ISSN 0027-1322, Moscow University Mathematics Bulletin, 2019, Vol. 74, No. 5, pp. 189–194. © Allerton Press, Inc., 2019.
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To Millionshchikov's Problem on the Baire Class of Central Exponents of Diffeomorphisms

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Received March 14, 2019; revised June 23, 2019; accepted June 23, 2019

Abstract—It is shown that central exponents of a local diffeomorphism of a Riemannian manifold treated as functions on the direct product of the manifold and the space of its local diffeomorphisms with C^1-compact-open topology belong to the fourth Baire class.

DOI: 10.3103/S0027132219050036

Let $n \in \mathbb{N}$ and M be an n-dimensional differentiable manifold of class C^1 with a countable base where a Riemannian metrics δ (of class C^0) be given. Thus, for each $x \in M$ the tangent space T_xM to M at the point x is endowed with the norm $|\xi| = \sqrt{\delta(\xi,\xi)}$, $\xi \in T_xM$. By TM we denote the variety of tangent vectors to the manifold M, and by $df\colon TM \to TM$ we denote the derivative (differential) of a smooth mapping $f\colon M \to M$. By S we denote the set of all mappings $M \to M$ of class C^1 whose derivative is not degenerate at each point $x \in M$.

Since we do not require on M any boundedness of derivatives of considered mappings, the values defined below are points of extended number axis $\overline{\mathbb{R}} \equiv \mathbb{R} \sqcup \{-\infty, +\infty\}$, which we endow with the standard order and order topology.

Define the *characteristic indicator* of a tangent vector $\xi \in TM$ under a mapping $f \in S$ by the equality [1]

$$\lambda(f,\xi) = \begin{cases} \overline{\lim\limits_{m\to+\infty}}\, m^{-1} \ln |df^m \xi| & \text{for } |\xi| \neq 0, \\ -\infty & \text{for } |\xi| = 0. \end{cases}$$

Lyapunov's exponents of a mapping $f \in S$ at a point $x \in M$ are defined by the equalities [1]

$$\lambda_i(f,x) = \inf_{L \in G_i(x)} \sup_{\xi \in L} \lambda(f,\xi), \quad i = 1,\dots,n,$$

where $G_q(x)$ is the set of all q-dimensional vector subspaces of the tangent space T_xM.

Further, for all $f \in S$, $x \in M$, and $i = 1,\dots,n$ assume $E_i(f,x) = \{\xi \in T_xM : \lambda(f,\xi) \le \lambda_i(f,x)\}$. It is well known and easily proved that $E_i(f,x)$ is a vector subspace of the space T_xM [2].

Central exponents of a mapping $f \in S$ at a point $x \in M$ are defined by the equalities [1]

$$\Omega^{(i)}(f,x) = \inf_{T\in\mathbb{N}} \overline{\lim_{m\to\infty}} \frac{1}{mT} \sum_{k=1}^{m} \ln \|df^T|_{df^{(k-1)T} E_i(f,x)}\|, \quad i = 1,\dots,n,$$

where $Y|_L$ is the restriction of the mapping Y onto the subspace L and the norm of the linear mapping of normed spaces is defined in the standard way (as the maximum of the norm of the image of the normed vector).

Remark 1. In the case of compact manifold M the values introduced above are finite and do not depend on the choice of Riemannian metrics on M (since any two Riemannian metrics on a compact manifold are equivalent).

We endow the space S with the C^1-compact-open topology. Recall its definition [3, §2.1]. Let $f \in S$ and (φ, U), (ψ, V) be maps of the manifold M. Further, let $K \subset U$ be a (not empty) compactum such that $f(K) \subset V$. For each $\varepsilon > 0$, define the set

$$\mathcal{K}(f;(\varphi,U),(\psi,V),K,\varepsilon) \tag{1}$$

as the set of $g \in S$ such that $g(K) \subset V$ and the following inequalities hold:

$$\max_{x\in\varphi(K)} |(\psi f\varphi^{-1})(x) - (\psi g\varphi^{-1})(x)| < \varepsilon, \quad \max_{x\in\varphi(K)} \|(\psi f\varphi^{-1})'(x) - (\psi g\varphi^{-1})'(x)\| < \varepsilon,$$

where $|\cdot|$ is the Euclidean norm on the space $\mathbb{R}^n$ and $\|\cdot\|$ is the corresponding operator norm. Since the functions $\psi f\varphi^{-1}$ and $\psi g\varphi^{-1}$ are determined on the open set $\varphi(U \cap g^{-1}(V) \cap f^{-1}(V)) \supset \varphi(K)$, the derivatives presenting in the latter inequality have sense. The function f is called the center of the set (1). A C^1-compact-open topology is generated by sets (1), i.e., finite intersections of indicated sets form the base of this topology.

Recall the definition of Baire's classes [4, §31.IX] of functions determined on a topological space X. Zero Baire's class consists of all continuous functions $f\colon X \to \overline{\mathbb{R}}$. If classes with numbers less than k have been already defined, then the kth Baire class consists of functions $f\colon X \to \overline{\mathbb{R}}$ admitting the representation $f(x) = \lim_{m\to\infty} f_m(x)$, $x \in X$, where the functions f_m, $m \in \mathbb{N}$, belong to classes with numbers less than k.

In [5], Millionshchikov formulated the problem on determination of the least Baire class containing the function $\Omega^{(i)}$ and pointed out that the function $\Omega^{(n)}$ belongs to the second class. After that, in [6], he formulated the assertion that the functions $\Omega^{(i)}$, $i = 1,\dots,n-1$, belong to the fourth class (in the case of diffeomorphisms of a compact manifold), however, in [7], this assertion was replaced by the author by a weaker one, namely, that those functions belong to the fifth class. Proofs of assertions mentioned here were not published.

The aim of this paper is the proof of the initial conjecture of Millionshchikov that the central exponent belongs to the fourth Baire class. The problem of lover estimate of the number of Baire class remains open.

Theorem. *Each of functions* $\Omega^{(i)}\colon S \times M \to \overline{\mathbb{R}}$, $i \in \{1,\dots,n-1\}$, *belongs to fourth Baire's class, and the function* $\Omega^{(n)}\colon S \times M \to \overline{\mathbb{R}}$ *belongs to the second one.*

Remark 2. A similar problem for Bohl exponents was considered in [8].

Preface the proof of the theorem with several lemmas. It is convenient to use another, coordinate-free definition of the C^1-compact-open topology in the space S. To do that, consider the following auxiliary construction.

Let X and Y be topological manifolds with atlases chosen on them (not necessarily maximal). Endow the space $C(X,Y)$ of continuous mappings from X to Y with a C^0-compact-open topology defined in the following way [3, §2.1]. Let $f \in C(X,Y)$ and (φ,U), (ψ,V) be maps belonging to the chosen atlases of the manifolds X and Y, respectively. Further, let $K \subset U$ be a compact set such that $f(K) \subset V$. For each $\varepsilon > 0$ we define the set (the function f is called its center)

$$\mathcal{N}(f; (\varphi,U), (\psi,V), K, \varepsilon) \tag{2}$$

as the set of $g \in C(X,Y)$ such that $g(K) \subset V$ and the following inequality holds:

$$\max_{x\in\varphi(K)} |(\psi f\varphi^{-1})(x) - (\psi g\varphi^{-1})(x)| < \varepsilon.$$

Sets of form (2) form the prebase of topology on the space $C(X,Y)$.

The following assertion shows how to define the same topology not using coordinate maps and also proves its independence on the choice of atlases of the manifolds X and Y.

Lemma 1. *Let X and Y be topological manifolds with atlases chosen on them (not necessarily maximal). In this case the C^0-compact-open topology on the space $C(X,Y)$ given by those atlases has the prebase consisting of the sets*

$$\nu(K,W) = \{f \in C(X,Y) : f(K) \subset W\}, \tag{3}$$

where $K \subset X$ is a compactum and $W \subset Y$ is an open set.

Proof. 1. First we prove that the topology generated by sets (3) does not change if we consider only the compact sets $K \subset X$ and open sets $W \subset Y$ lying in the domain of action of one of maps from the chosen atlases. Let $f \in \nu(K,W)$. Since the manifolds X and Y are compact and the function f is continuous, for each point $x \in K$ there exists its neighborhood $O(x)$ possessing the following properties: a) $f(\overline{O(x)}) \subset W$; b) the set $\overline{O(x)}$ lies in the domain of action of one map from the given atlas of the manifold X; c) the set $f(\overline{O(x)})$ lies in the domain of action $V(x)$ of one map from the given atlas of the manifold Y. Let $O(x_i)$, $i = 1,\dots,m$, be a finite covering of the compactum K. Assume $K_i = K \cap \overline{O(x_i)}$, $i = 1,\dots,m$. In this case, $f \in \bigcap_{i=1}^m \nu(K_i, W \cap V(x_i)) \subset \nu(K,W)$.

2. Let (φ, U) and (ψ, V) be maps of the manifolds X and Y, respectively. Further, let $f \in \nu(K, W)$, where $K \subset U$ is a compact set and $W \subset V$ is an open set. Show that the set $\nu(K, W)$ has a subset of form (2) containing the point f. Assume

$$d = \inf\{|x - y| : x \in \psi(f(K)),\ y \in \psi(V \setminus W)\}.$$

Taking into account that $\psi(f(K))$ is a compactum not intersecting the set $\psi(V \setminus W)$ closed with respect to $\psi(V)$, we get $d > 0$. In this case, $f \in \mathcal{N}(f; (\varphi, U), (\psi, V), K, d) \subset \nu(K, W)$.

3. Show that for any set P of form (2) and its point g there exists a set Q open in the topology generated by sets of form (3) such that $g \in Q \subset P$. Let $P = \mathcal{N}(f; (\varphi, U), (\psi, V), K, \varepsilon)$ and $g \in P$. Assume $\eta = (\varepsilon - \max_{x \in \varphi(K)} |(\psi f \varphi^{-1})(x) - (\psi g \varphi^{-1})(x)|)/4$. Below by $B_r(z)$ we denote the open ball with the center at the point z and the radius $r > 0$ lying in the corresponding subspace. Since the function $\psi g \varphi^{-1}$ is continuous, for each $x \in \varphi(K)$ there exists a ball $B_{r(x)}(x) \subset \varphi(U)$ such that $(\psi g \varphi^{-1})(B_{r(x)}(x)) \subset B_\eta((\psi g \varphi^{-1})(x))$. Take a finite subcovering $B_{r(x_i)/2}(x_i)$, $i = 1, \ldots, m$, from the covering of the compactum $\varphi(K)$ by the balls $B_{r(x)/2}(x)$. In this case we have

$$\varphi(K) = \bigcup_{i=1}^{m} C_i, \quad C_i = \overline{B}_{r(x_i)/2}(x_i) \cap \varphi(K) \subset B_{r(x_i)}(x_i).$$

Assume

$$Q = \bigcap_{i=1}^{m} \nu(\varphi^{-1}(C_i), \psi^{-1}(B_\eta((\psi g \varphi^{-1})(x_i)) \cap \psi(V))).$$

By construction, $g \in Q$. Show that $Q \subset P$. Let $h \in Q$ and $x \in \varphi(K)$. In this case, $x \in C_i$ for some $i \in \{1, \ldots, m\}$ and $(h\varphi^{-1})(x) \in V$. Thus, $h(K) \subset V$. The inclusions $(\psi h \varphi^{-1})(C_i) \subset B_\eta((\psi g \varphi^{-1})(x_i))$ and $(\psi g \varphi^{-1})(C_i) \subset B_\eta((\psi g \varphi^{-1})(x_i))$ imply $|(\psi h \varphi^{-1})(x) - (\psi g \varphi^{-1})(x)| < 2\eta$. Therefore, the inequality $|(\psi h \varphi^{-1})(x) - (\psi f \varphi^{-1})(x)| \le \varepsilon - 2\eta$ is valid, which gives $h \in P$. The lemma is proved.

By $\pi\colon TM \to M$ we denote the natural projection of the tangent bundle. In addition, if (φ, U) is a map of the manifold M, then by $T\varphi\colon \pi^{-1}(U) \to \mathbb{R}^n \times \mathbb{R}^n$ we denote the mappings acting according to the rule $T\varphi(\xi) = (\varphi(\pi(\xi)), \varphi'(\xi))$, $\xi \in \pi^{-1}(U)$. The pair $(T\varphi, \pi^{-1}(U))$ is a map on the manifold TM called the *natural* map [3, §1.2] corresponding to the map (φ, U). The *natural* atlas on TM is defined similarly.

The following lemma indicates the way to define a topology on the space S not using coordinate maps and also establishes some properties of this topology.

Lemma 2. *The following assertions are valid:*

1) *the mapping $\Upsilon\colon S \to C(TM, TM)$ acting by the rule $\Upsilon(f) = df$ is a homeomorphism onto its image;*
2) *the function acting from $S \times TM$ to $\mathbb{R}$ by the rule $(f, \xi) \mapsto |df\xi|$ is continuous;*
3) *for each $k \in \mathbb{N}$ the mapping acting from S to S by the rule $f \mapsto f^k$ is continuous.*

Proof. 1. Choose some atlas on the manifold M and the corresponding natural atlas on TM. Note that each set of form (1) contains together with each its point a set of the same form with the center at this point (with lesser value of the parameter ε). The same is true for sets of form (2). Further, the image of any set $\mathcal{K}(f; (\varphi, U), (\psi, V), K, \varepsilon)$ under the mapping Υ contains the set $\mathcal{N}(df; (T\varphi, \pi^{-1}(U)), (T\psi, \pi^{-1}(V)), \Sigma, \varepsilon) \cap \Upsilon(S)$, where $\Sigma = \{\xi \in \pi^{-1}(K) : |\varphi'(\xi)| = 1\}$. The equality $\Sigma = (T\varphi)^{-1}(\varphi(K) \times \overline{B_1(0)})$ implies that Σ is compact. Therefore, the image of each open set under the mapping Υ is an open set in the space $\Upsilon(S)$.

On the other hand, let (φ, U) and (ψ, V) be maps of the chosen atlas of the manifold M. Further, let a compactum $\Xi \subset \pi^{-1}(U)$, $\varepsilon > 0$, and a mapping $f \in S$ be given and $df(\Xi) \subset \pi^{-1}(V)$. Assume $r = 2\sup\{|\varphi'(\xi)| : \xi \in \Xi\} + 2$. Then the following inclusion holds:

$$\Upsilon(\mathcal{K}(f; (\varphi, U), (\psi, V), \pi(\Xi), \varepsilon/r)) \subset \mathcal{N}(df; (T\varphi, \pi^{-1}(U)), (T\psi, \pi^{-1}(V)), \Xi, \varepsilon).$$

Above arguments imply that the preimage of any open subset of the space $\Upsilon(S)$ under the mapping Υ is an open set in the space S. Finally, Υ is one-to-one because of $f = \pi \circ df \circ O$, $f \in S$, where O is a zero vector field on M. Thus, we have established that the mapping $\Upsilon\colon S \to \Upsilon(S)$ is a homeomorphism.

2. Note that the function acting from $S \times TM$ to $\mathbb{R}$ by the rule $(f, \xi) \mapsto |df\xi|$ is a composition of the following continuous mappings: a) the direct product of the mapping Υ by the identical mapping of the space TM; b) the calculation mapping acting from $C(TM, TM) \times TM$ to TM by the rule $(F, \xi) \mapsto F(\xi)$; c)

the function $|\cdot|\colon TM \to \mathbb{R}$. The continuity of the first of indicated mappings follows from clause 1, and for the second one it follows from Theorem 2.4 of [9, Ch. XII]. Assertion 2 is proved.

3. Let $k \in \mathbb{N}$ be given. Due to clause 1, it is sufficient to show that the mapping acting from $C(TM, TM)$ to $C(TM, TM)$ by the rule $F \mapsto F^k$ is continuous. The latter can be proved by induction from Theorem 2.2 of [9, Ch. XII]. Assertion 3 is proved. The lemma is proved.

Lemma 3. *The spaces S, M, and TM are metrizable.*

Proof. Any manifold is a locally compact space. Due to [10, §41.X, Theorem 2], such spaces are completely regular [4, §14.I] and, therefore [4, §14.I, Theorem 1], are regular [4, §5.X]. Therefore, by Urysohn's theorem [4, §22.II, Theorem 1], each manifold with a countable base is metrizable. In particular, each of the spaces M and TM is metrizable. The metrizability of the space S follows from assertion 1 of Lemma 2 and Theorem 1 of [10, §44.VII] (see also [3, Theorem 4.4]). The lemma is proved.

By $p\colon S \times TM \to S \times M$ we denote the direct product of the identical mapping 1_S of the space S and the natural projection π of the tangent bundle, i.e., the mapping acting by the rule $(f, \xi) \mapsto (f, \pi(\xi))$, $f \in S$, $\xi \in TM$.

Lemma 4. *Let $H \subset S \times TM$. In this case the following assertions are valid:*

1) *if the set H is open, then the set $p(H)$ is open too;*

2) *if the set H is of type F_σ, then $p(H)$ is also a set of type F_σ.*

Proof. 1. Let $\{(\varphi_i, U_i)\}_{i\in\mathbb{N}}$ be the atlas of the manifold M and $(T\varphi_i, \pi^{-1}(U_i))_{i\in\mathbb{N}}$ be the corresponding natural atlas of its tangent bundle. To prove the first assertion, it is sufficient to verify that for any $i \in \mathbb{N}$ the restriction of the natural projection π to the space $\pi^{-1}(U_i)$ is open. In fact, this restriction is a composition of open mappings $T\varphi_i$, projection of the product of $\varphi_i(U_i) \times \mathbb{R}^n$ on the first cofactor, and the mapping φ_i^{-1}.

2. Now let H be a set of type F_σ. Fix $i \in \mathbb{N}$ and show that $p(H) \cap (S \times U_i)$ is a set of type F_σ. Define the mapping $\theta_i\colon \varphi_i(U_i) \times \mathbb{R}^n \to U_i \times \mathbb{R}^n$ by assuming $\theta_i(x, v) = (\varphi_i^{-1}(x), v)$ for any $x \in \varphi_i(U_i)$ and $v \in \mathbb{R}^n$. In this case the mapping $h_i = 1_S \times (\theta_i \circ T\varphi_i)\colon S \times \pi^{-1}(U_i) \to S \times U_i \times \mathbb{R}^n$ is a homeomorphism. The set $H_i = H \cap (S \times \pi^{-1}(U_i))$ is a set of type F_σ in the space $S \times \pi^{-1}(U_i)$, therefore, the set $h_i(H_i)$ is a set of type F_σ in the space $S \times U_i \times \mathbb{R}^n$. Due to [10, §41.IV, Remark 2], the projection of the set $h_i(H_i)$ onto the product of first two cofactors of the product $S \times U_i \times \mathbb{R}^n$ is a set of type F_σ in the space $S \times U_i$. Note that this projection coincides with the set $p(H_i)$. The definition of the relative topology implies that there exists a set P of type F_σ relative to the entire space $S \times M$ such that $p(H_i) = P \cap (S \times U_i)$ [4, §5.V]. Since the space $S \times M$ is metrizable (see Lemma 3), each its open subset is a set of type F_σ [4, §21.IV]. Thus, $p(H_i)$ is a set of type F_σ in the space $S \times M$. The required assertion follows now from the equality $p(H) = \bigcup_{i\in\mathbb{N}} p(H_i)$. The lemma is proved.

Lemma 5. *For any $r \in \mathbb{R}$ the set $\{(f, \xi) \in S \times TM : \lambda(f, \xi) < r\}$ is a set of type F_σ.*

Proof. The required assertion follows from the equality (we assume $\ln 0 = -\infty$)

$$\{(f,\xi) \in S \times TM : \lambda(f,\xi) < r\} = \bigcup_{k\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} \bigcap_{m\geq k} \{(f,\xi) \in S \times TM : \ln|df^m\xi|^{1/m} \leq r - j^{-1}\}, \ r \in \mathbb{R}.$$

In fact, the functions acting from $S \times TM$ to $\overline{\mathbb{R}}$ by the rule $(f, \xi) \mapsto \ln|df^m\xi|^{1/m}$, $m \in \mathbb{N}$, are continuous due to assertions 2 and 3 of Lemma 2, therefore, the sets under the sign of intersection are closed as well as the result of intersection. The lemma is proved.

Proof of the theorem. 1. By $(TM)_*$ we denote the subset of tangent bundle TM obtained from it by throwing out the zero vector in each layer. Fix arbitrary $k, T \in \mathbb{N}$ and define the function $g^{k,T}\colon S \times (TM)_* \to \mathbb{R}$ by the equality

$$g^{k,T}(f,\xi) = |df^{kT}\xi| \cdot |df^{(k-1)T}\xi|^{-1}, \quad f \in S, \quad \xi \in (TM)_*.$$

Due to assertions 2 and 3 of Lemma 2, the function $g^{k,T}$ is continuous.

Fix $i \in \{1, \ldots, n-1\}$ and for each $j \in \mathbb{N}$ define the function $h_{i,j}^{k,T}\colon S \times M \to \mathbb{R}$ by the equality

$$h_{i,j}^{k,T}(f,x) = \sup_{\xi \in E_{i,j}(f,x)\setminus\{0\}} g^{k,T}(f,\xi), \quad f \in S, \quad x \in M,$$

where $E_{i,j}(f,x) = \{\xi \in T_xM : \Phi(\lambda(f,\xi)) < \Phi(\lambda_i(f,x)) + j^{-1}\}$, and $\Phi\colon \overline{\mathbb{R}} \to [-1, 1]$ is the increasing homeomorphism

$$\Phi(x) = \begin{cases} \frac{x}{|x|+1} & \text{for } x \in \mathbb{R}, \\ \operatorname{sgn} x & \text{for } x = \pm\infty. \end{cases}$$

Fix arbitrary $r \in \mathbb{R}$ and show that the set $C^{k,T}_{i,j,r} = \{(f,x) \in S \times M : h^{k,T}_{i,j}(f,x) > r\}$ is a set of type $G_{\delta\sigma}$. Assume $G^{k,T}_r = \{(f,\xi) \in S \times (TM)_* : g^{k,T}(f,\xi) > r\}$. Moreover, for each $q \in \mathbb{Q}$ assume $A_q = \{(f,\xi) \in S \times TM : \Phi(\lambda(f,\xi)) < q\}$, $B^{i,j}_q = \{(f,x) \in S \times M : \Phi(\lambda_i(f,x)) + j^{-1} > q\}$. In this case the following representation is valid:

$$C^{k,T}_{i,j,r} = \bigcup_{q\in\mathbb{Q}} p(G^{k,T}_r \cap A_q) \cap B^{i,j}_q. \tag{4}$$

Due to the continuity of the function $g^{k,T}$, the set $G^{k,T}_r$ is open in the metrizable (by Lemma 3) space $S\times TM$ and hence it is a set of type F_σ [4, §21.IV]. Due to Lemma 5, the set A_q is also a set of type F_σ. Therefore, their intersection possesses the same property and hence, by Lemma 4, its image under the mapping p does too. Each set of type F_σ in the metrizable (due to Lemma 3) space $S \times M$ is a set of type $G_{\delta\sigma}$ [4, §21.IV]. According to the result of [11], the function $\lambda_i \colon S \times M \to \overline{\mathbb{R}}$ belongs to the second Baire class. Therefore, $B^{i,j}_q$ is also a set of type $G_{\delta\sigma}$ [12, §39.2]. Thus, the sets standing in (4) under the sign of countable union are sets of type $G_{\delta\sigma}$ as well as the results of this union.

2. Define the functions $s_{m,T,j}$, $\sigma_{l,T,j}$, and $\Omega^{(i)}_{T,j}$, $m,T,l,j \in \mathbb{N}$, from $S \times M$ to $\overline{\mathbb{R}}$ by the equalities ($f \in S$, $x \in M$)

$$s_{m,T,j}(f,x) = \frac{1}{mT}\sum_{k=1}^{m} \ln h^{k,T}_{i,j}(f,x), \quad \sigma_{l,T,j}(f,x) = \sup_{m\ge l} s_{m,T,j}(f,x), \quad \Omega^{(i)}_{T,j}(f,x) = \inf_{l\in\mathbb{N}} \sigma_{l,T,j}(f,x). \tag{5}$$

For any $T \in \mathbb{N}$ define the function $\Omega^{(i)}_T \colon S \times M \to \overline{\mathbb{R}}$ by the equality

$$\Omega^{(i)}_T(f,x) = \varlimsup_{m\to\infty} \frac{1}{mT}\sum_{k=1}^{m} \ln \|df^T|_{df^{(k-1)T}E_i(f,x)}\|, \quad f \in S, \quad x \in M,$$

and show that

$$\Omega^{(i)}_T(f,x) = \inf_{j\in\mathbb{N}} \Omega^{(i)}_{T,j}(f,x), \quad f \in S, \quad x \in M. \tag{6}$$

Fix $T > 0$, $f \in S$, and $x \in M$. Note that for any $j,k \in \mathbb{N}$ the definition of the function $h^{k,T}_{i,j}$ implies the equality $h^{k,T}_{i,j}(f,x) = \|df^T|_{df^{(k-1)T}E_{i,j}(f,x)}\|$. Further, for each $j \in \mathbb{N}$ we have the inclusion $E_i(f,x) \subset E_{i,j}(f,x)$, and for certain $j_0 \in \mathbb{N}$ it turns to equality because the set of values of the restriction of the function λ onto the tangent space T_xM is finite [2]. Finally, due to the definition of the upper limit we have the equality $\Omega^{(i)}_{T,j}(f,x) = \varlimsup_{m\to\infty} s_{m,T,j}(f,x)$.

3. According to paragraph 1, for each of functions $\ln h^{k,T}_{i,j}$, $j,k,T \in \mathbb{N}$, the preimage of each ray $(r,\infty]$, $r \in \mathbb{R}$ is the set of type $G_{\delta\sigma}$, therefore, due to [12, §37.1.I], their sums $s_{m,T,j}$ possess the same property as well as the functions $\sigma_{l,T,j}$, $m,T,l,j \in \mathbb{N}$. The equality $[r,\infty] = \bigcap_{s\in\mathbb{N}}(r - 1/s, \infty]$ and properties of the preimage of a set imply that the preimage of any ray $[r,\infty]$ is a set of type $G_{\delta\sigma\delta}$ for each of the functions $\sigma_{l,T,j}$, $l,T,j \in \mathbb{N}$. Due to the theorem [12, §39.2], these functions belong to the third Baire class. The definition of central exponents and equalities (5), (6) imply the representation

$$\Omega^{(i)}(f,x) = \inf_{(l,T,j)\in\mathbb{N}^3} \sigma_{l,T,j}(f,x), \quad f \in S, \quad x \in M.$$

Rewriting it in the form

$$\Omega^{(i)}(f,x) = \lim_{k\to\infty} \min_{l+T+j\le k} \sigma_{l,T,j}(f,x), \quad f \in S, \quad x \in M,$$

we get that the function $\Omega^{(i)}$ is a limit of a nonincreasing sequence of functions of third Baire class and, therefore, it belongs to the fourth class.

4. Consider the case $i = n$ separately. Define the functions $h^{k,T}$, $s_{m,T}$, $\sigma_{l,T}$, $k,l,m,T \in \mathbb{N}$, from $S \times M$ to $\overline{\mathbb{R}}$ by the equalities ($f \in S$, $x \in M$)

$$h^{k,T}(f,x) = \sup_{\xi\in T_xM\setminus\{0\}} g^{k,T}(f,\xi), \quad s_{m,T}(f,x) = \frac{1}{mT}\sum_{k=1}^{m} \ln h^{k,T}(f,x), \quad \sigma_{l,T}(f,x) = \sup_{m\ge l} s_{m,T}(f,x).$$

As was pointed out in paragraph 1, for each $r \in \mathbb{R}$ the set $G_r^{k,T}$ is open, therefore, the first assertion of Lemma 4 implies that the set $p(G_r^{k,T}) = \{(f,x) \in S \times M : h^{k,T}(f,x) > r\}$ is open too. Thus, the functions $h^{k,T}$ are semicontinuous from below and hence the functions $\ln h^{k,T}$ possess the same property as well as their sums $s_{m,T}$ and the functions $\sigma_{l,T}$, $k,m,T,l \in \mathbb{N}$ due to [12, §37.1.I].The definition of the upper limit implies the chain

$$\Omega^{(n)}(f,x) = \inf_{T\in\mathbb{N}} \overline{\lim_{m\to\infty}} s_{m,T}(f,x) = \lim_{k\to\infty} \min_{l+T\leq k} \sigma_{l,T}(f,x), \quad f \in S, \quad x \in M.$$

Since each semicontinuous function belongs to the first Baire class [12, §38.I], taking into account [12, §37.III], we get that the function $\Omega^{(n)}$ is the limit of a nonincreasing sequence of functions from the first class and hence it belongs to the second class. The theorem is proved.

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Translated by V. Valedinskii