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On Some Finite Difference Scheme for Gas Dynamics Equations

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Abstract—A conservative difference scheme with linear dependence of the pressure on the density of gas is proposed for gas dynamics equations. The scheme allows us to simulate 1-D flows inside a cylindrical domain with time-variable cross-sections and guarantees the positive sign of the density function.

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1. Introduction. One of contemporary applied problems of hydro- and gas dynamics are flow calculations for the case of closing valves and dampers in pipelines [1, 2]. One-dimensional simulations of such flows usually apply the method of characteristics. A valve is modeled as a place where flow parameters (pressure and velocity) are discontinuous, i.e., have different values from both sides of the valve. The system of continuum mechanics equations becomes not closed in this case. To close it, two empirical relations are added, which links the velocity and pressure in front of the valve with the velocity and pressure behind it. These relations depend on technical characteristics of specific valves. In this paper we propose to consider a valve as part of a pipe with cross section being variable in time. Such a scheme allows us to simulate the closure of the valve in a one-dimensional flow.

This study is the continuation of [3, 4] were some difference schemes were proposed to ensure the positivity of the grid density function for one-dimensional equations of barotropic gas dynamics with a power dependence of the pressure on the density with the exponent exceeding one.

2. Formulation of the problem. Consider the system of equations describing the motion of a viscous barotropic compressible gas (fluid) in a straight pipe of variable radius:

$$\begin{aligned} &\frac{\partial(A\rho)}{\partial t} + \frac{\partial(A\rho u)}{\partial x} = 0, \\ &\frac{\partial(A\rho u)}{\partial t} + \frac{\partial(A\rho u^2)}{\partial x} + A\frac{\partial p}{\partial x} + \lambda u|u|L = 0, \end{aligned} \tag{1}$$

where ρ, u, and p are the density, velocity, and pressure of the fluid, respectively, $A = A(x,t)$ is the cross-section area of the pipe, $L = L(x,t)$ is the perimeter of the cross section of the pipe. The pressure and density are connected by the linear law $p = \kappa\rho$, where κ is some positive constant. The force of friction in the second equation of system (1) is modeled by the experimental formula $\lambda u|u|$, where λ is a positive constant dependent only on the Reynolds number. The equations are considered in the cylinder $Q_T = [X_1, X_2] \times [0, T]$.

3. A priori estimates. Supplement system (1) with the boundary

$$u(X_1) = u(X_2) = 0 \tag{2}$$

and initial

$$\rho(x,0) = \rho_0(x),\ \ u(x,0) = u_0(x) \tag{3}$$

conditions. Without loss of generality, simplify the notations assuming that $\kappa = 1$ and $\lambda = 1$. Taking the scalar products of the first equation of system (1) and $-\frac{1}{2}u^2$ and the second equation and u in L_2 and summing them, we get

$$-\frac{1}{2}\Big((A\rho)_t, u^2\Big) + ((A\rho u)_t, u) - \frac{1}{2}\Big((A\rho u)_x, u^2\Big) + \big(\big(A\rho u^2\big)_x, u\big) + (A\rho_x, u) + (u|u|L, u) = 0. \tag{4}$$

Elementary transformations give

$$\begin{gathered}-\frac{1}{2}\Big((A\rho)_t, u^2\Big) + ((A\rho u)_t, u) = \frac{1}{2}\frac{d}{dt}\left(A\rho, u^2\right),\\ -\frac{1}{2}\Big((A\rho u)_x, u^2\Big) + \left(\left(A\rho u^2\right)_x, u\right) = 0.\end{gathered} \tag{5}$$

Transforming the fifth term in the left-hand side of (4) and assuming that $\rho > 0$, we get

$$\begin{gathered}(A\rho_x, u) = \left(\frac{1}{\rho}\rho_x, A\rho u\right) = ((\ln\rho)_x, A\rho u) = (\ln\rho, (A\rho)_t) = \big((A\rho\ln\rho)_t - A\rho(\ln\rho)_t, 1\big)\\ = \big((A\rho\ln\rho)_t - A\rho_t, 1\big) = \big((A\rho\ln\rho)_t - (A\rho)_t + \rho A_t, 1\big) = \big((A\rho(\ln\rho - 1))_t + \rho A(\ln A)_t, 1\big).\end{gathered} \tag{6}$$

Thus, taking into account (5) and (6), equality (4) takes the form

$$\frac{1}{2}\frac{d}{dt}\left(A\rho, u^2\right) + \big((A\rho(\ln\rho - 1))_t, 1\big) + \big(\rho A(\ln A)_t, 1\big) + \big(|u|L, u^2\big) = 0. \tag{7}$$

If the function $(\ln A)_t$ is continuous on the segment $[X_1, X_2]$ for any fixed $t \geqslant 0$, then the penultimate summand of (7) can be estimated with the use of the inequality

$$\big(\rho A(\ln A)_t, 1\big) \geqslant (\rho A, 1)\min_{[X_1,X_2]}(\ln A)_t = (\rho_0 A_0, 1)\min_{[X_1,X_2]}(\ln A)_t. \tag{8}$$

Using relation (8), we formally integrate (7) over time on the segment $[0, T]$. We obtain the "energy" inequality

$$\begin{gathered}\frac{1}{2}\Big(A\rho, u^2\Big) + \big(A\rho(\ln\rho - 1), 1\big) + \int_0^T\int_{X_1}^{X_2} u^2|u|L\,dx\,dt + \int_{X_1}^{X_2}\rho_0 A_0\,dx\int_0^T \min_{[X_1,X_2]}(\ln A)_t\,dt\\ \leqslant \frac{1}{2}\Big(A_0\rho_0, u_0^2\Big) + \big(A_0\rho_0(\ln\rho_0 - 1), 1\big).\end{gathered} \tag{9}$$

Estimate the value $\int_0^T \min(\ln A)_t dt$ in some particular case. Let the function A satisfy the condition

$$A_t \geqslant -\theta \text{ for } x \in [X_1, X_2],\ t \in [0, T],$$

where θ is some positive constant. Denote the value $\min\limits_{[X_1,X_2]} A_0$ by $A_{0,\min}$. In this case we have $A(x,t) \geqslant A_{0,\min} - \theta t$ on the segment $[X_1, X_2]$ and

$$(\ln A)_t = \frac{A_t}{A} \geqslant -\frac{\theta}{A} \geqslant -\frac{\theta}{A_{0,\min} - \theta t}.$$

The latter summand in the left-hand side of (9) can be estimated here with the use of the inequality

$$\int_{X_1}^{X_2}\rho_0 A_0\,dx\int_0^T \min_{[X_1,X_2]}(\ln A)_t\,dt \geqslant \int_{X_1}^{X_2}\rho_0 A_0\,dx\int_0^T -\frac{\theta}{A_{0,\min} - \theta t}\,dt = \ln\frac{A_{0,\min} - T\theta}{A_{0,\min}}\int_{X_1}^{X_2}\rho_0 A_0\,dx. \tag{10}$$

In the case of function $A(x,t)$ linearly decreasing in time, expression (10) allows us to estimate the last term of the left-hand side of inequality (9) by the minimal area of the cross section of the channel at the time moment T.

4. Difference approximation. Introduce a spatial grid with the mesh size $h = (X_2 - X_1)/N$, namely, $\Omega_\rho = \{x_i = X_1 + ih,\ i = 0, \ldots, N-1\}$, $\Omega_u = \{x_i = X_1 + ih,\ i = 0, \ldots, N\}$. Set the time step $\tau = T/M$. Let at the time moment $n\tau$ the grid function ρ^n be determined on Ω_ρ and u^n be determined on Ω_u. Assume that these functions are continued by zero to the entire grid line. Construct the difference approximation of system (1)–(3) so that it possesses the following properties.

1. The difference scheme must be conservative, i.e., it must satisfy the grid analogue of the mass conservation law

$$\sum_{i=0}^{N-1} hA_i^n\rho_i^n = \sum_{i=0}^{N-1} hA_i^{n+1}\rho_i^{n+1},$$

where $A_i^n = A(ih, n\tau)$.

2. The density ρ^n must be strictly positive.

3. The grid approximations of $(A\rho u)_x$ and $(A\rho u^2)_x$ must be adjoint, i.e., they must satisfy the grid analogue of the equality

$$\big((A\rho u^2)_x, u\big) - \frac{1}{2}\Big((A\rho u)_x, u^2\Big) = 0.$$

4. The solution to the difference scheme must exist.

5. Relation (9) imply the boundedness of the norm $\|(A\rho)^{1/2}u\|_{L_2}$ on any finite time interval. The solution to the difference scheme must possess a similar property.

Approximation of the continuity equation. We approximate the continuity equation similar to [4], i.e.,

if u_i and u_{i+1} have the same signs, then we use the "upwind approximation"

$$(\rho u)_x \approx \begin{cases} \dfrac{\rho_i A_{i+1}u_{i+1} - \rho_{i-1}A_i u_i}{h} & \text{for } u_i,\ u_{i+1} > 0; \\ \dfrac{\rho_{i+1} A_{i+1}u_{i+1} - \rho_i A_i u_i}{h} & \text{for } u_i,\ u_{i+1} < 0; \end{cases}$$

if $u_i > 0$ and $u_{i+1} < 0$, then

$$(\rho A u)_x = \rho_x A u + \rho(Au)_x \approx \frac{\rho_{i+1}A_{i+1}u_{i+1} - \rho_i A_i u_i}{h} + \frac{(\rho_i - \rho_{i-1})A_i u_i}{h};$$

if $u_i < 0$ and $u_{i+1} > 0$, then

$$(\rho A u)_x = \rho_x A u + \rho(Au)_x \approx \frac{\rho_i(A_{i+1}u_{i+1} - A_i u_i)}{h}.$$

This approximation has the order $O(h)$. As usual, denote $\hat{u} = u^{n+1}$, $u = u^n$, $u_x = \frac{u_{i+1}-u_i}{h}$, $u_t = \frac{u^{n+1}-u^n}{\tau}$, in non-index notation we assume $u = u_i$, $u_{(-1)} = u_{i-1}$, and consider a completely implicit difference scheme for problem (1)–(3). For the sake of convenience, introduce the grid operator $\{\cdot\}$ as

$$\{\rho\}_i = \begin{cases} \rho_{i-1}, & \text{for } u_i > 0; \\ \rho_i, & \text{for } u_i < 0. \end{cases}$$

In these notations the approximation of the first equation of system (1) takes the form

$$(A\rho)_t + (\{\hat{\rho}\}\hat{A}\hat{u})_x = 0. \tag{11}$$

In the same manner as in [4], we can prove that approximation (11) ensures the fulfillment of a grid analogue of mass conservation law, i.e.,

$$\sum_{i=0}^{N-1} hA_i^0\rho_i^0 = \sum_{i=0}^{N-1} hA_i^n\rho_i^n, \tag{12}$$

and under the conditions $\rho_i^0 > 0$ and $A_i^n > 0$ for any $i = \overline{0, N-1}$ and $n \geqslant 0$ we have the inequality $\rho_i^n > 0$ for any $i = \overline{0, N-1}$ and $n > 0$.

Approximation of the motion equation. The second equation of system (1) is approximated in the following way:

$$(A\rho u)_t + \frac{1}{2}\Big(\{\hat{\rho}\}\hat{A}\hat{u}(\hat{u}_{(-1)} + \hat{u})\Big)_x + \kappa\frac{1}{h}\hat{A}\{\hat{\rho}\}\ln\frac{\hat{\rho}}{\hat{\rho}_{(-1)}} + \lambda\hat{u}|\hat{u}|\hat{L} = 0. \tag{13}$$

Prove that the norm $\|(A\rho)^{1/2}u\|_{L_2,h}$ is uniformly bounded with respect to time on any finite time interval. In fact, multiply equation (11) by $-\frac{1}{2}\tau\hat{u}^2$ and equation (13) by $\tau\hat{u}$ and sum the results. We obtain the equality

$$-\frac{1}{2}\tau\Big((A\rho)_t, \hat{u}^2\Big) + \tau\big((A\rho u)_t, \hat{u}\big) + \frac{1}{2}\tau\Big(\Big(\{\hat{\rho}\}\hat{A}\hat{u}(\hat{u}_{(-1)} + \hat{u})\Big)_x, \hat{u}\Big)$$

$$-\tau\frac{1}{2}\left(\left(\{\hat\rho\}\hat A\hat u\right)_x,\hat u^2\right)+\tau\kappa\left(\frac{1}{h}\hat A\{\hat\rho\}\ln\frac{\hat\rho}{\hat\rho_{(-1)}},\hat u\right)+\tau\lambda\left(\hat u|\hat u|\hat L,\hat u\right)=0. \tag{14}$$

It is not difficult to show that

$$-\frac{1}{2}\tau\Big((A\rho)_t,\hat u^2\Big)+\tau\big((A\rho u)_t,\hat u\big)=\frac{1}{2}\tau\Big(A\rho,u^2\Big)_t+\frac{1}{2}\Big(A\rho,(\hat u-u)^2\Big), \tag{15}$$

and the grid approximations of $(A\rho u)_x$ and $(A\rho u^2)_x$ are skew-symmetric, i.e.,

$$\frac{1}{2}\tau\left(\left(\{\hat\rho\}\hat A\hat u(\hat u_{(-1)}+\hat u)\right)_x,\hat u\right)-\tau\frac{1}{2}\left(\left(\{\hat\rho\}\hat A\hat u\right)_x,\hat u^2\right)=0. \tag{16}$$

Using the relation $\ln x\leqslant x-1$, simplify the penultimate summand in (14). We get

$$\tau\left(\frac{1}{h}\hat A\{\hat\rho\}\ln\frac{\hat\rho}{\hat\rho_{(-1)}},\hat u\right)=\tau\left((\ln\hat\rho)_{\bar x},\hat A\{\hat\rho\}\hat u\right)=\tau\left((A\rho)_t\ln\hat\rho,1\right)=\tau\left((A\rho\ln\rho)_t-A\rho(\ln\rho)_t,1\right)$$

$$=\tau\left((A\rho\ln\rho)_t,1\right)-\left(A\rho\ln\frac{\hat\rho}{\rho},1\right)\geqslant\tau\left((A\rho\ln\rho)_t,1\right)-\left(A(\hat\rho-\rho),1\right)=\tau\left((A\rho\ln\rho)_t-(A\rho)_t+\hat\rho A_t,1\right)$$

$$\geqslant\tau\left((A\rho(\ln\rho-1))_t,1\right)+\tau(\hat A\hat\rho,1)\min_{\Omega_\rho}\frac{A_t}{\hat A}=\tau\left((A\rho(\ln\rho-1))_t,1\right)+\tau(A_0\rho_0,1)\min_{\Omega_\rho}\frac{A_t}{\hat A}. \tag{17}$$

Summing relation (14) over all steps in time and using relations (15)–(17), we obtain the energy inequality. Thus, we have proved the following

Theorem. *The solution to difference problem* (11), (13) *satisfies the inequality*

$$\frac{1}{2}\left(A^n\rho^n,(u^n)^2\right)+\kappa(A^n\rho^n(\ln\rho^n-1),1)+\frac{1}{2}\sum_{k=0}^{n-1}\left(A^k\rho^k,\left(u^{k+1}-u^k\right)^2\right)$$

$$+\kappa(A_0\rho_0,1)\sum_{k=0}^{n-1}\min_{\Omega_\rho}\frac{A^{k+1}-A^k}{A^{k+1}}+\tau\lambda\sum_{k=1}^{n}(|u^k|L^k,(u^k)^2)$$

$$\leqslant\frac{1}{2}\left(A^0\rho^0,\left(u^0\right)^2\right)+\kappa(A^0\rho^0(\ln\rho^0-1),1).$$

Estimate the value $\sum_{k=0}^{n-1}\min\frac{A^{k+1}-A^k}{A^{k+1}}$ in the same way as in the differential case with some additional restrictions. Denote $\min_{\Omega_\rho}A_0$ by $A_{0,\min}$. Let the following relations hold:

$$\tau<\frac{A_{0,\min}}{\theta(n+1)}\quad\text{and}\quad\frac{A^{k+1}-A^k}{\tau}\geqslant-\theta\text{ for }k=\overline{0,n-1}, \tag{18}$$

where θ is some positive constant. Using the estimate $A^k\geqslant A_{0,\min}-\tau\theta k$, we get

$$\sum_{k=0}^{n-1}\frac{A^{k+1}-A^k}{A^{k+1}}\geqslant\sum_{k=0}^{n-1}\frac{-\tau\theta}{A^{k+1}}\geqslant\sum_{k=0}^{n-1}\frac{-\tau\theta}{A_{0,\min}-\tau\theta(k+1)}. \tag{19}$$

Note that the inequality $\frac{\tau\theta}{A_{0,\min}-\tau\theta(k+1)}<1$, $k=\overline{0,n-1}$ is valid under conditions (18). Estimate the right-hand side of (19) using the inequality $y\leqslant\ln\frac{1}{1-y}$. We have

$$\sum_{k=0}^{n-1}\frac{-\tau\theta}{A_{0,\min}-\tau\theta(k+1)}\geqslant\sum_{k=0}^{n-1}-\ln\frac{A_{0,\min}-\tau\theta(k+1)}{A_{0,\min}-\tau\theta(k+1)-\tau\theta}=\ln\frac{A_{0,\min}-\tau\theta(n+1)}{A_{0,\min}-\tau\theta}.$$

This relation allows us to estimate the penultimate summand in the left-hand side of the energy inequality in the case of function $A(x,t)$ linearly decreasing in time with the use of the minimal cross section area of the channel at the nth time step.

5. Existence of the solution. Let the following condition hold:

$$(A_0\rho_0, 1) - (A^n, 1) > 0 \text{ for any } n > 0. \tag{20}$$

Similar to [4], prove that system of equations (11), (13) with boundary conditions (2) and initial conditions (3) has a solution for any τ and h. Apply the change $r = A\rho$ and $v = A\rho u$. The system takes the form

$$\begin{aligned} &r_t + \Big(\{\hat r/\hat A\}\hat A\hat v/\hat r\Big)_x = 0,\\ &v_t + \frac{1}{2}\Big(\{\hat r/\hat A\}\hat A\hat v\left(\hat v_{(-1)}/\hat r_{(-1)} + \hat v/\hat r\right)/\hat r\Big)_x + \kappa\frac{1}{h}\hat A\{\hat r/\hat A\}\ln\frac{\hat r/\hat A}{\hat r_{(-1)}/\hat A_{(-1)}} + \lambda\hat v/\hat r|\hat v/\hat r|\hat L = 0. \end{aligned} \tag{21}$$

Thus, system (21) is representable as the operator equation

$$\hat r = F_1(\hat r, \hat v), \quad \hat v = F_2(\hat r, \hat v).$$

To prove the existence of solution, we use the Leray–Schauder theorem, namely, prove that the solution to the system

$$\hat r = \theta F_1(\hat r, \hat v), \quad \hat v = \theta F_2(\hat r, \hat v) \tag{22}$$

is uniformly bounded with respect to θ for $\theta \in (0, 1)$. Rewrite system (22) in the form

$$\begin{aligned} &\frac{1-\theta}{\tau\theta}\hat A\hat\rho + (A\rho)_t + \Big(\{\hat\rho\}\hat A\hat u\Big)_x = 0,\\ &\frac{1-\theta}{\tau\theta}\hat A\hat\rho\hat u + (A\rho u)_t + \frac{1}{2}\Big(\{\hat\rho\}\hat A\hat u(\hat u_{(-1)} + \hat u)\Big)_x + \kappa\frac{1}{h}\hat A\{\hat\rho\}\ln\frac{\hat\rho}{\hat\rho_{(-1)}} + \lambda\hat u|\hat u|\hat L = 0. \end{aligned}$$

Multiply the first equation by $-\frac{1}{2}\tau\hat u^2$ and the second one by $\tau\hat u$ and sum up the results. Since the functions $A(x,t)$ and $L(x,t)$ are positive and $\rho^k > 0$ under the condition $\rho^0 > 0$, then

$$\begin{aligned} &\frac{1-\theta}{2\theta}\left(\hat A\hat\rho, \hat u^2\right) + \kappa\frac{1-\theta}{\theta}(\hat A\hat\rho, \ln\hat\rho) + \kappa(A_0\rho_0, 1)\min_{\Omega_\rho}\frac{\hat A - A}{\hat A} + \frac{1}{2}\Big(\hat A\hat\rho, \hat u^2\Big) + \kappa(\hat A\hat\rho(\ln\hat\rho - 1), 1)\\ &\leqslant \frac{1}{2}\Big(A\rho, u^2\Big) + \kappa(A\rho(\ln\rho - 1), 1)\\ &\leqslant \frac{1}{2}\Big(A_0\rho_0, u_0^2\Big) + \kappa(A_0\rho_0(\ln\rho_0 - 1), 1) - \kappa(A_0\rho_0, 1)\sum_{k=0}^{n-1}\min_{\Omega_\rho}\frac{A^{k+1} - A^k}{A^{k+1}}. \end{aligned} \tag{23}$$

Estimate the second summand in the left-hand side of relation (23) taking into account condition (20). We have

$$\Big(\hat A\hat\rho, \ln\hat\rho\Big) \geqslant \left(\hat A\hat\rho\left(1 - \frac{1}{\hat\rho}\right), 1\right) = \Big(\hat A\hat\rho - \hat A, 1\Big) = (A_0\rho_0, 1) - \Big(\hat A, 1\Big) > 0. \tag{24}$$

The last summand in the left-hand side of (23) satisfies the relation

$$\Big(\hat A\hat\rho(\ln\hat\rho - 1), 1\Big) \geqslant \left(\hat A\hat\rho\left(1 - \frac{1}{\hat\rho}\right) - \hat A\hat\rho, 1\right) = -\Big(\hat A, 1\Big). \tag{25}$$

Using estimates (24) and (25) and also grid mass conservation law (12), we get

$$\begin{aligned} &\|\hat v\|^2_{L_{2,h}} = \Big(\hat A\hat\rho\hat u, \hat A\hat\rho\hat u\Big) \leqslant \max_{\Omega_\rho}\hat A\hat\rho\cdot\Big(\hat A\hat\rho, \hat u^2\Big) \leqslant \frac{1}{h}\|A_0\rho_0\|_{L_{1,h}}\Big(\hat A\hat\rho, \hat u^2\Big)\\ &\leqslant \frac{1}{h}\|A_0\rho_0\|_{L_{1,h}}\left[(A_0\rho_0, (u_0)^2) + 2\kappa\left((A_0\rho_0(\ln\rho_0 - 1), 1) + (\hat A, 1) - (A_0\rho_0, 1)\sum_{k=0}^{n}\min_{\Omega_\rho}\frac{A^{k+1} - A^k}{A^{k+1}}\right)\right]. \end{aligned}$$

The value $\hat r$ is bounded in the norm $L_{1,h}$ due to conservative property (12) of the difference scheme. Thus, we have fulfilled all the conditions of the Leray–Schauder theorem and hence the problem has the solution for any τ and h.

6. Numerical experiment. Consider the model of a closing valve in a cylindrical pipe. To stay in the class of differentiable functions, use the following model of gate valve:

$$
\begin{aligned}
&R(x,t) = 1 - \frac{k_0 t}{T_k}\cos\left(\frac{\pi x}{2l}\right),\\
&L(x,t) = 2\pi R(x,t),\\
&A(x,t) = \pi (R(x,t))^2,
\end{aligned}
\tag{26}
$$

where the parameters k_0 and T_k correspond to the speed of closing the valve and full duration of closing the valve, and l denotes typical length of the valve. Formulas (26) are valid on the segment $x \in [-l, l]$, outside of this segment we assume $L(x,t) = 2\pi$ and $A(x,t) = \pi$. We also assume that the boundary nodes and their neighbors lie outside of the valve. Specify the grid boundary conditions in the following way:

$$
\begin{aligned}
&(\rho u)_x (X_1, t) = (\rho u)_{\bar{x}} (X_2, t) = 0,\\
&\rho (X_1, t) = r_1 + F(u(X_1, t)),\ \rho(X_2, t) = r_2,
\end{aligned}
\tag{27}
$$

where $r_1 > r_2$ are some constants, and define the function $F(u)$ corresponding to the passport pump performance as

$$
F(u) = C_0 - C_1 u^2,\ \text{where } C_0,\ C_1 > 0. \tag{28}
$$

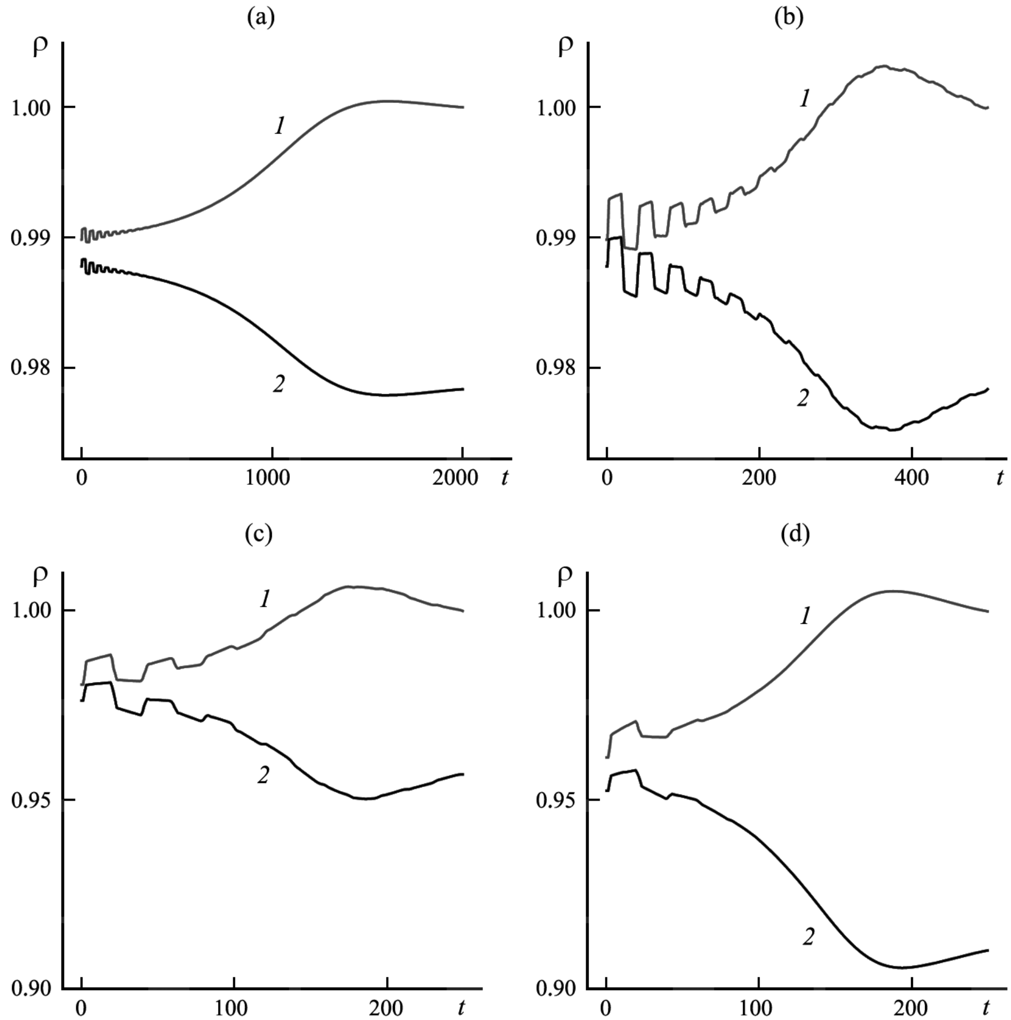
The initial conditions and the value r_2 can be determined by solving a stationary Cauchy problem, i.e., by assuming $\frac{\partial(A\rho u)}{\partial t} = 0$ and $\frac{\partial(A\rho)}{\partial t} = 0$. In this case the equations take the form

$$
\begin{aligned}
&\frac{d(\rho u)}{dx} = 0,\\
&\frac{d(\rho u^2)}{dx} + \kappa \frac{d\rho}{dx} + 2\lambda u|u| = 0,\\
&\rho(0) = r_1,\ u(0) = u_0.
\end{aligned}
\tag{29}
$$

Thus, we solve system (11), (13), (26) with the boundary conditions determined by formulas (27), (28) and the initial conditions determined by the solution to Cauchy problem (29). To calculate the solution (ρ^{n+1}, u^{n+1}) to the problem on each next step in time, we may use the following internal iterative process over k:

$$
\begin{aligned}
&\frac{A^{n+1}\rho^{k+1} - A^n\rho^n}{\tau} + \left(\{\rho^{k+1}\}A^{n+1}u^k\right)_x = 0,\\
&\frac{A^{n+1}\rho^{k+1}u^{k+1} - A^n\rho^n u^n}{\tau} + \frac{1}{2}\left(\{\rho^{k+1}\}A^{n+1}u^{k+1}\left(u^k_{(-1)} + u^k\right)\right)_x\\
&\qquad + \kappa\frac{1}{h}A^{n+1}\{\rho^{k+1}\}\ln\frac{\rho^{k+1}}{\rho^{k+1}_{(-1)}} + \lambda u^{k+1}|u^k|L^{n+1} = 0,\\
&\rho_0^{k+1} = r_1 + F(u_0^k),\ \rho_N^{k+1} = r_2,\\
&\rho_0^{k+1}u_0^{k+1} - \rho_1^{k+1}u_1^{k+1} = 0,\ \rho_N^{k+1}u_N^{k+1} - \rho_{N-1}^{k+1}u_{N-1}^{k+1} = 0.
\end{aligned}
$$

The figure presents the results of calculations in the domain $[-10, 10]$ on a grid of 500 nodes, step in time equals 10^{-3}. The graphs represent the dependence of the density on time in front of the valve (at the 225th node, curve *1*) and immediately after the valve (at the 275th node, curve *2*) for different values of the parameters k_0 and λ. In all the calculations we used the same values of $T_k = 100$, $l = 1$, $r_1 = 1$, $\kappa = 1$, $C_0 = 0$, and $C_1 = 0,1$.



Dependence of the density on time in front of the valve and after it for the following values of the parameters k_0 and λ: a — $k_0 = 0,05$; $\lambda = 0,05$; b — $0,2$; $0,05$; c — $0,4$; $0,1$; d — $0,4$; $0,2$.

According to Figure a, b, shock waves become stronger under increasing speed of valve closing. The graphs in Figure c, d correspond to different values of the friction coefficient λ. It is seen that shock waves fade faster under increasing coefficient of friction. Thus, the dependences presented here show that the calculated pressure behavior before and behind the valve agrees qualitatively well with the real one.

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