Mappings Preserving Relations Definable by Linear Order

A. L. Semenov[1, 2, 3*]

[1]*Department of Mathematical Logic and Theory of Algorithms, Faculty of Mechanics and Mathematics, Lomonosov Moscow State University, Moscow, 119234 Russia*

[2]*Axel Berg Institute of Cybernetics and Educational Computing, Federal Research Center "Computer Science and Control," Russian Academy of Sciences, Moscow, 119333 Russia*

[3]*Moscow Institute of Physics and Technology (National Research University), Dolgoprudny, Moscow oblast, 141701 Russia*

Abstract—The relations "between", "cycle", and "separation" were defined through the relation of linear order in the classical paper of Edward V. Huntington. In the current paper, the criteria for preserving these relations under injective mappings are obtained.

Homeomorphisms of the circle play an important role in the theory of dynamical systems (see, e.g., [1]). V. S. Kozyakin [2] considered mappings of the circle more general than homeomorphisms, namely, those preserving the cyclic order, and established that this more general property is sufficient for obtaining the main results about homeomorphisms used in the symbolic dynamics. In the present work we consider a list of such generalizations exhaustive in a sense and obtain the complete description of arising mappings.

By $\langle A, < \rangle$ we denote a linearly ordered set. The following relations can be defined via the order:

B is "between", that is, $xB(y, z)$ means "$y < x < z \lor z < x < y$";

C is "cycle", that is, $C(x, y, z)$ means "$x < y < z \lor z < x < y \lor y < z < x$";

S is "separation", that is, $(x, y)S(z, t)$ means "$zB(x, y) \not\equiv tB(x, y)$".

Lemma 1. *The relation S may be defined via C: $(x, y)S(z, t)$ is equivalent to*

$$C(x, y, z) \not\equiv C(x, y, t).$$

In work [3] the complete systems of axioms for each of the mentioned relations were provided. In work [4] it was proved that we may recover the order structure for each structure where some of the relations satisfying the corresponding system of axioms is given. The considered list of relations deflates all possibilities for determining order relations for rational numbers (see, e.g., [5]).

We will often use the following simple remark.

Lemma 2. *If x does not lie between a and b and x does not lie between a and c, then x does not lie between b and c.*

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[*]E-mail: alsemno@ya.ru

DEFINITIONS

We say that a mapping $f\colon A \to A$ *preserves a relation* P if $P(x, y, \ldots) \Leftrightarrow P(f(x), f(y), \ldots)$. A mapping preserving the order is called *increasing* (f increases). A mapping f for which $x < y \Rightarrow f(y) < f(x)$ is called *decreasing* (f decreases).

A cut of a structure $\langle A, < \rangle$ is a pair $\langle I_0, I_1 \rangle$ of sets for which the following two conditions are true

(1) $I_0 \cup I_1 = A$;

(2) $\forall x \in I_0, \quad y \in I_1 \quad x < y$.

Note that the case when one of cut elements is empty set is not excluded. (This allows simplifying some formulations.) We will call I_0 and I_1 the *cut segments*.

We will say that an injective mapping $f : A \to A$ *preserves the order of segments* I_0 and I_1, if $\forall x \in I_0, \; y \in I_1$ it is true that $f(x) < f(y)$; f *permutes segments* if $\forall x \in I_0, \; y \in I_1$ it is true that $f(x) > f(y)$. If one of the segments is empty, then f simultaneously preserves the order of segments and permutes them.

Theorem. *Suppose $f : A \to A$ is an injective mapping. Then f*

(B) preserves the relation B iff f increases or decreases;

(C) preserves the relation C iff, for some cut $\langle I_0, I_1 \rangle$ of the set $\langle A, < \rangle$, the mapping f increases on I_0 and on I_1 and permutes segments;

(S) preserves the relation S iff for some cut $\langle I_0, I_1 \rangle$ of the set $\langle A, < \rangle$

(S.1) f increases on I_0 and on I_1 and permutes segments

or

(S.2) f decreases on I_0 and on I_1 and preserves the order of segments.

PROOF

By (x, y) we denote an interval with ends $x, \; y$; here, it is not necessary that $x < y$.

(B) Checking the preservation of the relation B for increasing and decreasing mappings f is clear.

Now, suppose f preserves the relation B. If A consists of less than four elements, then the statement is clear. Otherwise, let us consider any four different elements of A. Due to the fact that f preserves the relation "between", after applying f two "intermediate" elements of the quadruple remain intermediate and two outermost elements remain outermost. Next, it is clear that the entire order of elements of the quadruple after application of f will be determined by the order of outermost elements and it preserves or changes to the opposite one. Therefore, if in some pair of elements f increases, then it increases in any pair of elements, and if in some pair it decreases, then it decreases in any pair. Hence, we obtain the desired statement (B).

(S) Suppose f satisfies the condition formulated in the theorem in terms of cut.

Suppose $a < c < b < d$ and $(a, b)S(c, d)$ is fulfilled. Next, suppose, for instance, that $a, c \in I_0, b, d \in I_1$. Then, in the case $(S.1)$ we have $f(b) < f(d) < f(a) < f(c)$. In the case $(S.2)$ we have $f(c) < f(a) < f(d) < f(b)$. In either of the cases the validity of S for images is clear. The remaining cases are also considered directly.

Suppose f preserves the relation S. If f preserves the relation B, then we should take $I_0 = A$ and $I_1 = \varnothing$, and the statement follows from the statement (B) of the theorem.

Now, suppose that f does not preserve the relation B. Then there exist three elements $a_0, a_1, b \in A$ such that b lies between a_0 and a_1, but $f(b)$ does not lie between $f(a_0)$ and $f(a_1)$. Consequently, for any c the preservation of the relation S implies the equality

$$\neg cB(a_0, a_1) \equiv (b, c)S(a_0, a_1) \equiv (f(b), f(c))S(f(a_0), f(a_1)) \equiv f(c)B(f(a_0), f(a_1)).$$

Thus, the following statements are true:

(*) for any element c lying between a_0 and a_1, its image $f(c)$ does not lie between the images $f(a_0)$ and $f(a_1)$;

(**) for any element c not lying between a_0 and a_1, its image $f(c)$ lies between the images $f(a_0)$ and $f(a_1)$.

In other words, f performs "inversion of A with respect to the segment (a_0, a_1)".

In the following, we assume that in our denotations $a_0 < a_1$. We define

$$I_0 = \{x | x < a_1 \wedge \neg f(a_1) B(f(a_0), f(x))\},$$

$$I_1 = \{x | x > a_0 \wedge \neg f(a_0) B(f(a_1), f(x))\}.$$

We prove several auxiliary statements.

1. From the definition it is clear that $a_0 \in I_0$ and $a_1 \in I_1$.

2. Let us check property 1 from the definition of cut.

Suppose x lies between a_0 and a_1. Then the first conditions in conjuctions for the sets I_0 and I_1 are satisfied. Due to "inversion" (*), the value of the function $f(x)$ does not lie between $f(a_0)$ and $f(a_1)$. Therefore, for exactly one pair of different $k, j \in \{0, 1\}$, the element $f(a_k)$ does not lie between $f(a_j)$ and $f(x)$. Hence, x belongs to exactly one of the sets I_0 and I_1.

Now, suppose x does not lie between a_0 and a_1. Due to "inversion" (**), the value of the function $f(x)$ lies between $f(a_0)$ and $f(a_1)$. Therefore, in the definitions of I_0 and I_1 the second term of the conjunction is satisfied. The first term of the conjunction is satsfied for exactly one of the sets I_0 and I_1.

Thus, $\langle I_0, I_1 \rangle$ is a partitioning of A into two nonintersecting subsets.

3. We take any $j \in \{0, 1\}, y, z \in I_j$ and any element x lying between y and z. We show that $f(x)$ lies between $f(y)$ and $f(z)$ and the set I_j does not contain foreign inclusions.

Indeed, we take $k \neq j$. Due to the first condition in the definition of I_j, the element a_k does not lie between y and z and it is true that $(y, z) S(x, a_k)$. According to the second condition from the definition of I_j, the value of the function $f(a_k)$ does not lie between $f(a_j)$ and $f(y)$ and does not lie between $f(a_j)$ and $f(z)$. By Lemma 2, the value of the function $f(a_k)$ does not lie between $f(y)$ and $f(z)$. Since f preserves the separation S, we have $(f(y), f(z)) S(f(x), f(a_k))$. Consequently, the value of the function $f(x)$ lies between $f(y)$ and $f(z)$.

We have established that the value of the function $f(a_k)$ does not lie between $f(y)$ and $f(a_j)$ and does not lie between $f(y)$ and $f(z)$; consequently, the value of the function $f(a_k)$ does not lie between $f(y)$ and $f(x)$. By Lemma 2, the value of the function $f(a_k)$ does not lie between $f(a_j)$ and $f(x)$, that is, for x the second term of conjunction in the definition of I_j is satisfied. The first term of the conjunction is satisfied due to the choice of y and z, and, therefore, $x \in I_j$.

4. We show that $\forall x \in I_0 \ y \in I_1 \ x < y$ is fulfilled. From points 1 and 3 it follows that all x less than a_0 lie in I_0 (otherwise, an element a_0 would lie between some element of I_1 and a_1); similarly, all x larger than a_1 lie in I_1. For x and y lying between a_0 and a_1, if $x \in I_0$, $y \in I_1$, then the relation $x > y$ is impossible: if it is valid, then an element y from I_1 would lie between two elements from I_0, precisely, between a_0 and x.

5. It follows from point 3 and from the already proved statement (B) of the theorem that, f increases or decreases on I_0 and on I_1 (but it is a priori possible that on one of these sets f increases and on another set it decreases). We prove that such "detuning" is impossible, that is, that the function f either increases on both sets, or decreases on both sets.

If $I_0 = \{a_0\}$ or $I_1 = \{a_1\}$, then f on this I_k simultaneously increases and decreases. The statement in these cases is proved. Therefore, in the following, we think that there are more than one element both in I_0 and in I_1. It is sufficient to point a pair of elements in each of segments so that in these pairs the function f behaves the same. Here, the properties of "inversion" (*) and (**) are used.

Suppose $f(a_0) < f(a_1)$. We show that f decreases on each segment of the cut.

We take an arbitrary element c different from a_0 in I_0. Due to the definition of I_0, the element c cannot be larger than a_1. If $c < a_0$, then, due to statement (**) about "inversion" the value $f(c)$ lies between $f(a_0)$ and $f(a_1)$, in particular, $f(a_0) < f(c)$. Thus, f decreases in pair c, a_0 and, therefore, in I_0. If c lies between a_0 and a_1, then due to statement (*) about "inversion" the value $f(c)$ cannot lie between $f(a_0)$ and $f(a_1)$. By the definition of I_0, the value $f(c)$ cannot be larger than $f(a_1)$; otherwise, the value $f(a_1)$ would lie between $f(a_0)$ and $f(c)$. Thus, $f(c) < f(a_0)$ and in this case f decreases on I_0.

For elements of I_1 the reasoning is analogous with replacement of $<$ with $>$. Thus, f decreases both on I_0 and in I_1.

It is proved in the similar manner that if $f(a_0) > f(a_1)$, then f increases on each segment of the cut.

(C) Suppose f increases on I_0 and on I_1 from some cut and $\forall x \in I_0$, $y \in I_1$ we have $f(y) < f(x)$. The validity of the relation C (as well as of all other considered relations) depends only on the order relations among arguments. The relation $C(x, y, z)$ is true in sets in which $x < y < z$, or $z < x < y$, or $y < z < x$; the second and third sets are obtained by cyclic permutations of the first one. The mapping f does not change the order of arguments lying in a single component of the cut. If exactly two elements lie in a single component of the cut and the third element lies in another one, then f permutes this element from the first place to the last one or from the last place to the first one and, therefore, preserves the relation C.

Now, suppose that f preserves the cycle C. Since the separation may be defined via the cycle, f preserves the separation as well. Therefore, we may use the statement (S) of the theorem being proved and choose I_0, I_1. It is clear that the case $(S.2)$ is impossible.

The third statement of the theorem and the theorem as a whole are proved.

EXAMPLES

We consider the case of separation automorphisms; the modifications for other relations are easy to obtain.

Finite Structures

The cut "passes" through any element a: we refer all elements less than a to I_0 and refer all elements not less than a to I_1. Each injection is an automorphism. The variants $(S.1)$ (increase) and $(S.2)$ (decrease) are possible and uniquely determined by the point of the cut.

Rational Numbers

For the set of all rational numbers $\mathbb{Q}$ (and for the interval $\mathbb{Q}(0; 1)$ isomorphic to it), the "cut point" is an arbitrary irrational number. The images of the cut elements may be separated by an arbitrary irrational number.

For the set of nonnegative rational numbers (and for the half-open interval $\mathbb{Q}[0; 1)$ isomorphic to it), the "cut point" is an arbitrary positive rational number; in the case of increase in f this point lies in I_1 and in the case of decrease in f this point lies in I_0. The image of zero may be an arbitrary positive rational number.

For the segment $\mathbb{Q}[0; 1]$ there are no nontrivial cuts.

Real Numbers

For the set of nonnegative real numbers (and for the half-open interval $\mathbb{S}^1[0; 1)$ isomorphic to it), the cut point may be any positive number (similar to the case of rational numbers). For such an ordered set the theorem in the case (C) was proved by V. S. Kozyakin in [2].

For the set of all real numbers $\mathbb{R}$ or for the segment $\mathbb{R}[0; 1]$ there are no nontrivial cuts.

The class of situations extends without requirement of one-to-oneness of the mapping.

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CONFLICT OF INTEREST

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