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On a Certain Mean Value Theorem

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Abstract—Asymptotics for mean values of complete rational trigonometric sums modulo a power of a prime number are obtained. For polynomials of one variable these asymptotics are not improvable in the degree of averaging of those sums.

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Vinogradov's method for evaluation of Weyl's trigonometric sums is based on the theorem on mean values of such sums [1–6]. Here we consider a similar theorem on mean values of complete rational trigonometric sums of the form

$$S(p^m; f(x)) = \sum_{x=1}^{p^m} e^{2\pi i f(x)}, f(x) = \sum_{s=1}^{n} \frac{a_s x^s}{p^{m_s}}, (a_s, p) = 1, m_s \le m.$$

The mean value $N(p^m)$ has the form

$$N(p^m) = p^{-mn} \sum_{\max\{m_n,\ldots,m_1\}\le m} \sum_{\substack{a_n=0\\(a_n,p)=1}}^{p^{m_n}-1} \cdots \sum_{\substack{a_1=0\\(a_1,p)=1}}^{p^{m_1}-1} |S(p^m; f(x))|^{2k}.$$

Assume $t = \max\{m_1, \ldots, m_n\}$. We get

$$N(p^m) = p^{-mn} \sum_{t=0}^{m} \sum_{\substack{a_n=0\\(a_n,\ldots,a_1,p)=1}}^{p^t-1} \cdots \sum_{a_1=0}^{p^t-1} \left| S\left(p^m; \frac{a_n x^n + \ldots + a_1 x}{p^t}\right)\right|^{2k}$$

$$= p^{2km-mn} \sum_{t=0}^{m} \sum_{\substack{a_n=0\\(a_n,\ldots,a_1,p)=1}}^{p^t-1} \cdots \sum_{a_1=0}^{p^t-1} \left| p^{-t} S\left(p^t; \frac{a_n x^n + \ldots + a_1 x}{p^t}\right)\right|^{2k} = p^{m(2k-n)}\sigma(p^m). \quad (1)$$

Write down all rational coefficients of the polynomial in the exponent of the sum as fractions with the denominator p^m. We obtain

$$N(p^m) = p^{-mn} \sum_{a_n=0}^{p^m-1} \cdots \sum_{a_1=0}^{p^m-1} \left| S\left(p^m; \frac{g(x)}{p^m}\right)\right|^{2k}, g(x) = \sum_{s=1}^{n} a_s x^s,$$

which is equal to the number of solutions to the following system of comparisons:

$$\begin{cases} x_1 + \ldots + x_k \equiv y_1 + \ldots + y_k \pmod{p^m}, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ x_1^n + \ldots + x_k^n \equiv y_1^n + \ldots + y_k^n \pmod{p^m}, \end{cases}$$

where the unknowns $x_1, \ldots, x_k, y_1, \ldots, y_k$ take values from the complete system of residues modulo p^m.

The following assertions are valid.

Theorem 1. *Let $n \ge 2, m$ be natural numbers, $p > n$ is a prime number. In this case for $2k > \frac{n(n+1)}{2} + 1$ and $m \to \infty$ we have*

$$N(p^m) = p^{m(2k-n)}(\sigma_p + O(m^n p^{((m-1)/n)(0,5n(n+1)+1-2k)}),$$

where

$$\sigma_p = 1 + \sum_{t=1}^{+\infty} A(p^t), A(p^t) = \sum_{\substack{a_n=0 \\ (a_n,\ldots,a_1,p)=1}}^{p^t-1} \ldots \sum_{a_1=0}^{p^t-1} |p^{-t} S(p^t; (a_n x^n + \ldots + a_1 x)/p^t)|^{2k},$$

$$S(p^t; (a_n x^n + \ldots + a_1 x)/p^t) = \sum_{x=1}^{p^t} e^{2\pi i \frac{a_n x^n + \ldots + a_1 x}{p^t}}.$$

Proof. Since the series σ_p converges for $2k > \frac{n(n+1)}{2} + 1$ and

$$A(p^t) \le n^{2k} (tp)^n p^{((t-1)/n)(0{,}5n(n+1)+1-2k)}$$

(see [4., p. 69]), then formula (1) implies

$$N(p^m) = p^{m(2k-n)} \sigma(p^m) = p^{m(2k-n)} (\sigma_p + O(m^n p^{((m-1)/n)(0{,}5n(n+1)+1-2k)})).$$

Theorem 1 is proved.

The assertion of the following Theorem 2 is based on the convergence of the series σ'_p for $2k > s+r+\ldots+n$ (see [3, p. 71, Theorem 5]).

Theorem 2. *Let $1 \le s < r < \ldots < n, m$ be natural numbers, the number of the numbers $s, r, \ldots, n$ be equal to l and $l < n$, let $p > n$ be a prime number, $N_l(p^m)$ be the number of solutions to the system of comparisons*

$$\begin{cases} x_1^s + \ldots + x_k^s \equiv y_1^s + \ldots + y_k^s \pmod{p^m}, \\ x_1^r + \ldots + x_k^r \equiv y_1^r + \ldots + y_k^r \pmod{p^m}, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ x_1^n + \ldots + x_k^n \equiv y_1^n + \ldots + y_k^n \pmod{p^m}, \end{cases}$$

where the unknowns $x_1, \ldots, x_k, y_1, \ldots, y_k$ take values from the complete system of residues modulo p^m. In this case for $2k > s + r + \ldots + n$ and $m \to \infty$ we have

$$N_l(p^m) = p^{m(2k-l)} (\sigma'_p + O(m^n p^{((m-1)/n)(s+r+\ldots+n-2k)})),$$

$$\sigma'_p = 1 + \sum_{t=1}^{+\infty} A_l(p^t), A_l(p^t) = \sum_{\substack{a_n=0 \\ (a_n,\ldots,a_r,a_s,p)=1}}^{p^t-1} \ldots \sum_{a_r=0}^{p^t-1} \sum_{a_s=0}^{p^t-1} |p^{-t} S(p^t; a_n x^n + \ldots + a_r x^r + a_s x^s)|^{2k},$$

$$S(p^t; (a_n x^n + \ldots + a_r x^r + a_s x^s)/p^t) = \sum_{x=1}^{p^t} e^{2\pi i \frac{a_n x^n + \ldots + a_r x^r + a_s x^s}{p^t}}.$$

Finally, formulate the mean value theorem for complete multiple rational trigonometric sums of the form

$$S\left(p^t; r; \frac{F(x_1, \ldots, x_r)}{p^t}\right) = \sum_{x_1=1}^{p^t} \ldots \sum_{x_1=1}^{p^t} e^{2\pi i \frac{F(x_1,\ldots,x_r)}{p^t}},$$

where $F(x_1, \ldots, x_r) = \sum_{t_1=0}^{n_1} \ldots \sum_{t_r=0}^{n_r} a(t_1, \ldots, t_r) x_1^{t_1} \ldots x_r^{t_r}$ is a polynomial with integer coefficients, $a(0, \ldots, 0) = 0$, and all the coefficients of the polynomial are prime in common with p. The number of coefficients of the polynomial $F(x_1, \ldots, x_r)$ equals $m = (n_1 + 1) \ldots (n_r + 1)$.

The mean value $N(p^s; r)$ of these sums is the number of solutions to the system of comparisons

$$\sum_{j=1}^{2k} (-1)^j x_{1,j}^{t_1} \ldots x_{r,j}^{t_r} \equiv 0 \pmod{p^s}$$

$$(0 \le t_1 \le n_1, \ldots, 0 \le t_r \le n_r, t_1 + \ldots + t_r \ge 1),$$

where the unknowns $x_{1,j}, \ldots, x_{r,j}, j = 1, \ldots, 2k$, take values from the complete system of residues modulo p^s.

In this case, $N(p^s; r) = p^{s(2kr-m+1)}\sigma(p^s; r)$, where

$$\sigma(p^s; r) = \sum_{t=0}^{s} \sum_{\substack{a(n_1,\ldots,n_r)=0 \\ (a(n_1,\ldots,n_r),\ldots,a(0,\ldots,1),p)=1}}^{p^t-1} \ldots \sum_{a(0,\ldots,1)=0}^{p^t-1} \left| p^{-tr} S\left(p^t; r; \frac{F(x_1,\ldots,x_r)}{p^t}\right)\right|^{2k}.$$

Assume $n = \max\{n_1,\ldots,n_r\}$. For $2k > nm$ the series $\sigma_p(r) = \lim_{s\to+\infty} \sigma(p^s; r)$ converges (see [3, p. 81, Theorem 7]).

Theorem 3. *The following asymptotic formula holds for $2k > nm$ and $s \to +\infty$:*

$$N(p^s; r) = p^{2kr-m+1}(\sigma_p(r) + o(1)),$$

where

$$\sigma_p(r) = \sum_{t=0}^{+\infty} \sum_{\substack{a(n_1,\ldots,n_r)=0 \\ (a(n_1,\ldots,n_r),\ldots,a(0,\ldots,1),p)=1}}^{p^t-1} \ldots \sum_{a(0,\ldots,1)=0}^{p^t-1} \left| p^{-tr} S\left(p^t; r; \frac{F(x_1,\ldots,x_r)}{p^t}\right)\right|^{2k}.$$

Note that Theorems 1 and 2 are not improvable because the borders for the value k are the factors of convergence for the corresponding series σ_p and σ'_p, the border for the value k in Theorem 3 is the best to date because for $r > 1$ the exact value of the convergence factor of the series $\sigma_p(r)$ is not found yet.

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