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BRIEF COMMUNICATIONS

Common Fixed Points of a Family of Commuting Mappings of Partially Ordered Sets.

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Abstract—The paper presents conditions providing the existence of a common fixed point of a family of commuting isotone multivalued mappings of a partially ordered set and the existence of the minimal element in the set of common fixed points. Additional conditions that guarantee the existence of the least element in that point set are also presented. Relations of the obtained results to well-known fixed point theorems are discussed.

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In this paper we consider existence issues for a common fixed point of a family of single-valued and multivalued commuting mappings of partially ordered sets. In the category of metric spaces the existence of a fixed point is as important as an estimate of a distance from a given element to some fixed point. Similarly, in the category of ordered sets the existence of the minimal element and the least element in the set of fixed points is of interest. The results on existence of fixed points for mappings of partially ordered sets have many applications. Among recent publications, let us mention paper [1] where an application of the Knaster–Tarski theorem in the computational geometry was demonstrated. In [2, Ch. 18], a relation between problems concerning fixed points of mappings of metric spaces and the corresponding results for mappings of partially ordered sets were presented. In particular, it was shown that the well-known Nadler theorem [3] can be derived from Smithson's theorem [4]. In collaboration with T. N. Fomenko, the author also studied the issues related to common fixed points and coincidence of non-commuting families of mappings of partly ordered sets.

In this paper we use standard definitions of a linearly ordered set, a partially ordered set, a chain, and a minimal element that can be found, for example, in [5, 6].

Introduce necessary definitions. Let we be given with a non-empty set A and a partially ordered set $(X, \preceq)$. In what follows, by $\rightrightarrows$ we always denote a multivalued mapping taking each element $x \in X$ to a non-empty subset $F(x) \subseteq X$. Besides, suppose a family of multivalued mappings is given, $\mathcal{G} = \{G_\alpha\}_{\alpha\in A}$, where $G_\alpha : X \rightrightarrows X$, $\alpha \in A$. For an arbitrary subset $Z \subseteq X$ and an index $\alpha \in A$ define the set $G_\alpha(Z) = \bigcup\limits_{z\in Z} G_\alpha(z)$. A neighborhood of a point $x_0 \in X$ in the set X is defined as a set $\mathcal{O}_X(x_0) = \{x \in X | x \preceq x_0\}$. For the intersection of neighborhoods of elements $x_1, x_2 \in X$ we use the standard notation $\Omega_X(x_1, x_2) = \mathcal{O}_X(x_1) \cap \mathcal{O}_X(x_2)$. An element $y \in Y$ such that $y \preceq y'$ for all $y' \in Y$ is called the least element of the subset $Y \subseteq X$.

Definition 1. A family of mappings $\mathcal{G} = \{G_\alpha\}_{\alpha\in A}$ is said to be *commuting* if for any pair of indices $\alpha, \beta \in A$ and any element $x \in X$ the equality $G_\alpha(G_\beta(x)) = G_\beta(G_\alpha(x))$ holds.

Definition 2. A multivalued mapping $F : X \rightrightarrows X$ is said to be *isotone* if for any $x_1, x_2 \in X$, $x_1 \preceq x_2$ and any $y_2 \in F(x_2)$ there exists an element $y_1 \in F(x_1)$ such that $y_1 \preceq y_2$. In particular, a single valued mapping $f : X \to X$ is said to be isotone if the relation $x_1 \preceq x_2$ implies $f(x_1) \preceq f(x_2)$.

Definition 3. The set $\{y_\alpha\}_{\alpha\in A} \subseteq X$ is called the *set of $\mathcal{G}$-values* at a point $x \in X$ if $y_\alpha \in G_\alpha(x)$, $\alpha \in A$.

Fix some element $x_0 \in X$. By $\mathcal{S}(x_0; \mathcal{G})$ we denote the set of elements $x \in \mathcal{O}_X(x_0)$ such that there exists a set $\{y_\alpha\}_{\alpha\in A}$ of $\mathcal{G}$-values at the point x such that $y_\alpha \preceq x$, $\alpha \in A$. In the case when $\mathcal{G}$ is a family of single-valued mappings we have $\mathcal{S}(x_0; \mathcal{G}) = \{x \in \mathcal{O}_X(x_0) | G_\alpha(x) \preceq x, \forall\alpha \in A\}$.

By $\mathrm{Comfix}(\mathcal{G}) = \{x \in X | x \in \bigcap\limits_{\alpha\in A} G_\alpha(x)\}$ we denote the set of common fixes points of a family of multivalued mappings $\mathcal{G}$. In the case when the family $\mathcal{G}$ consists of single-valued mappings we have $\mathrm{Comfix}(\mathcal{G}) = \{x \in X | x = G_\alpha(x) \text{ for all } \alpha \in A\}$.

Theorem 1. *Let we be given with an ordered set $(X, \preceq)$, a commuting family of single-valued isotone mappings $\mathcal{G} = \{G_\alpha\}_{\alpha\in A}$, where $G_\alpha : X \to X, \alpha \in A$, and a point $x_0 \in X$ such that $G_\alpha(x_0) \preceq x_0$ for all $\alpha \in A$. In addition, suppose for any chain $S \subseteq \mathcal{S}(x_0; \mathcal{G})$ there exists its lower bound $u \in X$ such that $G_\alpha(u) \preceq u$ for all $\alpha \in A$.*

In this case the set $\mathrm{Comfix}(\mathcal{G})$ *of common fixed points is not empty and contains a minimal element.*

Proof. To start with, note that the set $\mathcal{S}(x_0;\mathcal{G})$ is not empty. Indeed, in accordance with the assumptions for the element x_0, the inequality $G_\alpha(x_0) \preceq x_0$ holds for all $\alpha \in A$ and hence $x_0 \in \mathcal{S}(x_0;\mathcal{G})$. Consider the set $\mathcal{C}(x_0;\mathcal{G})$ consisting of all possible chains $S \subseteq \mathcal{S}(x_0;\mathcal{G})$. Order the set $\mathcal{C}(x_0;\mathcal{G})$ with respect to inclusion. By $\preceq_*$ we denote the ordering just defined. Due to the Hausdorff maximum principle, the ordered set $(\mathcal{C}(x_0;\mathcal{G}), \preceq_*)$ contains the maximal element S^*. In accordance with the assumptions of the theorem, there exists a lower bound $\xi \in X$ of the chain S^* such that $G_\alpha(\xi) \preceq \xi, \alpha \in A$.

Show that $G_\alpha(\xi) = \xi$ for all $\alpha \in A$. Suppose there exists an index $\beta \in A$ such that $G_\beta(\xi) \prec \xi$. Since the mapping G_β is isotone and $G_\alpha(\xi) \preceq \xi$, then we have $G_\beta(G_\alpha(\xi)) \preceq G_\beta(\xi)$ for all $\alpha \in A$. Taking into account the commutativity property of the family $\mathcal{G}$, we get the equality $G_\beta(G_\alpha(\xi)) = G_\alpha(G_\beta(\xi))$ and hence $G_\alpha(G_\beta(\xi)) \preceq G_\beta(\xi)$ for all $\alpha \in A$. Besides, $G_\beta(\xi) \prec x$ for each $x \in S^*$ because of $G_\beta(\xi) \prec \xi$ and the element ξ is a lower bound of the chain S^*. Thus, $S^* \cup \{\xi\} \in \mathcal{C}(x_0;\mathcal{G})$ and $S^* \prec_* S^* \cup \{\xi\}$, which contradicts the maximality of the chain S^*. As the result, we conclude that $G_\alpha(\xi) = \xi, \alpha \in A$, i.e., $\xi \in \mathrm{Comfix}(\mathcal{G})$.

It remains to show that ξ is the minimal element in the set $\mathrm{Comfix}(\mathcal{G})$. Suppose the contrary, i.e., there exists an element $\eta \in \mathrm{Comfix}(\mathcal{G})$ such that $\eta \prec \xi$. Since the equality $G_\alpha(\eta) = \eta$ is valid for all $\alpha \in A$ and $\eta \prec \xi$, then the element η is a lower bound of the chain S^* that is not contained in S^*. As the result, $S^* \cup \{\eta\} \in \mathcal{C}(x_0;\mathcal{G})$ and $S^* \prec_* S^* \cup \{\eta\}$, which contradicts the maximality of the chain S^* again. Thus, ξ is the minimal element of the set $\mathrm{Comfix}(\mathcal{G})$. The theorem is proved.

Compare Theorem 1 and Smithson's theorem, see [7, Theorem 2.1], presented below for dual ordering.

Theorem 2 (R. E. Smithson). *Let we be given with a partially ordered set* $(X, \preceq)$*, a point* $x_0 \in X$*, and a commuting family of isotone single-valued mappings* $\mathcal{G} = \{G_\alpha\}_{\alpha\in A}$*, where* $G_\alpha : X \to X$ *are such that for any* $\alpha \in A$ *the relation* $G_\alpha(x_0) \preceq x_0$ *holds and any chain containing* x_0 *possesses the infimum. In this case the set of common fixed points of the family of mappings is not empty.*

Assertion. *Theorem* 1 *is a generalization of Theorem* 2*.*

Proof. Let all the conditions of Theorem 2 be valid. Show that conditions of Theorem 1 are also valid. To do that, it suffices to show that any chain $S \subseteq \mathcal{S}(x_0;\mathcal{G})$ has a lower bound $u \in X$ such that $G_\alpha(u) \preceq u$ for all $\alpha \in A$. Consider an arbitrary chain $S_0 \subseteq \mathcal{S}(x_0;\mathcal{G})$ and the corresponding chain $C_0 = S_0 \cup \{x_0\}$. In accordance with conditions of Theorem 2, there exists an infimum $u \in X$ of the chain C_0, which is an infimum of the chain S_0 too. Due to the isotone property of the mapping G_α and $u \preceq x$, for any $\alpha \in A$ and any $x \in S_0$ we have $G_\alpha(u) \preceq G_\alpha(x) \preceq x$, i.e., $G_\alpha(u)$ is a lower bound of the chain S_0. Taking into account that u is the infimum of the chain S_0, we conclude that $G_\alpha(u) \preceq u$ for all $\alpha \in A$. Thus, all conditions of Theorem 1 are valid.

Now show that Theorem 1 does not follow from Theorem 2. To do that, consider the following example.

Example. Let $X = [-1, 0) \cup (0, 1]$ be endowed with the standard ordering of $\mathbb{R}$. Define the family $\mathcal{G} = \{G_\alpha\}_{\alpha\in A}$, where $A = \{x \in \mathbb{R} : x > 1\}$, as

$$G_\alpha(x) = \begin{cases} \frac{x}{\alpha}, & x \in (0,1]; \\ -1, & x = [-1;0). \end{cases}$$

It is easy to see that the mappings G_α are isotone. The family $\mathcal{G}$ is commuting because of $G_\alpha(G_\beta(x)) = G_\beta(G_\alpha(x)) = \frac{x}{\alpha\beta}$ for $x \in (0, 1]$, and $G_\alpha(G_\beta(x)) = G_\beta(G_\alpha(x)) = -1$ for $x \in [-1, 0)$.

Fix an arbitrary element $x_0 \in (0, 1]$. The chain $(0, x_0]$ contains x_0 and has no infimum. Thus, the conditions of Theorem 2 are not valid. Show that the conditions of Theorem 1 hold. To do that, it suffices to show that for any chain $S \subseteq \mathcal{S}(x_0;\mathcal{G})$ there exits its lower bound $u \in X$ such that $G_\alpha(u) \preceq u, \alpha \in A$. Since $\mathcal{S}(x_0;\mathcal{G}) \subseteq \mathcal{O}_X(x_0)$ and the element $u = -1$ is a lower bound of the set $\mathcal{O}_X(x_0)$ and $G_\alpha(-1) = -1, \alpha \in A$, then the element u is the lower bound we look for. Thus, all the conditions of Theorem 1 are valid.

Now consider the case of multivalued mappings.

Theorem 3. *Let we be given with a partially ordered set* $(X, \preceq)$*, a non-empty set* A*, and a family of commuting isotone multivalued mappings* $\mathcal{G} = \{G_\alpha\}_{\alpha\in A}$*,* $G_\alpha : X \rightrightarrows X$ *such that for any* $x \in X$ *and* $\alpha \in A$ *the set* $G_\alpha(x)$ *contains the least element. In addition, let there exist an element* $x_0 \in X$ *and a set* $\{y_\alpha\}_{\alpha\in A}$ *of* $\mathcal{G}$*-values at the point* x_0 *such that* $y_\alpha \preceq x_0$ *for all* $\alpha \in A$*. In this case if for any chain* $S \subseteq \mathcal{S}(x_0;\mathcal{G})$ *there exists its lower bound* z *and the set* $\{z_\alpha\}_{\alpha\in A}$ *of* $\mathcal{G}$*-values at the point* z *such that* $z_\alpha \preceq z$ *for all* $\alpha \in A$*, then the set* $\mathrm{Comfix}(\mathcal{G})$ *is not empty and contains the minimal element.*

Proof. Consider the family of single-valued mappings $g = \{g_\alpha\}_{\alpha\in A}, g_\alpha : X \to X$, where $g_\alpha(x) = \inf G_\alpha(x), x \in X, \alpha \in A$. In accordance with [7, Proposition 2.2], the family g is commuting and each mapping g_α from this family is isotone. Show that the family g satisfies all the conditions of Theorem 1. Consider the element $x_0 \in X$ and the corresponding set $\{y_\alpha\}_{\alpha\in A}$ of $\mathcal{G}$-values at the point x_0 such that

$y_\alpha \preceq x_0$ for all $\alpha \in A$. Since $g_\alpha(x_0) = \inf G_\alpha(x_0)$ and $y_\alpha \in G_\alpha(x_0)$, then $g_\alpha(x_0) \preceq y_\alpha \preceq x_0, \alpha \in A$. Similarly, one can show that for any chain $S \subseteq \mathcal{S}(x_0; g)$ there exists its lower bound $u \in X$ such that $g_\alpha(u) \preceq u, \alpha \in A$.

Thus, all the conditions of Theorem 1 are valid and hence the set $\mathrm{Comfix}(g)$ is not empty and contains the minimal element ξ that is the maximal lower bound of the chain S^* from $\mathcal{C}(x_0; g)$ with respect to the ordering $\preceq_*$, where the element ξ, the chain S^*, and the ordering $\preceq_*$ are described in the proof of Theorem 1. Therefore, $\mathrm{Comfix}(\mathcal{G}) \neq \varnothing$. Show that ξ is the minimal element in $\mathrm{Comfix}(\mathcal{G})$. Using the proof by contradiction, suppose there exists $\eta \in \mathrm{Comfix}(\mathcal{G})$ such that $\eta \prec \xi$. Since $\eta \in G_\alpha(\eta)$ and $g_\alpha(\eta) = \inf G_\alpha(\eta)$, then $g_\alpha(\eta) \preceq \eta$ for an arbitrary element $\alpha \in A$. Thus, $\eta \in \mathcal{S}(x_0; g)$ and $\eta \prec x, \forall x \in S^*$, i.e., $S^* \cup \{\eta\} \in \mathcal{C}(x_0; g)$ and $S^* \prec_* S^* \cup \{\eta\}$, which contradicts the maximality of the element S^*. Thus, we conclude that the inequality $\eta \prec \xi$ is impossible and hence ξ is the minimal element of the set $\mathrm{Comfix}(\mathcal{G})$.

Note that if all mappings of the family $\mathcal{G}$ from the conditions of Theorem 3 are single-valued, then we obtain exactly Theorem 1. It is easy to check that Theorem 3 is a generalization of [7, Theorem 2.3].

Now let us strengthen the conditions of Theorem 3 by adding requirements that guarantee the existence of the least elements in the set of common fixed points. Consider the set $\mathcal{S}(\mathcal{G}) = \bigcup_{x \in X} \mathcal{S}(x; \mathcal{G})$.

Theorem 4. *Let all the conditions of Theorem* 3 *be valid and for any pair* $x_1, x_2 \in X$ (*for* $x_1, x_2 \in \mathcal{O}_X(x_0)$, *respectively*) *we have* $\mathcal{S}(\mathcal{G}) \cap \Omega_X(x_1, x_2) \neq \varnothing$. *In this case the set* $\mathrm{Comfix}(\mathcal{G})$ (*the set* $\mathrm{Comfix}(\mathcal{G}) \cap \mathcal{O}_X(x_0)$, *respectively*) *contains the least element.*

Proof. Consider the case when for any pair $x_1, x_2 \in X$ the relation $\mathcal{S}(\mathcal{G}) \cap \Omega_X(x_1, x_2) \neq \varnothing$ holds. In the second case the proof is rather similar. Consider the family of single-valued mappings $g = \{g_\alpha\}_{\alpha \in A}$, $g_\alpha : X \to X, g_\alpha(x) = \inf G_\alpha(x)$, the maximal chain S^* from $\mathcal{C}(x_0; g)$ with respect to the ordering $\preceq_*$, and the lower bound $\xi \in X$ of the chain S^*, where the element ξ, the chain S^*, and the ordering $\preceq_*$ are constructed in the proof of Theorem 3. Since ξ is the minimal element in the set $\mathrm{Comfix}(\mathcal{G})$ according to the proof of Theorem 3, then to complete the proof it suffices to show that the element ξ is comparable with any element of $\mathrm{Comfix}(\mathcal{G})$. Suppose there exists an element $\zeta \in \mathrm{Comfix}(\mathcal{G})$ such that ζ and ξ are not comparable. Then, in accordance with the assumptions of the theorem, there exists an element $w \in \mathcal{S}(\mathcal{G}) \cap \Omega_X(\xi, \zeta)$. Since ξ and ζ are not comparable, then $w \prec \xi$. Since $w \in \mathcal{S}(\mathcal{G})$, then there exists a set $\{w_\alpha\}_{\alpha \in A}$ of $\mathcal{G}$-values at the point w such that $w_\alpha \preceq w$ for all $\alpha \in A$. And since $g_\alpha(w) = \inf G_\alpha(w)$, then the inequalities $g_\alpha(w) \preceq w_\alpha \preceq w$, $\alpha \in A$, are valid. Thus, $w \in \mathcal{S}(x_0; g)$ and $w \prec x, \forall x \in S^*$, i.e., $S^* \cup \{w\} \in \mathcal{C}(x_0; g)$ and $S^* \prec_* S^* \cup \{w\}$, which contradicts the maximality of the element S^* in the set of chains $\mathcal{C}(x_0; g)$. As the result, we get that any element $\zeta \in \mathrm{Comfix}(\mathcal{G})$ is comparable with ξ. Taking into account that ξ is the minimal element of the set $\mathrm{Comfix}(\mathcal{G})$, we conclude that ξ is the least element of the set $\mathrm{Comfix}(\mathcal{G})$. The theorem is proved.

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