ISSN 0027-1322, Moscow University Mathematics Bulletin, 2016, Vol. 71, No. 4, pp. 161–165. © Allerton Press, Inc., 2016.
Original Russian Text © S.V. Bykovskaya, 2016, published in Vestnik Moskovskogo Universiteta, Matematika. Mekhanika, 2016, Vol. 71, No. 4, pp. 33–38.

Complete Systems of Monadic Predicates for Post Classes

S. V. Bykovskaya

Moscow State University, Faculty of Mechanics and Mathematics,
Leninskie Gory, Moscow, 119991 Russia, e-mail: bykovskaya.svetlana@gmail.com

Received February 19, 2016

Abstract—The problem of completeness of arbitrary systems of monadic predicates defined on finite sets is considered. Completeness criteria are obtained for an arbitrary system of monadic predicates over arbitrary set of Boolean functions.

DOI: 10.3103/S0027132216040057

We consider a set $\mathfrak{P}_V$ of monadic predicates defined on a certain nonempty finite set V (see the definitions in [1]).

Let $\mathfrak{A}$ be some finite set of predicates, F be some set of Boolean functions. Define the notion of a *predicate formula* over $(\mathfrak{A}, F)$:

1) expressions $P(x)$, where $P(x) \in \mathfrak{A}$, are (atomic) formulas over $(\mathfrak{A}, F)$;

2) let $f(x_1, \ldots, x_n) \in F$ and $A_1, \ldots, A_n$ be formulas over $(\mathfrak{A}, F)$, then the expression $f(A_1, \ldots, A_n)$ is a formula over $(\mathfrak{A}, F)$.

The *closure of a system of predicates* $\mathfrak{A} \subseteq \mathfrak{P}_V$ over a system of Boolean functions F (notation $[\mathfrak{A}]_F$) is the set of all predicates expressed by formulas over $(\mathfrak{A}, F)$.

A system of predicates $\mathfrak{A}$ is called *complete over a system of functions* F if $[\mathfrak{A}]_F = \mathfrak{P}_V$. The theorem that a system of predicates $\mathfrak{A} = \{A_1(x), \ldots, A_s(x)\}$ is complete over a system $F = \{x \vee y, x\&y, \overline{x}, x \to y\}$ if and only if for any two distinct elements a and b of the set V there exists a predicate $A_i(x), 1 \leq i \leq s$, such that $A_i(a) \neq A_i(b)$ is known, see [1].

In this paper for the system of functions F we take arbitrary systems of Boolean functions. We obtain a completeness criterion for a system of predicates over F for an arbitrary system F of Boolean functions.

It is clear that if F and H are such systems of Boolean functions that $[F] = [H]$, then for an arbitrary system of predicates $\mathfrak{A}$ the completeness criteria over F and over H are the same. Therefore, in order to obtain a completeness criterion of a system $\mathfrak{A}$ over an arbitrary system F, it is sufficient to establish the completeness criterion over the closed class $[F]$.

Let $P(x) \in \mathfrak{P}_V$ and $V' \subseteq V, V' \neq \emptyset$. By $P(x)|_{V'}$ we denote the restriction of the predicate $P(x)$ onto the set V'. Further, let $\mathfrak{A} \subseteq \mathfrak{P}_V$. By $\mathfrak{A}|_{V'}$ we denote the set of all restrictions of predicates from $\mathfrak{A}$ onto the set V'.

Let $\mathfrak{A}$ be an arbitrary system of predicates defined on a set V such that $|V| \geq m$, where $m \geq 2$, and let $F \subseteq P_2$. We say that the system $\mathfrak{A}$ is *m-complete* over F if for any set $V' \subseteq V$ such that $|V'| = m$ the system $\mathfrak{A}|_{V'}$ is complete over F.

The definitions directly imply

Lemma 1. *Let V be an arbitrary set such that $|V| \geq 3$, $\mathfrak{A}$ be an arbitrary system of predicates, F be an arbitrary system of functions, and there exist a number $m, 2 \leq m \leq |V|$, such that $\mathfrak{A}$ is not m-complete over F. In this case $\mathfrak{A}$ is not complete over F.*

Introduce the notion of order of a closed class of Boolean functions. The order of a function f is the number of its essential variables (notation $\mathrm{ord}(f)$). The order of a finite system of functions A is the maximum of orders of its functions (notation $\mathrm{ord}(A)$). The order of a closed class F (notation $\mathrm{ord}(F)$) is defined as

$$\mathrm{ord}(F) = \min_A \mathrm{ord}(A),$$

where the minimum is taken over all possible bases A of the class F.

Below we consider all closed classes of Boolean functions (see description of closed classes in [2]). Classes of constants (order 0) and monadic functions (order 1) cannot generate new predicates differing from constants and negations of predicates. Therefore, we restrict ourselves with consideration of closed classes of order p, where $p \geq 2$. Those classes can be divided into several groups with its own completeness criterion for each group.

Let $a_1, \dots, a_k \in V, k \geq 1$, define the predicate $P_{a_1,\dots,a_k}(x)$ in the following way:

$$P_{a_1,\dots,a_k}(x) = \begin{cases} 1, & \text{for } x \in \{a_1, \dots, a_k\}; \\ 0, & \text{otherwise.} \end{cases}$$

We denote the identically-false predicate by P^0 and the identically-true predicate by P^1.

It is easy to see that the following result is valid.

Lemma 2. *Let $\mathfrak{A}$ be an arbitrary system of predicates, F be a system of functions, $P^c \in [\mathfrak{A}]_F$, where $c \in \{0, 1\}$. In this case, $[\mathfrak{A}]_F = [\mathfrak{A}]_{F \cup \{c\}}$.*

Consider an arbitrary predicate $P(x) \in \mathfrak{P}_V$. The *predicate dual to it* is the predicate $P^*(x) = \overline{P}(x)$.

Further, let $\mathfrak{A}$ be an arbitrary system of predicates. The *system dual to* $\mathfrak{A}$ is the system

$$\mathfrak{A}^* = \bigcup_{P(x) \in \mathfrak{A}} P^*(x).$$

It is easy to show that the following result is valid.

Lemma 3 (duality principle). *Let $P_1(x), \dots, P_n(x)$ be arbitrary predicates from $\mathfrak{P}_V$, $f(x_1, \dots, x_n)$ be a Boolean function. If $P(x) = f(P_1(x), \dots, P_n(x))$, then $P^*(x) = f^*(P_1^*(x), \dots, P_n^*(x))$.*

Lemma 4. *Let V be an arbitrary set, $|V| = n, n > 3$; F be a system of functions such that $x \vee y, x \& y \in [F]$; a system of predicates $\mathfrak{A}$ be $(n-1)$-complete over F. In this case the system $\mathfrak{A}$ is complete over F.*

Proof. Let $V = \{a_1, \dots, a_n\}$. Consider the sets $V_i = V \setminus \{a_i\}$ and the corresponding restrictions $\mathfrak{A}|_{V_i}, i = 1, \dots, n$. The system $\mathfrak{A}$ is $(n-1)$-complete over F, therefore, the systems $\mathfrak{A}|_{V_i}$ are complete over $F, i = 1, \dots, n$.

Consider an element a_1 from V and construct a predicate $P_{a_1}(x)$. The system $\mathfrak{A}|_{V_2}$ is complete over F and hence $P_{a_1}(x)|_{V_2}$ belongs to $[\mathfrak{A}|_{V_2}]_F$. Let $P_{a_1}(x)|_{V_2}$ be implemented by the formula $\Phi(A_1(x)|_{V_2}, \dots, A_p(x)|_{V_2})$, where $A_1(x), \dots, A_p(x)$ belongs to $\mathfrak{A}$. In this case the formula $\Phi(A_1(x), \dots, A_p(x))$ realizes either $P_{a_1}(x)$, or $P_{a_1,a_2}(x)$. In the first case we have $P_{a_1}(x) \in [\mathfrak{A}]_F$. In the second case we have $P_{a_1,a_2}(x) \in [\mathfrak{A}]_F$.

Consider the system $\mathfrak{A}|_{V_3}$ and the predicate $P_{a_1}(x)$. Applying similar arguments, we get that either $P_{a_1}(x) \in [\mathfrak{A}]_F$, or $P_{a_1,a_3}(x) \in [\mathfrak{A}]_F$. Thus, either $P_{a_1}(x) \in [\mathfrak{A}]_F$, or $P_{a_1,a_3}(x), P_{a_1,a_2}(x) \in [\mathfrak{A}]_F$. In the second case the predicate $P_{a_1}(x)$ is realized by the formula $P_{a_1,a_3}(x) \& P_{a_1,a_2}(x)$. Thus, for any $a \in V$ the predicate $P_a(x)$ belongs to $[\mathfrak{A}]_F$.

Let $P(x)$ be an arbitrary predicate from $\mathfrak{P}_V$. If $P(x)$ is the identically-false predicate, then it is implemented by formula $P_{a_1}(x) \& P_{a_2}(x)$. Now let $P(x)$ be not identically-false and $a_1, \dots, a_r$ be all the elements of the set V on which $P(x)$ takes the value 1, $r \geq 1$. In this case the formula $\Phi(x)$ of the form $\bigvee_{i=1}^r P_{a_i}(x)$ implements the predicate $P(x)$. Thus, any predicate $P(x)$ belongs to $[\mathfrak{A}]_F$. Therefore, system $\mathfrak{A}$ is complete over F. The lemma is proved.

Define the functions $d_p(x_1, \dots, x_p), p \geq 2$, in the following way:

$$d_p(x_1, \dots, x_p) = \bigvee_{1 \leq i < j \leq p} x_i x_j.$$

Lemma 5. *Let n and m be natural numbers such that $m \geq 2, n > m+1$; V be an arbitrary set such that $|V| = n$; F be a system of functions such that $d_{m+1} \in [F]$; the system of predicates $\mathfrak{A}$ be $(n-1)$-complete over F. In this case, $P^0, P^1 \in [\mathfrak{A}]_F$.*

Proof. Let $V = \{a_1, \dots, a_n\}$. Consider the sets $V_i = V \setminus \{a_i\}$ and the corresponding restrictions $\mathfrak{A}|_{V_i}, i = 1, \dots, n$. By the hypothesis, the system of predicates $\mathfrak{A}$ is $(n-1)$-complete over F. Therefore, the systems $\mathfrak{A}|_{V_i}$ are complete over $F, i = 1, \dots, n$. Construct the identically-false predicate.

The system $\mathfrak{A}|_{V_1}$ is complete over F, therefore, $P^0|_{V_1} \in [\mathfrak{A}|_{V_1}]_F$. Let the predicate $P^0|_{V_1}$ be implemented by the formula Φ of the following form: $\Phi(A_1(x)|_{V_1}, \dots, A_l(x)|_{V_1})$, where $A_1(x), \dots, A_l(x) \in \mathfrak{A}$. In this case the formula $\Phi(A_1(x), \dots, A_l(x))$ realizes either P^0, or $P_{a_1}(x)$. In the first case we have $P^0 \in [\mathfrak{A}]_F$. In the second case we have $P_{a_1}(x) \in [\mathfrak{A}]_F$. Since $n > m+1$, we can also consider the systems $\mathfrak{A}|_{V_2}, \dots, \mathfrak{A}|_{V_{m+1}}$. Applying similar arguments, we get either $P^0 \in [\mathfrak{A}]_F$, or $P_{a_1}(x), \dots, P_{a_{m+1}}(x) \in [\mathfrak{A}]_F$. In the latter case the identically-false predicate is realized by the formula $d_{m+1}(P_{a_1}(x), \dots, P_{a_{m+1}}(x))$. Thus, $P^0 \in [\mathfrak{A}]_F$. Applying similar arguments, we get $P^1 \in [\mathfrak{A}]_F$. The lemma is proved.

Theorem 1. *Let F be one of the classes P_2, M, M_0, M_1, M_{01}, T_0, T_1, T_{01}, S, S_{01}, SM, O^p, MO^p, O_0^p, MO_0^p, I^p, MI^p, I_1^p, MI_1^p, where $p \geq 2$, $\operatorname{ord}(F) = m$, V be an arbitrary finite set, $|V| \geq m$. In this case the system of predicates $\mathfrak{A}$ is complete over the class F if and only if it is m-complete over F.*

Proof. Necessity. Lemma 1 implies that if the system $\mathfrak{A}$ is complete over F, then it is m-complete over F.

Sufficiency. Consider the two following cases: $\mathrm{ord}(F) = 2$ and $\mathrm{ord}(F) = m, m > 2$.

Let $\mathrm{ord}(F) = 2$. In this case we have $F \in \{P_2, M, M_0, M_1, M_{01}, T_0, T_1\}$. Let the hypothesis be true. Prove the assertion by induction over the cardinality n of the set V.

The basis of induction. Let $|V|=2$. Obviously, if the system $\mathfrak{A}$ is 2-complete over F, then it is complete over F.

The inductive passage. Let the assertion be valid for all V such that $2 \leq |V| \leq k$. Prove it for all V such that $|V|=k+1$.

Let $V = \{a_1, \ldots, a_{k+1}\}$. Consider the sets $V_i = V \setminus \{a_i\}$ and the corresponding restrictions $\mathfrak{A}|_{V_i}, i = 1, \ldots, k+1$. The system $\mathfrak{A}$ is 2-complete over F and hence the systems $\mathfrak{A}|_{V_i}$ are 2-complete over $F, i = 1, \ldots, k+1$. In this case, according to the inductive hypothesis, the systems $\mathfrak{A}|_{V_i}$ are complete over F, $i = 1, \ldots, k+1$. Therefore, the system $\mathfrak{A}$ is k-complete over F. Since $x \vee y, x\&y \in F$ and the system $\mathfrak{A}$ is k-complete over F, then the system $\mathfrak{A}$ is complete over F by Lemma 4.

Now consider the second case. Let $\mathrm{ord}(F) = m, m > 2$, then F is one of the classes T_{01}, S, S_{01}, SM, O^p, MO^p, O^p_0, MO^p_0, I^p, MI^p, I^p_1, MI^p_1. Let the hypothesis be true. Prove the assertion by induction over the cardinality n of the set V.

The basis of induction. Let $|V|=m$. Obviously, if the system $\mathfrak{A}$ is m-complete over F, then it is complete over F.

The inductive passage. Let the assertion be valid for all V such that $m \leq |V| \leq k$. Prove it for all V such that $|V|=k+1$.

Let $V = \{a_1, \ldots, a_{k+1}\}$. Consider the sets $V_i = V \setminus \{a_i\}$ and the corresponding restrictions $\mathfrak{A}|_{V_i}, i = 1, \ldots, k+1$. The system $\mathfrak{A}$ is m-complete over F, therefore, the systems $\mathfrak{A}|_{V_i}$ are m-complete over F, $i = 1, \ldots, k+1$. According to the inductive hypothesis, the systems $\mathfrak{A}|_{V_i}$ are complete over F, $i = 1, \ldots, k+1$. Therefore, the system $\mathfrak{A}$ is k-complete over F.

For $F \in \{T_{01}, S, S_{01}, SM\}$ we have $\mathrm{ord}(F) = 3$ and $d_3 \in F$ (see [1]).

For $F \in \{O^p, MO^p, O^p_0, MO^p_0, I^p, MI^p, I^p_1, MI^p_1\}$ we have $\mathrm{ord}(F) = p+1$ and $d_{p+1} \in F$ (see [1]).

Thus, since $d_m \in F$ and the system $\mathfrak{A}$ is k-complete over F, by Lemma 5 we get $P^0, P^1 \in [\mathfrak{A}]_F$. According to Lemma 2, for completeness of $\mathfrak{A}$ over F it is sufficient to show that $\mathfrak{A}$ is complete over the system $F \cup \{0, 1\}$.

The following relations hold:

$$[T_{01} \cup \{0,1\}] = [S \cup \{0,1\}] = [S_{01} \cup \{0,1\}] = P_2,$$

$$[O^p \cup \{0,1\}] = [O^p_0 \cup \{0,1\}] = [I^p \cup \{0,1\}] = [I^p_1 \cup \{0,1\}] = P_2,$$

$$[SM \cup \{0,1\}] = [MO^p \cup \{0,1\}] = [MO^p_0 \cup \{0,1\}] = [MI^p \cup \{0,1\}] = [MI^p_1 \cup \{0,1\}] = M.$$

If the system $\mathfrak{A}$ is k-complete over P_2 (over M, respectively), then it is 2-complete over P_2 (over M, respectively). According to the arguments presented for the first case, the system $\mathfrak{A}$ is complete over P_2 (over M, respectively). Thus, $\mathfrak{A}$ is complete over $F \cup \{0, 1\}$ and hence over F too. The theorem is proved.

Lemma 6. *Let V be an arbitrary set such that $|V| \geq 2$. In this case the system of predicates $\mathfrak{A}$ is complete over the system $F = \{0, x \vee y\}$ if and only if for any element $a \in V$ the relation $P_a(x) \in \mathfrak{A}$ holds.*

Proof. Necessity. Consider an arbitrary predicate $P_a(x)$. Since the system $\mathfrak{A}$ is complete over F, then $P_a(x) \in [\mathfrak{A}]_F$. The definition of the set $[\mathfrak{A}]_F$ implies that the formula $\Phi(x)$ realizing $P_a(x)$ has the form $A_1(x) \vee A_2(x) \vee \ldots \vee A_k(x)$, where $A_1(x), \ldots, A_k(x) \in \{\mathfrak{A} \cup P^0\}, k \geq 1$. Obviously, the inequalities $P_a(x) \geq A_i(x)$ hold for all $1 \leq i \leq k$. This implies $A_i(x) \in \{0, P_a(x)\}$, $i = 1, \ldots, k$. If all $A_i(x)$ are identically-false, then the above formula realizes the identically-false predicate, but not $P_a(x)$. Therefore, at least one of the predicates $A_i(x)$ realizes the predicate $P_a(x)$, i.e., $P_a(x) \in \mathfrak{A}$.

Sufficiency. Let the relation $P_a(x) \in \mathfrak{A}$ hold for any element $a \in V$ and let $P(x)$ be an arbitrary predicate from $\mathfrak{P}_V$. If $P(x)$ is the identically-false predicate, then $P(x) \in [\mathfrak{A}]_F$ (because of $0 \in F$). Let $P(x)$ be not identically-false and let $a_1, \ldots, a_r$ be all the elements of the set V on which $P(x)$ takes the value 1, $r \geq 1$. In this case the formula $\Phi(x)$ of the form $\bigvee_{i=1}^{r} P_{a_i}(x)$ realizes the predicate $P(x)$. Thus, any predicate $P(x)$ belongs to $[\mathfrak{A}]_F$. The lemma is proved.

This lemma directly implies

Theorem 2. *Let $F \in \{D, D_0\}$, V be an arbitrary finite set such that $|V| \geq 2$. In this case the system of predicates $\mathfrak{A}$ is complete over the class F if and only if the relation $P_a(x) \in \mathfrak{A}$ holds for any element $a \in V$.*

Taking into account Theorem 2 and the duality principle, we obtain the following

Theorem 3. *Let $F \in \{K, K_1\}$, V be an arbitrary finite set such that $|V| \geq 2$. In this case the system of predicates $\mathfrak{A}$ is complete over the class F if and only if the relation $\overline{P}_a(x) \in \mathfrak{A}$ holds for any element $a \in V$.*

Lemma 7. *Let V be an arbitrary finite set, $|V| \geq 2$, $F \subseteq O^\infty$, and the system of predicates $\mathfrak{A}$ is complete over F. In this case, $P^0 \in \mathfrak{A}$.*

Proof. By the hypothesis, $\mathfrak{A}$ is complete over F. Therefore, there exists a formula implementing the identically-false predicate.

Since any function $f \in [F]$, where $F \subseteq O^\infty$, is represented in the form $f' \vee x$, where x is some variable, then any formula for P^0 takes values equal to $\Phi(x) \vee P(x)$, where $P(x) \in \mathfrak{A}$. This implies $P(x) = P^0 \in \mathfrak{A}$. The lemma is proved.

The following theorem presents a completeness criterion for an arbitrary system of predicates over the classes O^∞, $MO^\infty, MO_0^\infty, O_0^\infty, D_1, D_{01}$.

Theorem 4. *Let $F \in \{P_2, M, M_0, T_0, D, D_0\}$, V be an arbitrary finite set, $|V| \geq 2$. In this case the system of predicates $\mathfrak{A}$ is complete over the class $F \cap O^\infty$ if and only if $P^0 \in \mathfrak{A}$ and the system $\mathfrak{A}$ is complete over F.*

Proof. Necessity. Let the system of predicates $\mathfrak{A}$ be complete over $F \cap O^\infty$. By Lemma 7 we get $P^0 \in \mathfrak{A}$. Suppose the system $\mathfrak{A}$ is not complete over F. Since $[\mathfrak{A}]_{F\cap O^\infty} \subseteq [\mathfrak{A}]_F$, then the system $\mathfrak{A}$ is not complete over $F \cap O^\infty$, we have got a contradiction.

Sufficiency. Let the condition be valid. The following relations hold:

$$[P_2 \cap O^\infty \cup \{0\}] = [O^\infty \cup \{0\}] = P_2, \quad [T_0 \cap O^\infty \cup \{0\}] = [O_0^\infty \cup \{0\}] = T_0,$$

$$[M \cap O^\infty \cup \{0\}] = [MO^\infty \cup \{0\}] = M, \quad [M_0 \cap O^\infty \cup \{0\}] = [MO_0^\infty \cup \{0\}] = M_0,$$

$$[D \cap O^\infty \cup \{0\}] = [D_1 \cup \{0\}] = D, \quad [D_0 \cap O^\infty \cup \{0\}] = [D_{01} \cup \{0\}] = D_0.$$

We obtain $[F \cap O^\infty \cup \{0\}] = F$. Since $P^0 \in \mathfrak{A}$, then, according to Lemma 2, for the completeness of $\mathfrak{A}$ over $F \cap O^\infty$ it is sufficient to show that the system $\mathfrak{A}$ is complete over F. By the hypothesis, the system $\mathfrak{A}$ is complete over F, therefore, it is complete over $F \cap O^\infty$ too. The theorem is proved.

The following theorem presents a completeness criterion for an arbitrary system of predicates over the classes I^∞, $MI^\infty, MI_1^\infty, I_1^\infty, K_0, K_{01}$.

Theorem 5. *Let $F \in \{P_2, M, M_1, T_1, K, K_1\}$, V be an arbitrary finite set, $|V| \geq 2$. In this case the system of predicates $\mathfrak{A}$ is complete over the class $F \cap I^\infty$ if and only if $P^1 \in \mathfrak{A}$ and the system $\mathfrak{A}$ is complete over F.*

Proof. The theorem directly follows from the previous one and the duality principle.

Let $\mathfrak{A} = \{A_1(x), \ldots, A_n(x)\}$ be an arbitrary system of predicates defined on V. We say that the system $\mathfrak{A}$ is *linearly independent* if the equality

$$c_1 \cdot A_1(x) + \ldots + c_n \cdot A_n(x) + c_{n+1} \cdot P^1 = 0,$$

where $c_1, c_2, \ldots, c_n, c_{n+1} \in \{0, 1\}$, implies $c_1 = c_2 = \ldots = c_n = c_{n+1} = 0$ (here and below we mean the sum modulo 2). Let $\mathfrak{A}$ be an arbitrary system of predicates defined on V. We say that its subsystem $\mathfrak{A}_1$ is the *maximal* linearly independent subsystem if the following conditions hold:

1) $\mathfrak{A}_1$ is linearly independent,

2) for any predicate $P(x)$ from $\mathfrak{A} \setminus \mathfrak{A}_1$ the system $\mathfrak{A}_1 \cup P(x)$ is not linearly independent.

The following result is valid.

Lemma 8. *Let $F = \{x + y, 1\}$, $\mathfrak{A}$ be an arbitrary system of predicates, $\mathfrak{A}_1$ be the maximal linearly independent subsystem of the system $\mathfrak{A}$. In this case, $[\mathfrak{A}]_F = [\mathfrak{A}_1]_F$.*

Proof. Since $\mathfrak{A}_1 \subseteq \mathfrak{A}$, then $[\mathfrak{A}_1]_F \subseteq [\mathfrak{A}]_F$.

Suppose $[\mathfrak{A}_1]_F \subset [\mathfrak{A}]_F$. In this case there exists a predicate $P(x)$ such that $P(x) \in [\mathfrak{A}]_F$, but $P(x) \notin [\mathfrak{A}_1]_F$. The predicate $P(x)$ is realized by the formula

$$A_1(x) + \ldots + A_n(x) + B_1(x) + \ldots + B_k(x) + c \cdot P^1,$$

where $A_1(x), \ldots, A_n(x) \in \mathfrak{A} \setminus \mathfrak{A}_1$, $B_1(x), \ldots, B_k(x) \in \mathfrak{A}_1, c \in \{0, 1\}$.

Let $\mathfrak{A}_1 = \{D_1(x), \ldots, D_m(x)\}$. Since $\mathfrak{A}_1$ is the maximal linearly independent subsystem of the system $\mathfrak{A}$, the systems $\mathfrak{A}_1 \cup A_j(x)$, where $j = 1, \ldots, n$, are linearly dependent. Consider the system $\mathfrak{A}_1 \cup A_1(x)$. There exist $c_1, \ldots, c_m \in \{0, 1\}$ not all equal to zero and such that

$$c_1 \cdot D_1(x) + \ldots + c_m \cdot D_m(x) + c_{m+1} \cdot P^1 + A_1(x) = 0.$$

This implies that the predicate $A_1(x)$ is realized by the formula

$$c_1 \cdot D_1(x) + \ldots + c_m \cdot D_m(x) + c_{m+1} \cdot P^1.$$

We get $A_1(x) \in [\mathfrak{A}_1]_F$. Similarly, considering the systems $\mathfrak{A}_1 \cup A_j(x)$, where $j = 1, \ldots, n$, we get $A_1(x), \ldots, A_n(x) \in [\mathfrak{A}_1]_F$, therefore, $P(x) \in [\mathfrak{A}_1]_F$. We have got a contradiction. Thus, $[\mathfrak{A}_1]_F = [\mathfrak{A}]_F$. The lemma is proved.

Theorem 6. *Let V be an arbitrary finite set, $|V| \geq 2$. In this case the system of predicates $\mathfrak{A}$ is complete over the class L if and only if there exists a linearly independent system of predicates $\mathfrak{A}' \subseteq \mathfrak{A}$ such that $|\mathfrak{A}'| = |V| - 1$.*

Proof. Necessity. Consider the system $F = \{y + z, 1\}$. Since $[F] = L$, it is sufficient to prove the theorem for the system F. Suppose the system $\mathfrak{A}$ is complete over F, but the condition does not hold, i.e., the maximal linearly independent subsystem $\mathfrak{A}_1$ of the system $\mathfrak{A}$ has the cardinality less than $|V| - 1$. Let $\mathfrak{A}_1 = \{P_1(x), \ldots, P_m(x)\}$. In this case, $m < |V| - 1$. The system $\mathfrak{A}$ is complete over F, therefore, Lemma 8 implies that the system $\mathfrak{A}_1$ is also complete over F. In this case any predicate from $[\mathfrak{A}_1]_F$ is realized by the formula

$$c_1 \cdot P_1(x) + c_2 \cdot P_2(x) + \ldots + c_m \cdot P_m(x) + c_{m+1} \cdot P^1,$$

where $c_1, c_2, \ldots, c_m, c_{m+1} \in \{0, 1\}$. Due to the linear independence of the system $\mathfrak{A}_1$, we have $|[\mathfrak{A}_1]_F| = 2^{m+1}$. On the other hand, $|\mathfrak{P}_V| = 2^{|V|} > 2^{m+1}$. Therefore, the system $\mathfrak{A}_1$ and hence the system $\mathfrak{A}$ are not complete over F, we have got a contradiction.

Sufficiency. Let the condition hold and the maximal linearly independent subsystem of the system $\mathfrak{A}$ have the cardinality not less than $|V| - 1$. Denote this subsystem by $\mathfrak{A}_1$. Let

$$\mathfrak{A}_1 = \{P_1(x), \ldots, P_m(x)\}, m \geq |V| - 1.$$

Any predicate from $[\mathfrak{A}_1]_F$ is realized by the formula

$$c_1 \cdot P_1(x) + c_2 \cdot P_2(x) + \ldots + c_m \cdot P_m(x) + c_{m+1} \cdot P^1,$$

where $c_1, c_2, \ldots, c_m, c_{m+1} \in \{0, 1\}$. Due to the linear independence of the system $\mathfrak{A}_1$, we have $|[\mathfrak{A}_1]_F| = 2^{m+1}$. On the other hand, $|\mathfrak{P}_V| = 2^{|V|} \leq 2^{m+1}$. Therefore, $[\mathfrak{A}_1]_F = \mathfrak{P}_V$. Thus, the system $\mathfrak{A}_1$ is complete over F. According to Lemma 8, the system $\mathfrak{A}$ is also complete over F. The theorem is proved.

Theorem 7. *Let V be an arbitrary finite set, $|V| \geq 2$, $F \in \{L_0, L_1, L_{01}, SL\}$. In this case the system of predicates $\mathfrak{A}$ is complete over the class F if and only if $\mathfrak{A}$ is complete over L and $P^0, P^1 \in [\mathfrak{A}]_F$.*

Proof. Necessity. Let $\mathfrak{A}$ be complete over the class F, then $P^0, P^1 \in [\mathfrak{A}]_F$. Suppose the system $\mathfrak{A}$ is not complete over L. Since $[\mathfrak{A}]_F \subseteq [\mathfrak{A}]_L$, then the system $\mathfrak{A}$ is not complete over F, we have got a contradiction.

Sufficiency. Let $\mathfrak{A}$ be complete over L and $P^0, P^1 \in [\mathfrak{A}]_F$. The following relations are valid:

$$[L_0 \cup \{0, 1\}] = [L_1 \cup \{0, 1\}] = [L_{01} \cup \{0, 1\}] = [SL \cup \{0, 1\}] = L.$$

Lemma 2 implies that for the completeness of $\mathfrak{A}$ over F it is sufficient to show that $\mathfrak{A}$ is complete over L. By the hypothesis, the system $\mathfrak{A}$ is complete over the class L, therefore, it is complete over F too. The theorem is proved.

REFERENCES

1. *Summary of O. B. Lupanov's Lectures "Introduction to Mathematical Logic,"* Ed. by A. B. Ugol'nikov (Moscow State Univ., Moscow, 2007) [in Russian].
2. A. B. Ugol'nikov, *Post Classes* (Moscow State Univ., Moscow, 2008) [in Russian].

Translated by V. Valedinskii