u))}{\frac{\partial W_{\nu,\nu}(u)}{\partial u}}\right|_{u=\alpha_{k,\nu}}, \\ -\nu \in \mathbb{N} \setminus \{1,2\}; \\ \dfrac{\alpha^2_{k,2}}{r_0^2} - \delta^2 \dfrac{\alpha^3_{k,2}}{12 r_0^4} \left.\dfrac{(W_{4,2}(u)+W_{2,4}(u))}{\frac{\partial W_{2,2}(u)}{\partial u}}\right|_{u=\alpha_{k,2}}, \quad -\nu = 2; \\ \dfrac{\alpha^2_{k,1}}{r_0^2} - \delta^2 \dfrac{3\alpha^3_{k,1}}{16 r_0^4} \left.\dfrac{(W_{3,1}(u)+W_{1,3}(u))}{\frac{\partial W_{1,1}(u)}{\partial u}}\right|_{u=\alpha_{k,1}}, \quad -\nu = 1; \\ \dfrac{\alpha^2_{k,0}}{r_0^2} - \delta^2 \dfrac{\alpha^3_{k,0}}{4 r_0^4} \left.\dfrac{(W_{2,0}(u)+W_{0,2}(u))}{\frac{\partial W_{0,0}(u)}{\partial u}}\right|_{u=\alpha_{k,0}}, \quad \nu = 0; \\ \dfrac{\alpha^2_{k,1}}{r_0^2} - \delta^2 \dfrac{\alpha^3_{k,1}}{16 r_0^4} \left.\dfrac{(W_{3,1}(u)+W_{1,3}(u))}{\frac{\partial W_{1,1}(u)}{\partial u}}\right|_{u=\alpha_{k,1}}, \quad \nu = 1; \\ \dfrac{\alpha^2_{k,\nu}}{r_0^2} + \delta^2 \dfrac{\alpha^3_{k,\nu}}{2 r_0^4} \left.\dfrac{\frac{1}{4(\nu-1)}(W_{\nu-2,\nu}(u)+W_{\nu,\nu-2}(u)) - \frac{1}{4(\nu+1)}(W_{\nu+2,\nu}(u)+W_{\nu,\nu+2}(u))}{\frac{\partial W_{\nu,\nu}(u)}{\partial u}}\right|_{u=\alpha_{k,\nu}}, \\ \left[\begin{array}{l} \nu \in \mathbb{N} \setminus \{1\}, \\ \nu \notin \mathbb{Z}. \end{array}\right. \end{array}\right. \tag{6}$$

Remark. This expansion is equivalent to the expansion in terms of the eccentricity ε_s of the inner or outer ellipse with the major semiaxis $r_s, s = 0, 1$, which results from the substitution $\delta = \varepsilon_s r_s$.

Remark. The case $\delta = 0$ corresponds to the covering of the circular ring. It is easy to see that in this case the result of Theorem 3 (that is, zero terms of expansions) corresponds to the result of Theorem 1 after multiplication by $\dfrac{\hbar^2}{2M}$.

Remark. The derivative $\dfrac{\partial W_{\nu,\nu}(u)}{\partial u}$ admits expression through the Bessel functions of the first and second kinds. We have the identities (see [10]): $2\dfrac{\partial Y_\nu(u)}{\partial u} = Y_{\nu-1}(u) - Y_{\nu+1}(u)$, $2\dfrac{\partial J_\nu(u)}{\partial u} = J_{\nu-1}(u) - J_{\nu+1}(u)$. By direct differentiation of $W_{\nu,\nu}(u)$ we obtain

$$\frac{\partial W_{\nu,\nu}(u)}{\partial u} = \lambda(Y_\nu(u)J_{\nu-1}(\lambda u) + Y_{\nu+1}(\lambda u)J_\nu(u)) - (Y_\nu(\lambda u)J_{\nu-1}(u) + Y_{\nu+1}(u)J_\nu(\lambda u)).$$

Proof. By the previous theorem the eigenvalues $E_{k,m}$ of the operator $\hat{H}$ and, consequently, the numbers $\varkappa^2_{k,m} = \varkappa^2_{k,m}(\delta)$ are related with the zeros $\beta_{k,\nu}$ of the function $f(q)$. Here, the function $f(q)$ has one of the three possible types (see (3)).

We present two lemmas with which we are going to prove (6).

Let $\nu = \dfrac{m}{p}$, $m \geqslant 0$, $a = \lambda_\nu(q)$, $q = \dfrac{\varkappa^2\delta^2}{4}$ and suppose that $ce_\nu(\varphi, q)$ is the even solution to the angular Mathieu equation with the specified parameters a, q. Recall that for small q the expansion holds (see [11, Section 2.2, pp. 122–124]):

$$ce_\nu(\varphi, q) = c_\nu \cos \nu\varphi + qc_{\nu+2} \cos(\nu + 2)\varphi + qc_{\nu-2} \cos(\nu - 2)\varphi + o(q).$$

The possible values $\varkappa^2$ are determined from the condition that the radial Mathieu function vanishes on the boundary ellipses. Namely, we put $R(\rho)$ as the solution to the radial Mathieu equation with the same parameters and the boundary condition $R(\rho_0) = R(\rho_1) = 0$, $\rho_0 < \rho_1$.

Lemma 2. *Let $r_0 = \delta \cosh \rho_0, r_1 = \delta \cosh \rho_1, \lambda = \dfrac{r_1}{r_0}$ and suppose that $\alpha_{k,\nu}$ is the kth zero of the function $W_{\nu,\nu}(u)$. Then for small δ the value $\varkappa^2$ is as follows:*

$$\varkappa^2 = \frac{\alpha^2_{k,\nu}}{r_0^2} + \delta^2 \frac{\alpha^3_{k,\nu}}{2c_\nu r_0^4}$$

$$\times \left. \frac{c_{\nu+2}\left(W_{\nu+2,\nu}(u) + W_{\nu,\nu+2}(u)\right) + c_{\nu-2}\left(W_{\nu-2,\nu}(u) + W_{\nu,\nu-2}(u)\right)}{\dfrac{\partial W_{\nu,\nu}(u)}{\partial u}} \right|_{u=\alpha_{k,\nu}} + o(\delta^2).$$

Proof. At $a = \lambda_\nu(q)$, $\nu = \dfrac{m}{p}$, $m \geqslant 0$, the Fourier coefficients of the angular Mathieu function $ce_\nu(\varphi, q)$ are coupled [7], up to a constant factor, with the expansion of the radial Mathieu function $R_1(\rho, q) = \left[\begin{array}{l} Ce_\nu(\rho, q),\ \nu \in \{0\} \cup \mathbb{N}; \\ M^{(1)}_\nu(\rho, q),\ \nu \notin \mathbb{Z}, \end{array}\right.$ into an infinite sum of the Bessel functions as follows:

$$R_1(\rho, q) = c_\nu J_\nu(2\sqrt{q}\cosh\rho) - qc_{\nu-2}J_{\nu-2}(2\sqrt{q}\cosh\rho) - qc_{\nu+2}J_{\nu+2}(2\sqrt{q}\cosh\rho) + o(q).$$

The second solution $R_2(\rho, q)$ to the radial Mathieu equation can be obtained from $R_1(\rho, q)$ by replacing the Bessel functions of the first kind $J_\nu(x)$ with the Bessel functions of the second kind $Y_\nu(x)$. In particular, these are the functions $Fey_\nu(\rho, q)$ for $\nu \in \{0\} \cup \mathbb{N}$ and $M^{(2)}_\nu(\rho, q)$ for $\nu \in \mathbb{Q} \setminus \mathbb{Z}, \nu \geq 0$.

We recall the boundary condition $R_2(\rho_0)R_1(\rho_1) - R_2(\rho_1)R_1(\rho_0) = 0$. Note that the arguments have the form $2\sqrt{q}\cosh\rho_s = 2\sqrt{\frac{\varkappa^2\delta^2}{4}\frac{r_s}{\delta}} = \varkappa r_s, s = 0, 1$. Let us consider the first summand in the boundary condition:

$$\begin{aligned} R_2(\rho_0)R_1(\rho_1) &= (c_\nu Y_\nu(\varkappa r_0) - qc_{\nu-2}Y_{\nu-2}(\varkappa r_0) - qc_{\nu+2}Y_{\nu+2}(\varkappa r_0) + o(q)) \\ &\quad\times (c_\nu J_\nu(\varkappa r_1) - qc_{\nu-2}J_{\nu-2}(\varkappa r_1) - qc_{\nu+2}J_{\nu+2}(\varkappa r_1) + o(q)) \\ &= c_\nu^2 Y_\nu(\varkappa r_0)J_\nu(\varkappa r_1) - qc_\nu\Big(c_{\nu-2}(Y_{\nu-2}(\varkappa r_0)J_\nu(\varkappa r_1) + Y_\nu(\varkappa r_0)J_{\nu-2}(\varkappa r_1)) \\ &\quad + c_{\nu+2}(Y_{\nu+2}(\varkappa r_0)J_\nu(\varkappa r_1) + Y_\nu(\varkappa r_0)J_{\nu+2}(\varkappa r_1))\Big) + o(q). \end{aligned} \tag{7}$$

Dividing both sides of the last expression by c_ν^2, for convenience's sake we determine $u = \varkappa r_0, \lambda = \frac{r_1}{r_0}, \varkappa r_1 = \lambda u$. We write the full expression $R_2(\rho_0)R_1(\rho_1) - R_2(\rho_1)R_1(\rho_0) = 0$: because the terms differ from each other only by permutation of the arguments u and λu, using formula (7) leads to appearance of functions (5). Thus,

$$\begin{aligned} 0 &= R_2(\rho_0)R_1(\rho_1) - R_2(\rho_1)R_1(\rho_0) \\ &= W_{\nu,\nu}(u) - q\left(\frac{c_{\nu-2}}{c_\nu}(W_{\nu-2,\nu}(u) + W_{\nu,\nu-2}(u)) + \frac{c_{\nu+2}}{c_\nu}(W_{\nu+2,\nu}(u) + W_{\nu,\nu+2}(u))\right) + o(q). \end{aligned} \tag{8}$$

Suppose that $\alpha_{k,\nu}$ is the kth zero of the function $W_{\nu,\nu}(u)$. Then in a sufficiently small its neighborhood it is true that

$$W_{\nu,\nu}(u) = (u - \alpha_{k,\nu})\left.\frac{\partial W_{\nu,\nu}(u)}{\partial u}\right|_{u=\alpha_{k,\nu}} + \frac{(u-\alpha_{k,\nu})^2}{2}\left.\frac{\partial^2 W_{\nu,\nu}(u)}{\partial u^2}\right|_{u=\alpha_{k,\nu}} + o((u-\alpha_{k,\nu})^2).$$

We put $u = \alpha_{k,\nu} + u_1\delta + u_2\delta^2 + o(\delta^2)$ and substitute $q = \frac{\varkappa^2\delta^2}{4} = \frac{u^2\delta^2}{4r_0^2}$ into expression (8):

$$\begin{aligned} &(u_1\delta + u_2\delta^2 + o(\delta^2))\left.\frac{\partial W_{\nu,\nu}(u)}{\partial u}\right|_{u=\alpha_{k,\nu}} + \frac{u_1^2\delta^2 + o(\delta^2)}{2}\left.\frac{\partial^2 W_{\nu,\nu}(u)}{\partial u^2}\right|_{u=\alpha_{k,\nu}} - \frac{\alpha_{k,\nu}^2\delta^2 + o(\delta^2)}{4r_0^2} \\ &\times\left[\left.\left(\frac{c_{\nu-2}}{c_\nu}(W_{\nu-2,\nu}(u) + W_{\nu,\nu-2}(u)) + \frac{c_{\nu+2}}{c_\nu}(W_{\nu+2,\nu}(u) + W_{\nu,\nu+2}(u))\right)\right|_{u=\alpha_{k,\nu}} + o(\delta)\right] + o(\delta^2) = 0. \end{aligned}$$

Because the equality must hold at each power of δ, we firstly obtain $u_1 = 0$ and then, by equating the coefficients at δ^2, arrive at

$$u_2 = \frac{\alpha_{k,\nu}^2}{4r_0^2}\left.\frac{\left(\frac{c_{\nu-2}}{c_\nu}(W_{\nu-2,\nu}(u) + W_{\nu,\nu-2}(u)) + \frac{c_{\nu+2}}{c_\nu}(W_{\nu+2,\nu}(u) + W_{\nu,\nu+2}(u))\right)}{\frac{\partial W_{\nu,\nu}(u)}{\partial u}}\right|_{u=\alpha_{k,\nu}}.$$

Thus,

$$\begin{aligned} u &= \alpha_{k,\nu} + \delta^2\frac{\alpha_{k,\nu}^2}{4r_0^2} \\ &\times\left.\frac{\left(\frac{c_{\nu-2}}{c_\nu}(W_{\nu-2,\nu}(u) + W_{\nu,\nu-2}(u)) + \frac{c_{\nu+2}}{c_\nu}(W_{\nu+2,\nu}(u) + W_{\nu,\nu+2}(u))\right)}{\frac{\partial W_{\nu,\nu}(u)}{\partial u}}\right|_{u=\alpha_{k,\nu}} + o(\delta^2). \end{aligned}$$

Because $u = \varkappa r_0$, we obtain

$$\varkappa^2 = \frac{\alpha_{k,\nu}^2}{r_0^2} + \delta^2 \frac{\alpha_{k,\nu}^3}{2c_\nu r_0^4} \times \left. \frac{c_{\nu+2}\left(W_{\nu+2,\nu}(u) + W_{\nu,\nu+2}(u)\right) + c_{\nu-2}\left(W_{\nu-2,\nu}(u) + W_{\nu,\nu-2}(u)\right)}{\dfrac{\partial W_{\nu,\nu}(u)}{\partial u}} \right|_{u=\alpha_{k,\nu}} + o(\delta^2).$$

The lemma is proved.

Let $a = \lambda_{-\nu}(q), \nu \in \mathbb{N}, q = \dfrac{\varkappa^2\delta^2}{4}$ and suppose that $se_\nu(\varphi, q)$ is the odd solution to the angular Mathieu equation with the specified parameters a, q. Recall that for small q the expansion holds (see [11, Section 2.2, pp. 122–124]):

$$se_\nu(\varphi, q) = c_\nu \sin \nu\varphi + qc_{\nu+2} \sin{(\nu + 2)\varphi} + qc_{\nu-2} \sin{(\nu - 2)\varphi} + o(q).$$

The possible values $\varkappa^2$ are determined from the condition that the radial Mathieu function vanishes on the boundary ellipses. Namely, we put $R(\rho)$ as the solution to the radial Mathieu equation with the same parameters and the boundary condition $R(\rho_0) = R(\rho_1) = 0$, $\rho_0 < \rho_1$.

Lemma 3. *Let $r_0 = \delta \cosh \rho_0, r_1 = \delta \cosh \rho_1, \lambda = \dfrac{r_1}{r_0}$ and suppose that $\alpha_{k,\nu}$ is the kth zero of the function $W_{\nu,\nu}(u)$, then $\varkappa^2$ depends on δ as*

$$\varkappa^2 = \frac{\alpha_{k,\nu}^2}{r_0^2} + \delta^2 \frac{\alpha_{k,\nu}^3}{2\nu c_\nu r_0^4} \times \left. \frac{(\nu-2)c_{\nu-2}\left(W_{\nu-2,\nu}(u) + W_{\nu,\nu-2}(u)\right) + (\nu+2)c_{\nu+2}\left(W_{\nu+2,\nu}(u) + W_{\nu,\nu+2}(u)\right)}{\dfrac{\partial W_{\nu,\nu}(u)}{\partial u}} \right|_{u=\alpha_{k,\nu}} + o(\delta^2).$$

Proof. At $a = \lambda_{-\nu}(q), \nu \in \mathbb{N}$, the Fourier coefficients of the odd angular Mathieu function $se_\nu(\varphi, q)$ are coupled [7], up to a constant factor, with the expansion of the radial Mathieu function $Se_\nu(\rho, q)$ into an infinite sum of Bessel functions as follows:

$$\begin{aligned} Se_\nu(\rho, q) = \nu c_\nu J_\nu(2\sqrt{q}\cosh\rho) - q(\nu-2)c_{\nu-2}J_{\nu-2}(2\sqrt{q}\cosh\rho) \\ - q(\nu+2)c_{\nu+2}J_{\nu+2}(2\sqrt{q}\cosh\rho) + o(q). \end{aligned}$$

We can obtain the second solution to the radial Mathieu equations $R_2(\rho) = Gey_\nu(\rho, q)$ from the solution $R_1(\rho) = Se_\nu(\rho, q)$ by replacing the Bessel functions of the first kind $J_\nu(x)$ with the Bessel functions of the second kind $Y_\nu(x)$.

In the boundary condition $R_2(\rho_0)R_1(\rho_1) - R_2(\rho_1)R_1(\rho_0) = 0$ the arguments have the form $2\sqrt{q}\cosh\rho_s = 2\sqrt{\dfrac{\varkappa^2\delta^2}{4}}\dfrac{r_s}{\delta} = \varkappa r_s$, $s = 0, 1$. Consider the first summand:

$$\begin{aligned} R_2(\rho_0)R_1(\rho_1) &= (\nu c_\nu Y_\nu(\varkappa r_0) - q(\nu-2)c_{\nu-2}Y_{\nu-2}(\varkappa r_0) - q(\nu+2)c_{\nu+2}Y_{\nu+2}(\varkappa r_0) + o(q)) \\ &\quad \times (\nu c_\nu J_\nu(\varkappa r_1) - q(\nu-2)c_{\nu-2}J_{\nu-2}(\varkappa r_1) - q(\nu+2)c_{\nu+2}J_{\nu+2}(\varkappa r_1) + o(q)) \\ &= \nu^2 c_\nu^2 Y_\nu(\varkappa r_0)J_\nu(\varkappa r_1) - q\nu c_\nu\Big((\nu-2)c_{\nu-2}\left(Y_{\nu-2}(\varkappa r_0)J_\nu(\varkappa r_1) + Y_\nu(\varkappa r_0)J_{\nu-2}(\varkappa r_1)\right) \\ &\quad + (\nu+2)c_{\nu+2}\left(Y_{\nu+2}(\varkappa r_0)J_\nu(\varkappa r_1) + Y_\nu(\varkappa r_0)J_{\nu+2}(\varkappa r_1)\right)\Big) + o(q). \end{aligned}$$

We divide both sides of the expression by $\nu^2 c_\nu^2$ and, for convenience's sake, determine $u = \varkappa r_0$, $\lambda = \frac{r_1}{r_0}\varkappa r_1 = \lambda u$. We write the full expression $R_2(\rho_0)R_1(\rho_1) - R_2(\rho_1)R_1(\rho_0) = 0$ in the same manner as in Lemma 2:

$$0 = R_2(\rho_0)R_1(\rho_1) - R_2(\rho_1)R_1(\rho_0) = W_{\nu,\nu}(u) - \frac{q}{\nu c_\nu}\Big((\nu-2)c_{\nu-2}\left(W_{\nu-2,\nu}(u) + W_{\nu,\nu-2}(u)\right) + (\nu+2)c_{\nu+2}\left(W_{\nu+2,\nu}(u) + W_{\nu,\nu+2}(u)\right)\Big) + o(q). \tag{9}$$

Suppose that $\alpha_{k,\nu}$ is the kth zero of the function $W_{\nu,\nu}(u)$. Then in its sufficiently small neighborhood it is true that

$$W_{\nu,\nu}(u) = (u - \alpha_{k,\nu})\left.\frac{\partial W_{\nu,\nu}(u)}{\partial u}\right|_{u=\alpha_{k,\nu}} + \frac{(u-\alpha_{k,\nu})^2}{2}\left.\frac{\partial^2 W_{\nu,\nu}(u)}{\partial u^2}\right|_{u=\alpha_{k,\nu}} + o((u-\alpha_{k,\nu})^2).$$

We put $u = \alpha_{k,\nu} + u_1\delta + u_2\delta^2 + o(\delta)$ and substitute $q = \frac{\varkappa^2\delta^2}{4} = \frac{u^2\delta^2}{4r_0^2}$ into expression (9):

$$(u_1\delta + u_2\delta^2 + o(\delta^2))\left.\frac{\partial W_{\nu,\nu}(u)}{\partial u}\right|_{u=\alpha_{k,\nu}} + \frac{u_1^2\delta^2 + o(\delta^2)}{2}\left.\frac{\partial^2 W_{\nu,\nu}(u)}{\partial u^2}\right|_{u=\alpha_{k,\nu}}$$
$$- \frac{\alpha_{k,\nu}^2\delta^2 + o(\delta^2)}{4r_0^2}\left[\left(\frac{(\nu-2)c_{\nu-2}}{\nu c_\nu}\left(W_{\nu-2,\nu}(u) + W_{\nu,\nu-2}(u)\right)\right.\right.$$
$$\left.\left.\left. + \frac{(\nu+2)c_{\nu+2}}{\nu c_\nu}\left(W_{\nu+2,\nu}(u) + W_{\nu,\nu+2}(u)\right)\right)\right|_{u=\alpha_{k,\nu}} + o(\delta)\right] + o(\delta^2) = 0.$$

We equate the coefficients at each power and obtain

$$u = \alpha_{k,\nu} + \delta^2\frac{\alpha_{k,\nu}^2}{4r_0^2}\frac{1}{\frac{\partial W_{\nu,\nu}(u)}{\partial u}}\left(\frac{(\nu-2)c_{\nu-2}}{\nu c_\nu}\left(W_{\nu-2,\nu}(u) + W_{\nu,\nu-2}(u)\right)\right.$$
$$\left.\left. + \frac{(\nu+2)c_{\nu+2}}{\nu c_\nu}\left(W_{\nu+2,\nu}(u) + W_{\nu,\nu+2}(u)\right)\right)\right|_{u=\alpha_{k,\nu}} + o(\delta^2),$$

which, by the definition $u = \varkappa r_0$, implies the equality

$$\varkappa^2 = \frac{\alpha_{k,\nu}^2}{r_0^2} + \delta^2\frac{\alpha_{k,\nu}^3}{2\nu c_\nu r_0^4}$$
$$\times \left.\frac{(\nu-2)c_{\nu-2}\left(W_{\nu-2,\nu}(u) + W_{\nu,\nu-2}(u)\right) + (\nu+2)c_{\nu+2}\left(W_{\nu+2,\nu}(u) + W_{\nu,\nu+2}(u)\right)}{\frac{\partial W_{\nu,\nu}(u)}{\partial u}}\right|_{u=\alpha_{k,\nu}} + o(\delta^2).$$

The lemma is proved.

Let us return to the proof of Theorem 3.

Case 1: $a = \lambda_\nu(q)$, $\nu = \dfrac{m}{p}$, $m \geqslant 0$. Then for small q the even solution to the angular Mathieu equation $ce_\nu(\varphi, q)$ can be represented as (see [7])

$$ce_\nu(\varphi, q) = \begin{cases} \cos\nu\varphi + \dfrac{q}{4(\nu-1)}\cos(\nu-2)\varphi - \dfrac{q}{4(\nu+1)}\cos(\nu+2)\varphi + o(q), \\ \nu \in \mathbb{N}\setminus\{1\} \text{ or } \nu \in \mathbb{Q}\setminus\mathbb{Z}; \\ \\ \cos\varphi - \dfrac{q}{8}\cos 3\varphi + o(q), \quad \nu = 1; \\ \\ \dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}\dfrac{q}{2}\cos 2\varphi + o(q), \quad \nu = 0. \end{cases}$$

Let $\nu \in \mathbb{N}\setminus\{1\}$ or $\nu \in \mathbb{Q}\setminus\mathbb{Z}$. Then $c_\nu = 1, c_{\nu+2} = \dfrac{-1}{4(\nu+1)}, c_{\nu-2} = \dfrac{1}{4(\nu-1)}$. We apply Lemma 2:

$$\varkappa^2 = \frac{\alpha_{k,\nu}^2}{r_0^2} + \delta^2\frac{\alpha_{k,\nu}^3}{2r_0^4}$$
$$\times \left.\frac{\dfrac{1}{4(\nu-1)}\left(W_{\nu-2,\nu}(u) + W_{\nu,\nu-2}(u)\right) - \dfrac{1}{4(\nu+1)}\left(W_{\nu+2,\nu}(u) + W_{\nu,\nu+2}(u)\right)}{\dfrac{\partial W_{\nu,\nu}(u)}{\partial u}}\right|_{u=\alpha_{k,\nu}} + o(\delta^2).$$

Suppose that $\nu = 1$. Then $c_\nu = 1, c_{\nu+2} = \dfrac{-1}{8}, c_{\nu-2} = 0$, and from Lemma 2 we obtain

$$\varkappa^2 = \frac{\alpha_{k,\nu}^2}{r_0^2} - \delta^2\frac{\alpha_{k,\nu}^3}{16r_0^4}\left.\frac{\left(W_{\nu+2,\nu}(u) + W_{\nu,\nu+2}(u)\right)}{\dfrac{\partial W_{\nu,\nu}(u)}{\partial u}}\right|_{u=\alpha_{k,\nu}} + o(\delta^2).$$

Let $\nu = 0$. Then $c_\nu = \dfrac{1}{\sqrt{2}}, c_{\nu+2} = \dfrac{-1}{2\sqrt{2}}, c_{\nu-2} = 0$, and by Lemma 2 we have

$$\varkappa^2 = \frac{\alpha_{k,\nu}^2}{r_0^2} - \delta^2\frac{\alpha_{k,\nu}^3}{4r_0^4}\left.\frac{\left(W_{\nu+2,\nu}(u) + W_{\nu,\nu+2}(u)\right)}{\dfrac{\partial W_{\nu,\nu}(u)}{\partial u}}\right|_{u=\alpha_{k,\nu}} + o(\delta^2).$$

Case 2: $a = \lambda_{-\nu}(q), \nu \in \mathbb{N}$. In this case for small q the odd solution $se_\nu(\varphi, q)$ to the angular Mathieu equation can be represented as (see [7])

$$se_\nu(\varphi,q) = \begin{cases} \sin\nu\varphi + \dfrac{q}{4(\nu-1)}\sin(\nu-2)\varphi - \dfrac{q}{4(\nu+1)}\sin(\nu+2)\varphi + o(q), \\ \nu \in \mathbb{N}\setminus\{1,2\}; \\ \sin 2\varphi - \dfrac{q}{12}\sin 4\varphi + o(q), \quad \nu = 2; \\ \sin\varphi - \dfrac{q}{8}\sin 3\varphi + o(q), \quad \nu = 1. \end{cases}$$

Let $\nu \in \mathbb{N}\setminus\{1,2\}$. Then $c_\nu = 1, c_{\nu-2} = \dfrac{1}{4(\nu-1)}, c_{\nu+2} = \dfrac{-1}{4(\nu+1)}$, and from Lemma 3 we obtain

$$\varkappa^2 = \frac{\alpha_{k,\nu}^2}{r_0^2} + \delta^2 \frac{\alpha_{k,\nu}^3}{8\nu r_0^4} \left. \frac{\dfrac{\nu-2}{\nu-1}\left(W_{\nu-2,\nu}(u) + W_{\nu,\nu-2}(u)\right) - \dfrac{\nu+2}{\nu+1}\left(W_{\nu+2,\nu}(u) + W_{\nu,\nu+2}(u)\right)}{\dfrac{\partial W_{\nu,\nu}(u)}{\partial u}} \right|_{u=\alpha_{k,\nu}} + o(\delta^2).$$

Suppose that $\nu = 2$. Then $c_\nu = 1, c_{\nu-2} = 0, c_{\nu+2} = \dfrac{-1}{12}$, and by Lemma 3

$$\varkappa^2 = \frac{\alpha_{k,\nu}^2}{r_0^2} - \delta^2 \frac{\alpha_{k,\nu}^3(\nu+2)}{24\nu r_0^4} \left. \frac{\left(W_{\nu+2,\nu}(u) + W_{\nu,\nu+2}(u)\right)}{\dfrac{\partial W_{\nu,\nu}(u)}{\partial u}} \right|_{u=\alpha_{k,\nu}} + o(\delta^2).$$

Let $\nu = 1$. Then $c_\nu = 1, c_{\nu-2} = 0, c_{\nu+2} = \dfrac{-1}{8}$, and from Lemma 3 we arrive at

$$\varkappa^2 = \frac{\alpha_{k,\nu}^2}{r_0^2} - \delta^2 \frac{\alpha_{k,\nu}^3(\nu+2)}{16\nu r_0^4} \left. \frac{\left(W_{\nu+2,\nu}(u) + W_{\nu,\nu+2}(u)\right)}{\dfrac{\partial W_{\nu,\nu}(u)}{\partial u}} \right|_{u=\alpha_{k,\nu}} + o(\delta^2).$$

The proof of Theorem 3 is completed.

5. CONCLUDING REMARKS

Let us consider another problem formulation. Suppose that a circular ring is given bounded by concentric circles with radii $r_1 > r_0 > \dfrac{c}{2}$.

We consider the mapping $F_c(z) = z + \dfrac{c^2}{4z}$ (an analog of the Zhukovsky function). It transfers our circular ring to an elliptic one bounded by ellipses with foci at the points $(\pm\dfrac{c}{2}, 0)$ and major semiaxes $F_c(r_0) = r_0 + \dfrac{c^2}{4r_0}, F_c(r_1) = r_1 + \dfrac{c^2}{4r_1}$.

Using the function $F_c(z)$ the Laplacian ∇^2 in the elliptic ring is transferred to the circular ring. The obtained operator ∇_c^2 can be considered a perturbation of the original Laplacian $\nabla^2 = \nabla_0^2$.

The asymptotic of the eigenvalues of ∇_c^2 has additional correction terms compared with our original problem, because under the action of $F_c(z)$ the semiaxes change. We will dedicate a section in a further publication to the discussion of asymptotics of eigenvalues of ∇_c^2.

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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