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Robust Utility Maximization in Terms of Supermartingale Measures

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Abstract—We consider a dual description of the optimal value of robust utility in the abstract financial market model $(\Omega, \mathcal{F}, \mathsf{P}, \mathcal{A}(x))$, where $\mathcal{A}(x) = x\mathcal{A}$, $x \geqslant 0$, is the set of the investor's terminal capitals corresponding to strategies with the initial capital x. The main result of the paper addresses the question of the transition in the definition of the dual problem from the polar of the set $\mathcal{A}$ to a narrower set of limit values of supermartingale densities.

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1. INTRODUCTION

In this paper, as a robust utility maximization problem with a penalty function, we mean the problem of maximizing the functional

$$\xi \rightsquigarrow \inf_{\mathsf{Q}\in\mathcal{Q}}\big(\mathsf{E}_{\mathsf{Q}}U(\xi) + \gamma(\mathsf{Q})\big), \quad \xi \in \mathcal{A},$$

over some convex set $\mathcal{A}$ of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$.

Assumption 1 (on a utility function): $U : \mathbb{R} \to [-\infty, +\infty)$ is a monotonically nondecreasing concave function such that $U(x) = -\infty$ for $x < 0$ and $U(x) \in \mathbb{R}$ for $x > 0$.

Let $\mathcal{Q}$ be some convex set of probability measures on $(\Omega, \mathcal{F})$, and let the penalty function γ be convex (see [1]).

We introduce the function V conjugate to U by the relation

$$V(y) = \sup_{x>0}\big(U(x) - xy\big), \quad y \in \mathbb{R}.$$

For a function $f\colon X \to \mathbb{R} \cup \{+\infty\}$, the effective set $\operatorname{dom} f$ is defined as

$$\operatorname{dom} f := \{x \in X\colon f(x) < +\infty\}.$$

By Assumption 1, $\operatorname{dom} V \subseteq \mathbb{R}_+$, the function V is not monotonically increasing, and

$$\lim_{y\to+\infty}\frac{V(y)}{y} = 0.$$

By the standard utility maximization problem we mean the case where $\mathcal{Q} = \{\mathsf{P}\}$.

Denote by ba the space of bounded finitely additive set functions $\mu\colon \mathcal{F} \to \mathbb{R}$ such that

$$A \in \mathcal{F}, \quad \mathsf{P}(A) = 0 \;\Rightarrow\; \mu(A) = 0,$$

with the total variation norm. It is well known that ba is dual to the space L^∞, and the duality is given by the relation

$$\langle \xi, \mu\rangle := \mu(\xi) := \int_\Omega \xi\, d\mu, \quad \mu \in ba, \quad \xi \in L^\infty.$$

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A subspace of the space ba consisting of countably additive measures is denoted by ca. For $\mu \in ba$, there exists a unique decomposition $\mu = \mu^r + \mu^s$ into a countably additive measure $\mu^r \in ca$ and a purely finitely additive set function $\mu^s \in ba$. The space ca is naturally identified with L^1 by the relations $\xi \in L^1$ and $\xi \rightsquigarrow \xi \cdot \mathsf{P} \in ca$, where $\xi \cdot \mathsf{P}$ is a measure with density ξ in P.

The proof of the results of this paper uses the notion of f-divergence. Let us give its formal definition. Let $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and convex function with $\operatorname{dom} f \subseteq \mathbb{R}_+$. In [2], Gushchin gave a definition of the f-divergence $\mathcal{J}_f(\mu, \nu)$ of finitely additive functions μ and ν given on $(\Omega, \mathcal{F})$. For $\mu, \nu \in ba$, this definition is equivalent to the following:

$$\mathcal{J}_f(\mu, \nu) = \sup_{\xi, \eta \in L^\infty :\, \eta + f^*(\xi) \leqslant 0} \big(\mu(\xi) + \nu(\eta)\big),$$

where f^* is the Fenchel transform of the function f. It follows from the definition that the function $\mathcal{J}_f(\mu, \nu)$ on $ba \times ba$ takes values in $\mathbb{R} \cup \{+\infty\}$ and is convex and lower semicontinuous in the topology $\sigma(ba \times ba, L^\infty \times L^\infty)$. The properties used in this paper were proved in [2, Theorem 1].

It will be convenient for us to extend the domain of the penalty function γ to the space ba by setting it equal to $+\infty$ outside $\mathcal{Q}$. Then $\mathcal{Q}$ is characterized as the effective domain $\operatorname{dom} \gamma$.

Assumption 2 (on a penalty function): $\gamma \colon ba \to \mathbb{R} \cup \{+\infty\}$ is a proper convex function such that $\operatorname{dom} \gamma =: \mathcal{Q}$ is a subset of the set of all probability measures on $(\Omega, \mathcal{F})$, $\inf_{\mathsf{Q} \in \mathcal{Q}} \gamma(\mathsf{Q}) \geqslant 0$, and the set

$$\{d\mathsf{Q}/d\mathsf{P} \colon\ \mathsf{Q} \in \mathcal{Q},\ \gamma(\mathsf{Q}) \leqslant c\}$$

is closed in L^1 and uniformly integrable with respect to P for any $c \geqslant 0$.

Denote by L^0 the space of P-a.s. equivalence classes of equal random variables with real values. When we speak of random variables, we mean the equivalence classes that they generate.

Assumption 3 (on the set of terminal wealths): $\mathcal{A}$ is a convex subset L^0 containing a random variable $\xi_0 \geqslant \varkappa$ for some $\varkappa > 0$.

The cone of nonnegative random variables is denoted by L^0_+. We define

$$\mathcal{D} := \{\eta \in L^0_+ \colon\ \mathsf{E}_\mathsf{P} \eta \xi \leqslant 1 \text{ for any } \xi \in \mathcal{A}\}. \tag{1}$$

It is clear that $\mathcal{D} \subseteq L^1_+$ since $\mathsf{E}_\mathsf{P} \eta \leqslant \varkappa^{-1}$ for any $\eta \in \mathcal{D}$. For $x > 0$ and $y \geqslant 0$, we put

$$\mathcal{A}(x) := x\mathcal{A}, \quad \mathcal{D}(y) := y\mathcal{D}.$$

We define primal and dual optimization problems:

$$u(x) := \sup_{\xi \in \mathcal{A}(x)} \inf_{\mathsf{Q} \in \mathcal{L}} \big(\mathsf{E}_\mathsf{Q} U(\xi) + \gamma(\mathsf{Q})\big), \quad x > 0; \tag{2}$$

$$v(y) := \inf_{\eta \in \mathcal{D}(y),\, \mathsf{Q} \in \mathcal{L}} \left(\mathsf{E}_\mathsf{Q} V\left(\frac{\eta}{d\mathsf{Q}/d\mathsf{P}}\right) + \gamma(\mathsf{Q})\right), \quad y \geqslant 0. \tag{3}$$

We have the equalities (see [3]):

$$u(x) = \min_{y \geqslant 0} \big(v(y) + xy\big), \quad x > 0; \tag{4}$$

$$v(y) = \sup_{x > 0} \big(u(x) - xy\big), \quad y \geqslant 0. \tag{5}$$

The main result of this paper is new and answers the question: when the set $\mathcal{D}$ defined in (1) can be replaced by a convex set $\widetilde{\mathcal{D}} \subseteq \mathcal{D}$ in the definition of the dual function v (see (3))? This situation is considered in abstract form in Lemma 1 and in a more concrete form in Theorem 1. A similar result for the nonrobust case was obtained in the joint work of Kramkov and Schachermayer (see [4]).

2. AUXILIARY RESULTS

Given a probability measure $\mathsf{Q} \ll \mathsf{P}$, we define the functions

$$v_{\mathsf{Q}}(y) := \inf_{\eta\in\mathcal{D}} \mathsf{E}_{\mathsf{Q}} V\left(\frac{y\eta}{d\mathsf{Q}/d\mathsf{P}}\right), \quad y \geqslant 0;$$

$$\widetilde{v}_{\mathsf{Q}}(y) := \inf_{\eta\in\widetilde{\mathcal{D}}} \mathsf{E}_{\mathsf{Q}} V\left(\frac{y\eta}{d\mathsf{Q}/d\mathsf{P}}\right), \quad y \geqslant 0.$$

It can be seen from (3) that it suffices to consider whether or not the functions v_{Q} and $\widetilde{v}_{\mathsf{Q}}$ coincide.

Definition 1. For a set $\mathcal{E} \subseteq L^0_+$, we define its *polar* $\mathcal{E}^\circ$ by

$$\mathcal{E}^\circ := \{\xi \in L^0_+ \colon \mathsf{E}_{\mathsf{P}}\eta\xi \leqslant 1 \text{ for any } \eta \in \mathcal{E}\}.$$

Using these terms, the definition of the set $\mathcal{D}$ in (1), in which $\mathcal{A}$ can be replaced by $\mathcal{C}_+ := (\mathcal{A} - L^0_+) \cap L^\infty_+$, is written as $\mathcal{D} = \mathcal{C}^\circ_+$; $\overline{\mathcal{C}}^0_+$ denotes the closure of the set $\mathcal{C}_+$ in L^0.

Lemma 1. *Suppose that the set $\mathcal{A}$ satisfies Assumption 3, $\mathcal{A} \subseteq L^0_+$, the set $\mathcal{D}$ is defined in (1) and $\widetilde{\mathcal{D}} \subseteq \mathcal{D}$, and the set $\widetilde{\mathcal{D}}$ is convex and not empty. We introduce the following conditions:*

(i) *For any $\eta \in \mathcal{D}$, there exists $\widetilde{\eta} \in \widetilde{\mathcal{D}}$ such that $\eta \leqslant \widetilde{\eta}$.*

(ii) *$v_Q(y) = \widetilde{v}_Q(y)$ for all $Q \ll P$ and $y \geqslant 0$ for any function U satisfying Assumption 1.*

(iii) *$v_Q(y) = \widetilde{v}_Q(y)$ for all $Q \ll P$ and $y \geqslant 0$ for some strictly increasing function U satisfying Assumption 1.*

(iv) *For any $f \in L^0_+$,*

$$\sup_{g\in\mathcal{D}} \mathsf{E}fg = \sup_{g\in\widetilde{\mathcal{D}}} \mathsf{E}fg.$$

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). If the closure $\overline{\widetilde{\mathcal{D}}}^0_+$ of the set $\widetilde{\mathcal{D}}$ in L^0 lies in $\widetilde{\mathcal{D}} - L^0_+$, then all four conditions are equivalent.

Remark 1. We have $\mathcal{D}^\circ = \left(\mathcal{C}^\circ_+\right)^\circ$. As is easily seen, condition (iv) of Lemma 1 is equivalent to the fact that $\mathcal{D}^\circ = \widetilde{\mathcal{D}}^\circ$. On the other hand, since $\mathcal{C}$ is convex and solid, the Brannath–Schachermayer bipolar theorem [5] states that $\left(\mathcal{C}^\circ_+\right)^\circ$ coincides with the closure $\overline{\mathcal{C}}^0_+$ of the set $\mathcal{C}_+$ in L^0.

Proof of Lemma 1. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious. Suppose that condition (iii) holds, while condition (iv) is not satisfied. Then there are $f \in L^0_+$ and $\eta \in \mathcal{D}$ such that

$$\mathsf{E}f\eta > \sup_{g\in\widetilde{\mathcal{D}}} \mathsf{E}fg. \tag{6}$$

Cutting off f and η from above, we can consider that f and η are bounded, and, adding a small constant to f (recall that $\mathsf{E}g \leqslant \varkappa^{-1}$ for any $g \in \mathcal{D} \supseteq \widetilde{\mathcal{D}}$), we have $f \geqslant \varepsilon > 0$. Let us now set $q = y\eta/U'_+(f)$, where $y > 0$ is chosen from the normalization condition $\mathsf{E}q = 1$, and $\mathsf{Q} = q \cdot \mathsf{P}$. Note that here U'_+ is the right derivative of the utility function U. It exists since, by Assumption 1, the utility function never goes to infinity on the positive semiaxis $\mathbb{R}_+$. Note that P and Q are probability measures, i.e., countably additive measures: $\mathsf{P} = \mathsf{P}^r$, $\mathsf{Q} = \mathsf{Q}^r$, and $\mathsf{P}^s = \mathsf{Q}^s = 0$. We have [2, Theorem 1]

$$\mathsf{E}_{\mathsf{Q}} V\left(\frac{yg}{d\mathsf{Q}/d\mathsf{P}}\right) = \mathsf{E}_{\mathsf{Q}} V\left(yg\frac{d\mathsf{P}/d\mathsf{P}}{d\mathsf{Q}/d\mathsf{P}}\right) = \left[\mathcal{J}_V(0,0) = 0\right] = \mathcal{J}_V\left((yg)\cdot\mathsf{P},\mathsf{Q}\right) + \mathcal{J}_V(0,0)$$
$$= \mathcal{J}_V\left((yg)\cdot\mathsf{P}^r,\mathsf{Q}^r\right) + \mathcal{J}_V\left((yg)\cdot\mathsf{P}^s,\mathsf{Q}^s\right) = \mathcal{J}_V\left((yg)\cdot\mathsf{P},\mathsf{Q}\right) = \sup_{\xi\in L^\infty \colon U(\xi)\in L^\infty}\left(\mathsf{E}_{\mathsf{Q}}U(\xi) - y\mathsf{E}g\xi\right).$$

Note that here $\mathcal{J}_V$ is the V-divergence. The last equality follows from the definition of the V-divergence in terms of mathematical expectation.

Note that, for $g = \eta$, the upper bound is attained on f:

$$\mathsf{E}_{\mathsf{Q}} U(\xi) - y\mathsf{E}\eta\xi = y\mathsf{E}\eta\left(\frac{U(\xi)}{U'_+(f)} - \xi\right) \leqslant y\mathsf{E}\eta\left(\frac{U(f)}{U'_+(f)} - f\right),$$

the inequality follows from the concavity of U, since the local maximum is global for a concave function. Therefore,

$$\mathsf{E}_{\mathsf{Q}} V\left(\frac{y\eta}{d\mathsf{Q}/d\mathsf{P}}\right) = \mathsf{E}_{\mathsf{Q}} U(f) - y\mathsf{E}\eta f,$$

while

$$\mathsf{E}_{\mathsf{Q}} V\left(\frac{yg}{d\mathsf{Q}/d\mathsf{P}}\right) \geqslant \mathsf{E}_{\mathsf{Q}} U(f) - y\mathsf{E}gf.$$

Hence,

$$v_{\mathsf{Q}}(y) \leqslant \mathsf{E}_{\mathsf{Q}} V\left(\frac{y\eta}{d\mathsf{Q}/d\mathsf{P}}\right) = \mathsf{E}_{\mathsf{Q}} U(f) - y\mathsf{E}\eta f < \mathsf{E}_{\mathsf{Q}} U(f) - y\sup_{g\in\widetilde{\mathcal{D}}} \mathsf{E}gf$$
$$\leqslant \inf_{g\in\widetilde{\mathcal{D}}} \mathsf{E}_{\mathsf{Q}} V\left(\frac{yg}{d\mathsf{Q}/d\mathsf{P}}\right) = \widetilde{v}_{\mathsf{Q}}(y),$$

where the strict inequality follows from (6). We come to the required contradiction.

Let now $\overline{\widetilde{\mathcal{D}}}^0_+ \subseteq \widetilde{\mathcal{D}} - L^0_+$, and let condition (iv) hold. It follows from Remark 1 that (iv) implies $\mathcal{D} = \left(\widetilde{\mathcal{D}}^\circ\right)^\circ$. On the other hand, since the set $\widetilde{\mathcal{D}}$ is convex and bounded in L^1, standard arguments based on the transition to convex combinations show that the set $(\widetilde{\mathcal{D}} - L^0_+) \cap L^0_+$ is closed in L^0 and, consequently, there is the smallest subset of L^0_+ containing $\widetilde{\mathcal{D}}$ that is convex, solid and closed in L^0. By the Brannath–Schachermayer bipolar theorem [5], $\left(\widetilde{\mathcal{D}}^\circ\right)^\circ = (\widetilde{\mathcal{D}} - L^0_+) \cap L^0_+$. Thus, condition (i) is satisfied.

3. MAIN RESULTS

Assume that the probability space $(\Omega, \mathcal{F}, \mathsf{P})$ is endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ satisfying the usual conditions. We have $\mathcal{F} = \sigma(\cup_{t\geqslant 0}\mathcal{F}_t)$ and $\mathcal{F}_0$ contains only sets of P-measure 0 or 1. We denote by $\mathbb{D}$ the set of real consistent random processes $X = (X_t)_{t\geqslant 0}$ whose trajectories are continuous on the right and have finite limits on the left; let $\mathbb{D}_+ = \{X \in \mathbb{D} : \ X \geqslant 0\}$ and $\mathbb{D}_{++} = \{X \in \mathbb{D} : \ \mathsf{P}(\inf_t X_t > 0) = 1\}$. If $X \in \mathbb{D}$ and there P-a.s. exists a finite limit $\lim_{t\to\infty} X_t$, then the element L^0 corresponding to this limit is denoted by X_∞.

We assume that a family of processes $\mathcal{X} \subseteq \mathbb{D}_+$ is given such that its elements are interpreted as wealth processes corresponding to all possible investment strategies, with a unit initial capital. If an investor has an initial capital $x > 0$, then the wealth processes corresponding to his different strategies form the family $\mathcal{X}(x) = x\mathcal{X}$.

Assumption 4 (on a family of wealth processes): the set $\mathcal{X} \subseteq \mathbb{D}_+$ is convex, $X_0 = 1$ for any process $X \in \mathcal{X}$, $1 \in \mathcal{X}$, and there P-a.s. exists a finite limit $\lim_{t\to\infty} X_t$ for any $X \in \mathcal{X}$.

We set $\mathcal{A} = \{X_\infty : X \in \mathcal{X}\}$. If $\mathcal{X}$ satisfies Assumption 4, then $\mathcal{A}$ satisfies Assumption 3 and $\mathcal{A} \subseteq L^0_+$. We define $\mathcal{D}$ by (1).

Definition 2. A process $Y \in \mathbb{D}_+$ is called the *supermartingale density* for the class of processes $\mathcal{X}$ if $Y_0 = 1$ and YX is a P-supermartingale for any $X \in \mathcal{X}$.

The class of all supermartingale densities is denoted by $\mathcal{Y}$; let $\widetilde{\mathcal{D}} := \{Y_\infty : Y \in \mathcal{Y}\}$.

The next lemma is standard.

Lemma 2. *The set $\widetilde{\mathcal{D}}$ is convex, $\widetilde{\mathcal{D}} \subseteq \mathcal{D}$, and $\overline{\widetilde{\mathcal{D}}}^0_+ \subseteq \widetilde{\mathcal{D}} - L^0_+$.*

Proof of Lemma 2. The convexity of $\widetilde{\mathcal{D}}$ is obvious. If $Y \in \mathcal{Y}$, then, for any process $X \in \mathcal{X}$, due to Fatou's lemma and the supermartingale property, we have

$$\mathsf{E}Y_\infty X_\infty \leqslant \lim_{t\to\infty} \mathsf{E}Y_t X_t \leqslant \mathsf{E}Y_0 X_0 = 1;$$

therefore, $Y_\infty \in \mathcal{D}$.

Let now a sequence (Y^n) of $\mathcal{Y}$ be given, and let Y^n_∞ converge P-a.s. to η. By Lemma 5.2 of [6], there are a sequence $Z^n \in \operatorname{conv}(Y^n, Y^{n+1}, \dots)$ and a supermartingale Z with $Z_0 \leqslant 1$ such that Z^n are *Fatou convergent* on a countable everywhere dense subset of $\mathbb{R}_+$ (we refer the reader to the mentioned paper [6] for the definition of Fatou convergence); in this case, we can assume that $Z^n_\infty \to Z_\infty$ P-a.s. (by a deterministic change of time, we can reduce the processes Y^n to $[0,1)$ and continue them to $[1,\infty)$ by Y^n_∞). Since XZ^n are Fatou convergent to XZ for $X \in \mathbb{D}_+$ and the Fatou convergence retains the supermartingale property, the process XZ is a supermartingale for any $X \in \mathcal{X}$. Since it is obvious that $\xi = Z_\infty$, it remains to note that, in the case $0 < Z_0 \leqslant 1$, we have $Z/Z_0 \in \mathcal{Y}$ and the quantity Z_∞/Z_0 majorizes ξ, and the case $Z_0 = 0$ is trivial.

Recall that we are interested in the following question: under what assumptions the set $\mathcal{Y}$ is nonempty and conditions (i)–(iv) of Lemma 1 are satisfied for $\widetilde{\mathcal{D}}$, i.e., when the solution of the robust utility maximization problem (2) satisfies equalities (4) and (5) with the dual function v, in the definition (3) of which the set $\{Y_\infty : Y \in \mathcal{Y}\}$ stands instead of the set $\mathcal{D}$?

Definition 3. A family $\mathcal{X} \subseteq \mathbb{D}_+$ is called *forked* if, for any $X^i \in \mathcal{X} \cap \mathbb{D}_{++}$, $i = 1, 2, 3$, for any $s \geqslant 0$ and every $B \in \mathcal{F}_s$, the process

$$X_t = X^1_t \mathbb{1}_{\{t<s\}} + X^1_s \left(\mathbb{1}_B \frac{X^2_t}{X^2_s} + \mathbb{1}_{\Omega\setminus B} \frac{X^3_t}{X^3_s} \right) \mathbb{1}_{\{t\geqslant s\}}$$

belongs to $\mathcal{X}$.

This definition is very close to the definition of the *fork-convex* family (see [7]), in which $\mathbb{1}_B$ and $\mathbb{1}_{\Omega\setminus B}$ are replaced by h and $1-h$, respectively, where h is a $\mathcal{F}_s$-measurable random variable with values in $[0,1]$. Even when combined with convexity, our forking property is rather weaker than the property of fork-convexity.

Obviously, for any family $\mathcal{X} \in \mathbb{D}_+$ there is the smallest forked family containing $\mathcal{X}$, which we denote by $\operatorname{fork}(\mathcal{X})$.

Theorem. *Suppose that Assumption* 4 *holds true,* $\mathcal{A} = \{X_\infty : X \in \mathcal{X}\}$, $\mathcal{D} \neq \{0\}$, *where the set* $\mathcal{D}$ *is defined in* (1)*, and* $\widetilde{\mathcal{D}} := \{Y_\infty : Y \in \mathcal{Y}\}$. *In order that the set* $\widetilde{\mathcal{D}}$ *be nonempty and conditions* (i)–(iv) *of Lemma* 1 *hold for it, it is necessary and sufficient that*

$$\{X_\infty : X \in \operatorname{fork}(\mathcal{X})\} \subseteq \overline{\mathcal{C}}^0_+. \tag{7}$$

Proof. It is easier to prove necessity than sufficiency. Assume

$$\mathcal{X}_0 := \operatorname{fork}(\mathcal{X}) \cap \{X \in \mathbb{D}_+ :\ XY \text{ is a supermartingale for any } Y \in \mathcal{Y}\}.$$

It is easy to verify that the set $\mathcal{X}_0$ is forked. Hence, $\mathcal{X}_0 = \operatorname{fork}(\mathcal{X})$ and the process XY is a supermartingale for any $X \in \operatorname{fork}(\mathcal{X})$ and $Y \in \mathcal{Y}$.

We take $X \in \text{fork}(\mathcal{X})$ and let $\eta \in \mathcal{D}$. By condition (i) of Lemma 1, there exists a process $Y \in \mathcal{Y}$ such that $Y_\infty \geqslant \eta$. Then

$$\mathsf{E}X_\infty \eta \leqslant \mathsf{E}X_\infty Y_\infty \leqslant \mathsf{E}X_0 Y_0 = 1,$$

i.e., $X_\infty \in \mathcal{D}^\circ = \overline{\mathcal{C}}^0_+$.

Let us prove the sufficiency of condition (7). Take an arbitrary variable $\eta \in \mathcal{D}$, $\eta \neq 0$. We have $\mathsf{E}X_\infty \eta \leqslant 1$ for any process $X \in \text{fork}(\mathcal{X})$. For $t \in \mathbb{R}_+$, we define a random variable Y_t by the equality

$$Y_t = \operatorname*{ess\,sup}_{X \in \text{fork}(\mathcal{X}) \cap \mathbb{D}_{++}} \frac{\mathsf{E}(\eta X_\infty | \mathcal{F}_t)}{X_t}.$$

Further, the proof of Lemma 4 from [8] is repeated almost verbatim, which requires only forking but not fork-convexity of the set $\mathcal{X}_> := \text{fork}(\mathcal{X}) \cap \mathcal{D}_{++}$. We first prove that, for each $t \in \mathbb{R}_+$, there is a sequence (X^n) of $\mathcal{X}_>$ such that the random variables $\dfrac{\mathsf{E}(\eta X^n_\infty | \mathcal{F}_t)}{X^n_t}$ are monotonically increasing towards Y_t. Then we verify the supermartingale property of the process YX for any process $X \in \mathcal{X}_>$. Finally, we check that the mathematical expectation $\mathsf{E}Y_t$ is right-continuous. This implies that the process Y has a modification from $\mathbb{D}_+$, which we denote by Y from now on.

Let now $X \in \mathcal{X}$. Then, for any positive integer n, the process $X^n := (1 - 1/n)X + 1/n$ belongs to $\mathcal{X} \cap \mathcal{X}_>$; therefore, $X^n Y$ is a supermartingale. Whence it follows in an elementary way that XY is a supermartingale. Obviously, $Y_t \geqslant \mathsf{E}(\eta | \mathcal{F}_t)$ for every t, which implies $Y_\infty \geqslant \eta$. On the other hand, $Y_0 = \sup_{X \in \mathcal{X}_>} \mathsf{E}\eta X_\infty \leqslant 1$. Therefore, $Z := Y/Y_0 \in \mathcal{D}$ and $Z_\infty \geqslant \eta$.

As a corollary, we obtain a slight generalization of the result of Rokhlin (see [7]), where a fork-convex family of random processes was considered.

Corollary. *Let $\mathcal{W} \subseteq \mathbb{D}_+$ be a convex and forked family of random processes, $1 \in \mathcal{W}$, and let $X_0 = 1$ for any process $X \in \mathcal{W}$. The following conditions are equivalent:*

(i) *The set $\{X_t : X \in \mathcal{W},\ t \in \mathbb{R}_+\}$ is bounded in probability.*

(ii) *There exists a supermartingale density Y for the family $\mathcal{W}$ with $P(Y_\infty > 0) = 1$.*

Proof. The implication (ii) $\Rightarrow$ (i) is elementary and can be proved in the same way as in [7]. Let us prove (i) $\Rightarrow$ (ii). To this end, we introduce a family of processes

$$\mathcal{X} := \{X^t : X \in \mathcal{W},\ t \in \mathbb{R}_+\},$$

where X^t denotes the process stopped at time t: $X^t_s = X_{s \wedge t}$. Since the family $\mathcal{W}$ is forked and $1 \in \mathcal{W}$, we have $\mathcal{X} \subseteq \mathcal{W}$. It is obvious that the family $\mathcal{X}$ satisfies Assumption 4 and is forked.

Condition (i) means that the set $\mathcal{A} = \{X_\infty : X \in \mathcal{X}\}$ is bounded in probability. By Yan's theorem [9, Theorem 1], there exists $\eta \in L^\infty_+$ with $\mathsf{P}(\eta > 0) = 1$ and $\sup_{X \in \mathcal{X}} \mathsf{E}\eta X_\infty \leqslant 1$. The theorem implies the existence of a supermartingale density Y for the family $\mathcal{X}$ with $Y_\infty \geqslant \eta$. But it can be easily seen that Y is a supermartingale density for the family $\mathcal{W}$ as well.

CONFLICT OF INTEREST

The author declares that she has no conflicts of interest.

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