ISSN 0027-1322, Moscow University Mathematics Bulletin, 2017, Vol. 72, No. 3, pp. 121–128. © Allerton Press, Inc., 2017.
Original Russian Text © D.S. Timonina, 2017, published in Vestnik Moskovskogo Universiteta, Matematika. Mekhanika, 2017, Vol. 72, No. 3, pp. 35–43.

Liouville Classification of Integrable Geodesic Flows on a Torus of Revolution in a Potential Field

D. S. Timonina

Moscow State University, Faculty of Mechanics and Mathematics, Leninskie Gory, Moscow, 119991 Russia, e-mail: darik93@yandex.ru.

Received November 9, 2016

Abstract—A Liouville classification of integrable Hamiltonian systems being geodesic flows on a two-dimensional torus of revolution in an invariant potential field is obtained in the case of linear integral. This classification is obtained using the Fomenko–Zieschang invariant (so called marked molecules) of the systems under consideration. All types of bifurcation curves are described. A classification of singularities of the system solutions is also obtained.

DOI: 10.3103/S0027132217030056

1. INTRODUCTION

Many papers are focused on the theory of Liouville classification of integrable Hamiltonian systems developed by A. T. Fomenko and his school, see [1]. The essence of Fomenko's theory is that each integrable system with two degrees of freedom restricted to a three-dimensional non-singular compact constant-energy manifold is associated with some invariant being a graph with numeric marks at edges. By means of this invariant called a marked molecule or a Fomenko–Zieschang invariant one can obtain a complete description of Liouville foliation of a given system on constant-energy surfaces (up to fiberwise equivalence).

E. N. Selivanova obtained the Liouville classification of integrable geodesic flows on a torus (see [1, v. 2, §3.1]), Nguyen Tien Zung, L. S. Polyakova, and V. S. Matveev classified integrable geodesic flows on a two-dimensional sphere (see [1, v. 2, §3.3]). E. O. Kantonistova (see [2]) continued this study and constructed a complete classification of integrable geodesic flows in a potential field on manifolds of revolution in the case of linear integral.

In this paper we generalize the theorem of E. N. Selivanova. Namely, a Liouville classification of integrable geodesic flows in a potential field on a torus is obtained in the case of linear integral. This classification is based on calculation of Fomenko–Zieschang invariants (marked molecules) of the systems considered here (see details in Theorems 2 and 3).

The fundamentals of the topological classification theory of integrable Hamiltonian systems and its applications to the study of mechanical systems were expounded in [3–14].

2. FORMULATION OF THE PROBLEM

Describe the problem studied and solved in this paper.

Definition 1. A manifold M with a given metric g is called a *manifold of revolution* or a *manifold with a metric of revolution* if an effective action of the circle S^1 on M by isometries is defined.

Consider a Riemannian manifold M diffeomorphic to a two-dimensional torus ($M = T^2$) endowed with standard angular coordinates $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and a metric g of the following form:

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + f^2(\theta)\mathrm{d}\varphi^2, \tag{1}$$

where $f(\theta) : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$ is a smooth positive function.

Remark. The manifold $M = T^2$ endowed with metric (1) is a manifold of revolution.

Consider a natural mechanical system on the cotangent bundle T^*M of M endowed with the standard symplectic structure $\omega = \mathrm{dp} \wedge \mathrm{dq}$ and a Hamiltonian function

$$H = \frac{1}{2}g^{ij}(\mathrm{q})p_i p_j + V(\mathrm{q}), \tag{2}$$

where q $= (q^1, q^2)$ are local coordinates on $M = T^2$, p $= (p^1, p^2)$ are the corresponding impulses. i.e., coordinates in $T^*_q M$, and g^{ij} denote the components of the matrix inverse to the matrix g of the metric.

Assertion 1. *The Hamiltonian system with Hamiltonian function* (2) *on a manifold of revolution $M = T^2$ is completely integrable in the Liouville sense for any pair of functions $(f(\theta), V(\theta))$.*

Proof. The system has two first integrals, the energy integral

$$H = \frac{p_\theta^2}{2} + \frac{p_\varphi^2}{2f^2(\theta)} + V(\theta)$$

and the additional integral $K = p_\varphi$ (because $\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0$). Assertion 1 is proved.

The aim of the paper is to classify considered systems up to Liouville equivalence, in particular, to describe all possible types of bifurcation curves, to construct rough molecules, and to calculate numeric marks on them.

3. RESULTS OF THE PAPER

3.1. The bifurcation diagram of momentum mapping.

Assertion 2. *A natural mechanical system on the manifold $M = T^2$ such that $(f'(\theta))^2 + (V'(\theta))^2 \neq 0$ has critical points $(p_\theta, p_\varphi, \theta, \varphi) = (0, k(\theta), \theta, \varphi)$ of rank* 1*, where $k(\theta) := \pm f(\theta)\sqrt{\frac{f(\theta)V'(\theta)}{f'(\theta)}}$, $\theta \in I \cup \{\theta : V'(\theta) = 0\}$, $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, where I is an open subset of $\mathbb{R}/2\pi\mathbb{Z}$ defined by the inequality $V'(\theta)f'(\theta) > 0$ (the set I is a finite or countable set of pairwise non-intersecting intervals $I_i = (\theta_1^i, \theta_2^i)$). At each end of each interval I_i the equality $V'(\theta)f'(\theta) = 0$ holds.*

The system has no points of rank 0*.*

The bifurcation diagram is symmetric with respect to the axis h. It consists of the curves that are images of rank 1 *singular points' family under the momentum mapping and permitting a parameterization of the form*

$$h(\theta) := \frac{f(\theta)V'(\theta)}{2f'(\theta)} + V(\theta), \qquad k(\theta) := \pm f(\theta)\sqrt{\frac{f(\theta)V'(\theta)}{f'(\theta)}},$$

where $\theta \in I \cup \{\theta : V'(\theta) = 0\}$. Each such parameterized curve consists of a finite or a countable number of arcs, and the $(2i-1)$th and $(2i)$th arcs, $i = 1, 2, \ldots$ (these two arcs differ in the sign of $k(\theta)$ only) correspond to the interval I_i of variation of the parameter θ.

Proof. The explicit form of points of rank 1 follows from the condition of linear dependence for dH and dp_φ, i.e.,

$$adH + bdp_\varphi = 0,$$

where $a^2 + b^2 \neq 0$. Assertion 2 is proved.

Lemma 1. *For $k > 0$ the velocity vector at the points of the bifurcation diagram lies either in the first quadrant, or in the third quadrant of the coordinate plane (h, k), or it is equal to zero.*

Proof. The proof is similar to that of the corresponding lemma of Kantonistova [2] proved for the case of a sphere.

Definition 2. We say that a curve γ has a singularity of "non-degenerate cusp" type at a point θ^* if $\gamma'(\theta^*) = 0$ and the vectors $\gamma'''(\theta^*)$ and $v := \frac{\gamma''(\theta^*)}{|\gamma''(\theta^*)|}$ are non-collinear.

It is not difficult to verify that in this case $\lim_{\theta \to \theta^*_-} \frac{\gamma'(\theta^*)}{|\gamma'(\theta^*)|} = -v$, and $\lim_{\theta \to \theta^*_+} \frac{\gamma'(\theta^*)}{|\gamma'(\theta^*)|} = v$.

Now prove that all irregular (singular) points of bifurcation curves are non-degenerate cusps. To do that, recall the definition of a non-degenerate singular point. Let x be a rank 1 critical point of the momentum mapping. Let $\mathrm{d}K(x) \neq 0$. In this case, due to Darboux theorem, in a neighborhood of the point x there exists a canonical coordinate system (p_1, q_1, p_2, q_2), where $K = p_1$. Since K and H commute, then the function H does not depend on q_1, i.e., $H = H(p_1, p_2, q_2)$.

The point x is a critical point of the momentum mapping, therefore, $\frac{\partial H}{\partial p_2} = \frac{\partial H}{\partial q_2} = 0$.

Definition 3. Let H and K be the Hamiltonian function and the additional integral, respectively. A singular rank 1 point x of the momentum mapping such that $\mathrm{d}K \neq 0$ is called *non-degenerate* if in some symplectic coordinates the matrix

$$J := \begin{pmatrix} \frac{\partial^2 H}{\partial p_2 \partial p_2} & \frac{\partial^2 H}{\partial p_2 \partial q_2} \\ \frac{\partial^2 H}{\partial p_2 \partial q_2} & \frac{\partial^2 H}{\partial q_2 \partial q_2} \end{pmatrix}$$

is non-degenerate at the point x.

Apply Definition 3 to the case considered here. The system possesses a canonical coordinates $(p_\theta, p_\varphi, \theta, \varphi)$ of the form described above, therefore,

$$|J| = \det \begin{pmatrix} \frac{\partial^2 H}{\partial p_\theta \partial p_\theta} & \frac{\partial^2 H}{\partial p_\theta \partial \theta} \\ \frac{\partial^2 H}{\partial p_\theta \partial \theta} & \frac{\partial^2 H}{\partial \theta \partial \theta} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ 0 & \frac{\partial^2 W_{p_\varphi}}{\partial^2 \theta} \end{pmatrix} = \frac{\partial^2 W_{p_\varphi}}{\partial^2 \theta},$$

where $W_{p_\varphi}(\theta) := \frac{p_\varphi^2}{2f^2(\theta)} + V(\theta)$.

Lemma 2. *For any $\theta \in I_i$ the following conditions are equivalent:*

1) *a singular point $(0, k(\theta), \theta, \varphi)$ of rank 1 of the momentum mapping is non-degenerate;*

2) *the parameterized bifurcation curve is regular at the point θ;*

3) *the value $k'(\theta)$ is nonzero;*

4) *the point θ is a Morse type critical point of the function $W_{k(\theta)}(\theta)$, i.e., the value $W''_{k(\theta)}(\theta)$ is nonzero, where*

$$W''_{k(\theta)}(\theta) = \frac{\left(3f'(\theta)^2 - f(\theta)f''(\theta)\right) V'(\theta)}{f(\theta)f'(\theta)} + V''(\theta).$$

Proof. The implication $1 \Rightarrow 2$ is valid, see [1, v. 1, p. 48]).

The implication $3 \Leftrightarrow 4$ can be verified by direct calculations.

The implication $3 \Rightarrow 2$ is evident.

The implication $4 \Leftrightarrow 1$ follows from Definition 3.

The implication $2 \Rightarrow 3$ can be verified by direct calculations.

The lemma is proved.

Lemma 3. *For any interval I_i the following conditions are equivalent:*

1) *the function $k(\theta)$ is a Morse function on I_i;*

2) *at any critical point θ_0 of the function $k(\theta)$ on I_i the following conditions hold:*

$$\begin{cases} W'_{k(\theta_0)}(\theta_0) = 0, \\ W''_{k(\theta_0)}(\theta_0) = 0, \\ W'''_{k(\theta_0)}(\theta_0) \neq 0; \end{cases}$$

3) *any non-regular point θ of an arc of the bifurcation curve parameterized by the interval I_i is a non-degenerate cusp.*

Proof. The proof is similar to the proof of the corresponding lemma from Kantonistova in [2] proved for the case of a sphere.

We include the keystone of the equivalence of Conditions 2 and 3 proof only. In accordance with Definition 2, the point θ_0 is a non-degenerate cusp point of a parametric curve $(h(\theta), k(\theta))$ if and only if the following conditions hold:

$$h'(\theta_0) = 0,\ k'(\theta_0) = 0,\ h'''(\theta_0)k''(\theta_0) - h''(\theta_0)k'''(\theta_0) \neq 0.$$

We can verify that these conditions are equivalent to Conditions 3 by direct calculations.

Definition 4. A "parabola" type arc is said to be a continuous arc symmetric with respect to the axis h, having a single point belonging to the axis h, consisting of smooth sub-arcs separated by cusp points (no cusp points could be), and tending to plus infinity with respect to both coordinates in the upper half plane. The velocity vector of those sub-arcs lies either in the first quadrant, or in the third quadrant of the coordinate plane.

Definition 5. A "lune" type arc is said to be a closed continuous arc symmetric with respect to the axis h, having two points lying on the axis h, and consisting of smooth sub-arcs separated by cusp points. The velocity vector of those sub-arcs lies either in the first quadrant, or in the third quadrant of the coordinate plane.

Definition 6. A "beak" type arc is said to be a continuous arc consisting of smooth sub-arcs separated by cusp points and whose both ends tends to plus infinity with respect to both coordinates in the upper half plane. The velocity vector of those sub-arcs lies either in the first quadrant, or in the third quadrant of the coordinate plane.

Examples of arcs described in Definitions 4, 5, and 6, are shown in Figure 1, *a*, *d*; *b*, *e*; *c*, *f*, respectively.

Theorem 1. *Let a natural mechanical system on the manifold $M = T^2$ be given by a pair of functions $(f(\theta), V(\theta))$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Let the following conditions be valid:*

1) *$f(\theta)$, $V(\theta)$, and $k(\theta)$ are Morse functions,*

2) $(f'(\theta))^2 + (V'(\theta))^2 \neq 0$.

Let I be an open subset of $\mathbb{R}/2\pi\mathbb{Z}$ defined by the inequality $V'(\theta)f'(\theta) > 0$ and consisting of pairwise non-intersecting intervals $I_i = (\theta_1^i, \theta_2^i)$.

In this case the following statements are valid:

1) *a parameterized bifurcation arc is regular in a neighborhood of any end point of any interval $\left(\theta_1^i, \theta_2^i\right)$;*

2) *a parameterized bifurcation arc has no accumulation points of irregular points, and each its irregular point is a non-degenerate cusp point;*

3) *bifurcation arcs are smooth at the points $\theta = \theta_j^i$, where $V'(\theta) = 0$;*

4) *the bifurcation diagram consists of points and arcs of "parabola", "beak", and "lune" types, i.e.,*

a) if $f'\left(\theta_1^i\right) = f'\left(\theta_2^i\right) = 0$, then the arc parameterized by the interval I_i is of "beak" type;

b) if $f'\left(\theta_1^i\right) = V'\left(\theta_2^i\right) = 0$ (or $f'\left(\theta_2^i\right) = V'\left(\theta_1^i\right) = 0$), then the arc parameterized by the interval I_i is a half of "parabola" type arc;

c) if $V'\left(\theta_1^i\right) = V'\left(\theta_2^i\right) = 0$, then the arc parameterized by the interval I_i is a half of "lune" type.

Proof. The proof of Theorem 1 is similar to that of the corresponding theorem from Kantonistova [2] proved for the case of a sphere. Therefore, we include the proof for Items 1, and 4 only.

1) Since

$$k'(\theta) = \frac{V'(\theta)\left(3f^2(\theta)f'(\theta)^2 - f^3(\theta)f''(\theta)\right) + f^3(\theta)V''(\theta)f'(\theta)}{\left(f'(\theta)\right)^2},$$

then $k'(\theta)$ does not tend to zero for $\theta \to \theta_j^i$ (because at any end of any interval $\left(\theta_1^i, \theta_2^i\right)$ the relations $V'(\theta)f'(\theta) = 0$, $(f'(\theta))^2 + (V'(\theta))^2 \neq 0$ are valid).

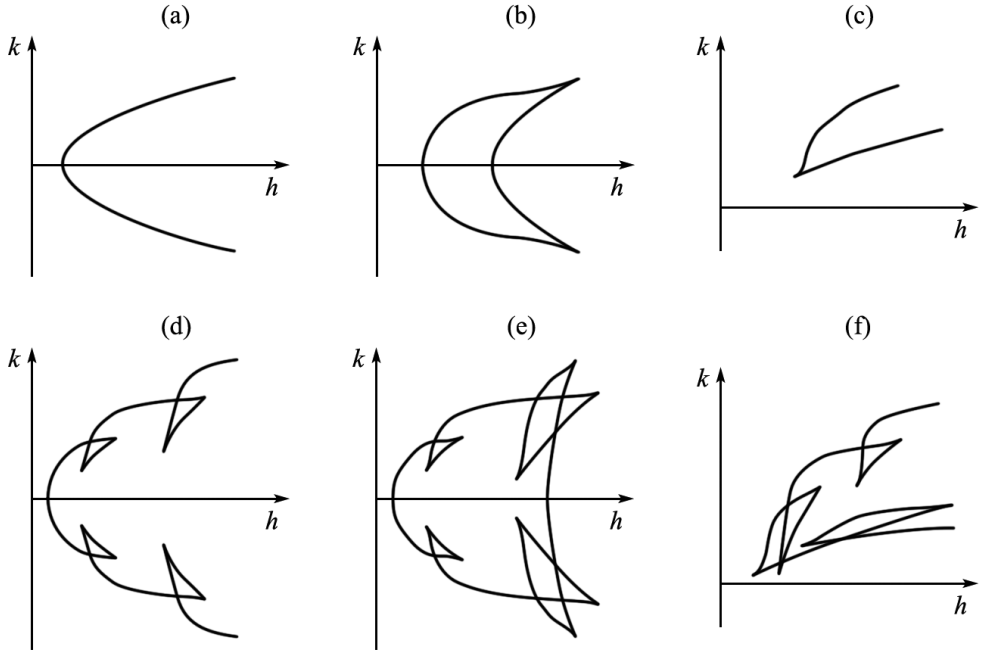


Fig 1.

4) Suppose the relation $f'\left(\theta_1^i\right) = f'\left(\theta_2^i\right) = 0$ holds at the end of the interval $\left(\theta_1^i, \theta_2^i\right)$. Then $h(\theta) \to \infty$ and $k(\theta) \to \infty$ for $\theta \to \theta_1^i\left(\theta_2^i\right)$ and hence the arc parameterized by the interval $\left(\theta_1^i, \theta_2^i\right)$ is of "beak" type.

Suppose the relation $V'\left(\theta_1^i\right) = V'\left(\theta_2^i\right) = 0$ holds at an end of the interval $\left(\theta_1^i, \theta_2^i\right)$. Then $h(\theta) \to V\left(\theta_1^i\right)\left(V\left(\theta_2^i\right)\right)$ and $k(\theta) \to 0$ for $\theta \to \theta_1^i\left(\theta_2^i\right)$, therefore, the arc parameterized by the interval $\left(\theta_1^i, \theta_2^i\right)$ is a half of "lune."

Suppose the relation $f'\left(\theta_1^i\right) = V'\left(\theta_2^i\right) = 0$ holds at the end of the interval $\left(\theta_1^i, \theta_2^i\right)$. Then $h(\theta) \to \infty$ and $k(\theta) \to \infty$ for $\theta \to \theta_1^i$, and $h(\theta) \to V\left(\theta_2^i\right)$ and $k(\theta) \to 0$ for $\theta \to \theta_2^i$, therefore, the arc parameterised by the interval $\left(\theta_1^i, \theta_2^i\right)$ is a half of "parabola." (Similarly, for $f'\left(\theta_2^i\right) = V'\left(\theta_1^i\right) = 0$.)

3.2. Rough molecule construction.

Construct the rough molecule for the system under consideration.

Assertion 3. *A constant energy surface Q_h^3 is non-singular if and only if $h \neq V(\theta_i)$, where $V'(\theta_i) = 0$.*

Lemma 4. *The function K is a Bott function on a non-singular constant energy surface Q^3_h if and only if the function $U_h(\theta) := 2f^2(\theta)\,(h - V(\theta))$ called an effective potential is a Morse function.*

Remark. If Q^3_h is a non-singular constant energy surface, then K is a Bott function on it if and only if the straight line $h = \text{const}$ does not pass through cusp points on the bifurcation diagram.

List definitions of some 3-atoms.

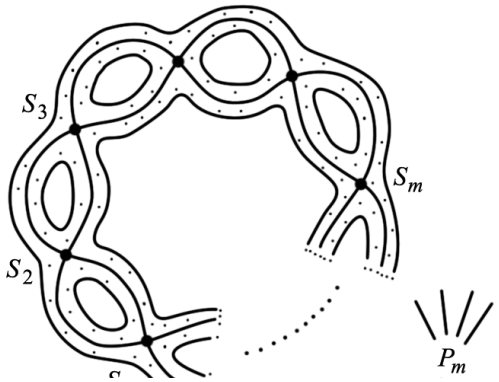
1. The 3-atom A. Topologically, this 3-atom is a solid torus foliated into concentric tori shrinking to the axis of the solid torus.

2. The 3-atom V_s ($s \geq 2$, $V_2 = B$). This 3-atom is the direct product of the 2-atom shown in Figure 2 and the circle S^1. The Liouville foliation is naturally induced by the 2-atom.

Fig 2.

3. The 3-atom P_m ($m \geq 1, P_1 = B$). This 3-atom is the direct product of the 2-atom shown in Figure 3 and the circle S^1. The Liouville foliation is naturally induced by the 2-atom.

Lemma 5. *Let $Q \subseteq Q^3_h$ be a connected component of a non-singular constant energy surface such that the integral K is a Bott function on it. Then the rough molecule corresponding to Q can contain atoms of the following three types only: V_s, ($s \geq 2$), P_m, ($m \geq 1$), and A. In the case $K > 0$ each atom V_s has a single incoming edge and s outgoing edges, each atom P_m has two incoming edges and m outgoing edges (for $K < 0$ the situation is antisymmetric, i.e., each atom V_s has s incoming edges and a single outgoing one, and the atom P_m has m incoming edges and two outgoing ones). The molecule itself is symmetric (not taking the orientation into consideration).*



Fix the values of both the integrals $H = h$, $K = k$. We get the points $(p_\theta, k, \theta, \varphi)$, $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$ in T^*M satisfying the following equation:

$$\frac{p_\theta^2}{2} + \frac{k^2}{2f^2(\theta)} + V(\theta) = h.$$

Rewrite this equation in the form

$$p_\theta = \pm\frac{1}{f(\theta)}\sqrt{U_h(\theta) - k^2},$$

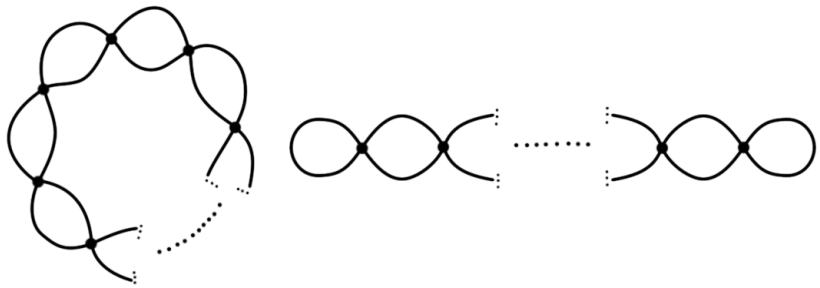


Fig 4.

where the periodic function $U_h(\theta) := 2f^2(\theta)\,(h - V(\theta))$ with the period 2π is called the *effective potential*.

On each segment $[\theta_1; \theta_2]$ such that $U_h(\theta) - k^2 \geq 0$ the set $p_\theta(\theta)$ is either a point, or a circle, or an open chain of circles, or a closed chain of circles (Figure 4). Changing the value k, we see that neighborhoods of those singular fibers correspond to 2-atoms shown in Figures 2 and 3. Therefore, we conclude that the bifurcation arcs correspond either to the atom A, or to the atom V_s (where $V_2 = B$), or to the atom P_m (where $P_1 = B$, $P_2 = C_2$), and the momentum mapping preimage of a point that does not belong to the bifurcation arcs is either a single Liouville torus, or several Liouville tori. In the case when the expression $U_h(\theta) - k^2$ is positive for any θ the pre-image of the point (h, k) is a union of two tori.

Thus, changing continuously the value k of the second first integral, we can use the following algorithm to construct the rough molecule from the graph of the function $U_h(\theta)$.

1. Take the graph of the function $U_h(\theta)$ and construct horizontal lines corresponding to different values of k^2. The segments of these lines lying strictly "lower" the graph of the function $U_h(\theta)$ correspond to Liouville tori (each segment corresponds to one torus except for the case when the graph of the function $U_h(\theta)$ is positive on the whole domain. Then the segments located lower the global minimum of the function $U_h(\theta)$ correspond to two Liouville tori).

2. If the function $U_h(\theta)$ is positive on the whole domain, then the segment passing through the global minimum of $U_h(\theta)$ corresponds to the atom P_m. Otherwise there are no atoms P_m $(m > 1)$ in the molecule. If the function $U_h(\theta)$ takes negative values, then Q^3_h could have several connected components, otherwise Q^3_h is a connected manifold.

3. Consider a single connected component Q^3_h. Each segment passing through a local minimum (minima) of the function $U_h(\theta)$ corresponds to the atom V_l, where $(l-1)$ is the number of local minima passed by the segment.

4. Each local maximum of the function $U_h(\theta)$ corresponds to the atom A.

Therefore, we construct this way a half of the rough molecule of the connected component of Q^3_h. The second half is symmetric. The lemma is proved.

Lemma 6. *Let $Q \subseteq Q^3_h$ be a connected component of a non-singular constant energy surface such that K is a Bott function on it. In this case the value $K = 0$ cannot correspond to a bifurcation of Liouville tori (i.e., 0 is not a critical value of the function K).*

Theorem 2. *Let a system be given on the manifold of rotation $M = T^2$ by a pair of functions $(V(\theta), f(\theta))$. Let $Q \subseteq Q^3_h$ be a connected component of a non-singular constant energy surface such that K is a Bott function on it. In this case the molecule of the system is symmetric with respect to the axis h (not taking into account the orientations of edges), and the edges are oriented in accending order of k. The molecule has the following form depending on the graph of the effective potential:*

1) *if the function $U_h(\theta)$ is positive on the whole domain, then $W \equiv P_m = P_m \equiv W$, where "$\equiv$" stands for m Liouville tori. The number m is equal to the number of global minima of the function $U_h(\theta)$. In this case Q^3_h has only one connected component*;

2) *if the function $U_h(\theta)$ takes negative values, then the molecule corresponding to Q has the form $W - W$. In this case the number of connected components of the manifold Q^3_h is equal to the half of the number of intersections of the graph of the function $U_h(\theta)$ with the axis $U_h(\theta) = 0$ (i.e., each segment $[\theta_i, \theta_j]$ such that $U_h(\theta) \geq 0$ corresponds to a single connected component of Q^3_h).*

Each molecule W is either a single atom A, or a tree, or a forest. All non-ending vertices of the tree are atoms V_l, and the ending vertices correspond to atoms A. In the case $K > 0$ each atom V_l has a single incoming edge and l outgoing edges (and for $K < 0$ the situation is antisymmetric, i.e., not taking into account the orientations on the edges, the molecule is symmetric with respect to the level $K = 0$, and the orientations on the pieces $W_+ = W(K > 0)$ and $W_- = W(K < 0)$ are opposite).

The rough molecule is constructed by the graph of the function $U_h(\theta)$ in accordance with the algorithm described above.

Proof. Assertions of the theorem follow from Lemmas 5, and 6.

Lemma 7. *Consider the half-plane $k > 0$. In a neighbourhood of a cusp point, a regular fragment of the bifurcation curve consisting of points corresponding to the atom A lies above the regular fragment of the bifurcation curve consisting of points corresponding to the atoms V_k and P_m. In the half-plane $k < 0$ the situation is antisymmetric, i.e., in a neighbourhood of a cusp point a regular fragment of the bifurcation curve consisting of points corresponding to the atoms A lies below a regular fragment of the bifurcation curve consisting of points corresponding to the atoms V_k and P_m.*

Proof. The assertion of the lemma follows from the fact that the local minimum of the effective potential (it can correspond to atoms V_k and P_m) lies below the neighbouring local maximum (corresponding to the atom A).

Assertion 4. *Let Q be a connected component of a non-singular constant energy surface Q^3_h. Cut it into two parts along the surface $K = 0$. It falls into two connected pieces. If $h < \max V(\theta)$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, then each component is a solid torus and for $h > \max V(\theta)$ these components are $T^2 \times D^1$.*

Proof. Cut all Q^3_h along the surface $K = 0$. Consider the part of Q^3_h where $p_\varphi \geq 0$. It is given by the system

$$\begin{cases} \frac{p_\theta^2}{2} + \frac{p_\varphi^2}{2f^2(\theta)} + V(\theta) = h, \\ p_\varphi \geq 0. \end{cases}$$

Transform it into the form

$$\begin{cases} p_\varphi^2 = f^2(\theta)\left(2h - 2V(\theta) - p_\theta^2\right), \\ p_\varphi \geq 0, \end{cases}$$

and get the required assertion.

3.3. Calculation of marks.

A rough molecule contains a lot of information on the structure of Liouville foliation, however, this information is incomplete in the sense that the molecule does not define the diffeomorphic type of the manifold Q^3_h and the Liouville structure on it (up to fiberwise diffeomorphism). Therefore, it is necessary to add an information on glueing of individual atoms contained in numeric marks (see details in [1]).

Theorem 3. *Let Q^3 be a connected component of a non-singular constant energy surface Q^3_h. In this case the molecule corresponding to Q^3 satisfies the following properties.*

1) *The marks on the molecule's edges of type $A - V_l$ have the form $r = 0$, $\varepsilon = 1$.*

2) *The marks on the molecule's edges of type $A - P_k$ have the form $r = 0$, $\varepsilon = 1$.*

3) *The marks on the molecule's edges of type $V_k - V_l$ have the form $r = \infty$, $\varepsilon = 1$, where both saddle atoms are located in a single half-plane (either $k > 0$, or $k < 0$).*

4) *The marks on the molecule's edges of type $P_k - V_l$ have the form $r = \infty$, $\varepsilon = 1$.*

5) *If the rough molecule has the form $A - A$, then the marks on the edge have the form $r = \infty$, $\varepsilon = +1$.*

6) *The marks on the central edge of the molecule (it is either $P_m - P_m$, or $V_s - V_s$ depending on the form of the rough molecule) have the form $r = \infty$, $\varepsilon = -1$.*

7) *If the molecule is distinct from the molecule $A-A$, then it contains a single family that can be obtained by discarding all atoms A. The mark n is equal to zero in this case.*

Proof. To calculate the marks, we have to chose admissible coordinate systems on boundary tori of the atoms. Recall the rules of their choice. Let a 3-atom be of type A, i.e., a solid torus. In this case for the first basic cycle λ we take the meridian of the solid torus, i.e., the cycle contracting to a point in the solid torus. In our case we have

$$\left\{p_\theta = \pm\frac{1}{f(\theta)}\sqrt{U_h(\theta) - k^2}, p_\varphi = \text{const}, \theta \in [\theta_1, \theta_2], \varphi = \text{const}\right\}.$$

For the second cycle we can take any cycle μ completing λ to a base. Take the fiber given as $\{p_\theta = \text{const}$, $p_\varphi = \text{const}$, $\theta = \text{const}$, $\varphi \in \mathbb{R}\text{mod}2\pi\mathbb{Z}\}$. It is a fiber of Seifert fibration (its orientation is naturally defined by the orientation of the Hamiltonian vector flow on the critical circle).

Let the 3-atom be a saddle one (V_l or P_m). It has the structure of trivial S^1-fibration over a 2-atom. Take a fiber of this fibration as the first basic cycle λ. In our case we have

$$\{p_\theta = \text{const}, p_\varphi = \text{const}, \theta = \text{const}, \varphi \in \mathbb{R}\text{mod}2\pi\mathbb{Z}\}.$$

Now consider a cross-section of the 3-atom over which this atom is represented as a trivial S^1-fibration. It intersects each boundary torus by some cycle that can be taken as the second basic cycle μ. In our case it is either the cycle

$$\left\{p_\theta = \pm\frac{1}{f(\theta)}\sqrt{U_h(\theta) - k^2}, p_\varphi = \text{const}, \theta \in [\theta_1, \theta_2], \varphi = \text{const}\right\},$$

or the cycle

$$\{p_\theta = \text{const}, p_\varphi = \text{const}, \theta \in \mathbb{R}\text{mod}2\pi\mathbb{Z}, \varphi = \text{const}\}$$

depending on the atom (V_l or P_m) and the boundary torus.

Notice that $\dot\varphi = f^2(\theta)p_\varphi$. Therefore, if two atoms lie above the level $k = 0$ simultaneously (or below this level simultaneously), then the derivative of the angle coordinate φ along the vector field $\text{sgrad}H$ has the same sign and hence the orientations of the cycles

$$\{p_\theta = \text{const}, p_\varphi = \text{const}, \theta = \text{const}, \varphi \in \mathbb{R}\text{mod}2\pi\mathbb{Z}\}$$

of these atoms are the same. Otherwise, these cycles have opposite orientations.

Write down the gluing matrix at the edge $A-V_l$. Since the second basic cycle around the atom A coincides with the first basic cycle around the atom V_l and these atoms lie in the same half-plane ($k > 0$ or $k < 0$), then the gluing matrix has the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, $r = 0$, $\varepsilon = 1$.

The remaining gluing matrices can be obtained similarly.

If a molecule $W - W$ does not equal $A - A$, then the molecule has a unique family obtained by discarding all atoms A. The mark n in this case is equal to zero because all θ_i (from the definition of the mark n) turn out to be equal to zero. The theorem is proved.

Corollary. *If each rough molecule corresponding to Q^3 has the form $A - A$, then $Q^3 = S^1 \times S^2$.*

ACKNOWLEDGMENTS

The author is grateful to A. T. Fomenko for formulation of the problem an to A. A. Oshemkov and E. O. Kudryavtseva for many fruitful discussions.

REFERENCES

1. A. V. Bolsinov and A. T. Fomenko, *Integrable Hamiltonian Systems* Vol. 1, 2 (Udmurt. Univer., Izhevsk, 1999; Chapman and Hall, CRC Press, Boca Raton, L., N.Y., Washington, 2004).
2. E. O. Kantonistova, "Topological Classification of Integrable Hamiltonian Systems in a Potential Field on Surfaces of Revolution," Matem. Sbornik **207** (3), 47 (2016) [Sbornik: Math. **207** (3), 358 (2016)].
3. A. T. Fomenko, "The Symplectic Topology of Completely Integrable Hamiltonian Systems," Uspekhi Matem. Nauk **44** (1), 145 (1989) [Russ. Math. Surv. **44** (1), 181 (1989].
4. A. V. Bolsinov and A. T. Fomenko, "Orbital Classification of Geodesic Flows on Two-Dimensional Ellipsoids. The Jacobi Problem is Orbitally Equivalent to the Integrable Euler Case in Rigid Body Dynamics," Funk. Anal. Prilozh. **29** (3), 1 (1995) [Funct. Anal. Appl. **29** (3) 149 (1995)].
5. A. V. Bolsinov and A. T. Fomenko, *Integrable Hamiltonian systems. Geometry, Topology, Classification* (Chapman and Hall, CRC Press, Boca Raton, L., N.Y., Washington, 2004).
6. A. T. Fomenko, "A Bordism Theory for Integrable Nondegenerate Hamiltonian Systems with Two Degrees of Freedom. A New Topological Invariant of Higher-Dimensional Integrable Systems," Izvestiya Akad. Nauk SSSR, Ser. Matem. **55** (4), 747 (1991) [Math. of the USSR–Izvestiya **39** (1), 731 (1992)].
7. A. T. Fomenko, "A Topological Invariant which Roughly Classifies Integrable Strictly Nondegenerate Hamiltonians on Four-Dimensional Symplectic Manifolds," Funk. Anal. Prolozh. **25** (4) 23 (1991) [Func. Anal. Appl. **25** (4), 262 (1991)].
8. A. T. Fomenko and P. V. Morozov, "Some New Results in Topological Classification of Integrable Systems in Rigid Body Dynamics," in *Proc. Workshop "Contemporary Geometry and Related Topics," Belgrade, Yugoslavia, 15–21 May 2002* (World Sci. Publ. Co., Belgrade, 2004), pp. 201–222.
9. A. V. Bolsinov, A. T. Fomenko, and A. A. Oshemkov, *Topological Methods in the Theory of Integrable Hamiltonian Systems* (Cambridge Sci. Publ., Cambridge, 2006).
10. E. A. Kudryavtseva, I. M. Nikonov, and A. T. Fomenko, "Maximally Symmetric Cell Decompositions of Surfaces and their Coverings," Matem. Sbornik **199** (9), 3 (2008) [Sbornik: Math. **199** (9), 1263 (2008)].
11. E. A. Kudryavtseva and A. T. Fomenko, "Symmetries Groups of Nice Morse Functions on Surfaces," Doklady Russ. Akad. Nauk, Ser. Matem. **446** (6), 615 (2012) [Doklady Math. **86** (2), 691 (2012)].
12. A. Yu. Konyaev and A. T. Fomenko, "Algebra and Geometry through Hamiltonian Systems," in *Continuous and Disturbed Systems. Theory and Applications*, Vol. 211. Ed. by V. Z. Zgurovsky and V. A. Sadovnichii (Springer Int. Publ., Berlin, N.Y., Zurich, 2014), pp. 3–21.
13. V. V. Fokicheva and A. T. Fomenko, "Integrable Billiards Model Important Integrable Cases of Rigid Body Dynamics," Doklady Russ. Akad. Nauk, Ser. Matem. **465** (2), 150 (2015) [Doklady Math. **92** (3), 682 (2015)].
14. V. V. Fokicheva and A. T. Fomenko, "Billiard Systems as the Models for the Rigid Body Dynamics," in *Studies in Systems, Decision and Control; Advances in Dynamical Systems and Control*, Vol. 69, Ed. by V. A. Sadovnichii and V. Z. Zgurovsky (Springer Int. Publ., Kiev, 2016), pp. 13–32.

Translated by A. Ivanov