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Optimal Stopping for Absolute Maximum of Homogeneous Diffusion

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Abstract—The optimal stopping problem is studied in the paper for functions dependent on the absolute maximum of some homogeneous diffusion. The cases of infinite and finite time horizons are considered. A differential equation for the optimal stopping boundary is obtained in both the cases. A principle of maximum is proved for a function satisfying a single-crossing condition.

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In this paper we consider the optimal stopping problem for target functions dependent on the absolute maximum of homogeneous diffusion. Similar problems were considered in many papers. The problem of maximization of the ratio of the value of the process to the value of its absolute maximum was considered in [1] in the model of geometrical Brownian motion. In this case the answer to that problem was not obtained for certain values of parameters. The missing case was considered in [2]. In addition, the problem for the inverse relation was also solved in that paper. Another generalization was proposed in [3], the cases of searching for the minimal value and the formulation for the absolute minimum of the process were considered. The problems of minimization of the square and arbitrary power of the difference of the current value and the maximum in the case of Brownian motion were solved in [4, 5] and [6], respectively. Note that in all papers mentioned above the principal part of the solution consisted in the reduction of the problem to the case of a one-dimensional Markov process and the function in the form of Maier, or Lagrange. An alternative approach was proposed in [7] to the optimal stopping problem in the case of an arbitrary functions in the LS-form on an infinite time horizon and for arbitrary homogeneous diffusion. A differential equation was obtained in this formulation so that the boundary of the optimal stopping domain must satisfy this equation. It was shown that the "maximum principle" holds, namely, one should take the maximal solution to the equation not crossing the diagonal, i.e., the set of points where the value of the process coincides with its current maximum. In this paper we apply a similar approach and extend it to the case of finite horizon, which allows us to generalize the results of [1–6].

Consider the case of an arbitrary homogeneous diffusion

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \tag{1}$$

where B_t is the standard Brownian motion coordinated with the filtration $(\mathcal{F}_t)_{t\geq 0}$ and the functions b and σ are Lipschitzian, i.e.,

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq C\,|x - y|$$

for some constant C, which ensures the existence and P-a.s. uniqueness of the strong solution to (1). We also suppose the diffusion X_t is regular, this does not restrict the generality because any diffusion can be represented as a composition of regular ones.

As is known, the characteristic operator for diffusion (1) has the form

$$\mathbb{L}_X f(x) = b(x)\frac{\partial f}{\partial x}(x) + \frac{\sigma^2(x)}{2}\frac{\partial^2 f}{\partial x^2}(x).$$

By $\mathcal{J}$ we denote the set of values which the process (X_t, M_t) can take and consider the optimal stopping problem

$$V_*(x, s) = \sup_{\tau\in\mathfrak{M}_X(f)} \mathbf{E}_{xs} f(X_\tau, M_\tau), \tag{2}$$

where we assume that the function $f(x, s) \in C^{2,1}(\mathcal{J})$ estimating the "closeness" of the value of the process at the stopping moment to the value of the current maximum satisfies the condition $f'_s(x, s)|_{x=s} = 0$, and $\mathfrak{M}_X(f)$ is the set of all stopping moments τ satisfying the condition $\mathbf{E}_{xs}\sup_{t\leq\tau} |f(X_t, M_t)| < \infty, \forall (x, s) \in \mathcal{J}$.

At last, suppose that if $\mathsf{P}(\lim_{t\to\infty} X_t = l_I) > 0$, then $\forall s \in I$ $\limsup_{x\to l_I} f(x,s) < \infty$. The violation of this condition for certain s means that $V_*(x,s) = \infty$ for $x < s$.

Problem (2) is a particular case of the problem $V_*(x,s) = \sup_{\tau\geq 0} \mathbf{E}_{xs} h(X_\tau, M_{\tau+\varepsilon})$, where $\varepsilon > 0$ is some constant (probably equal to infinity) for the function $f(X_\tau, M_\tau) = \mathbf{E}(h(X_\tau, M_{\tau+\varepsilon}|X_\tau, M_\tau))$.

Recall that the *scale function* for a homogeneous diffusion satisfying stochastic equation (1) is given by the formula

$$L(x) = \int^x \exp\left(-\int^y \frac{2b(z)}{\sigma^2(z)}\right).$$

Denote $\mu(x) = \frac{b(x)}{\sigma^2(x)} = -\frac{1}{2}(\ln L'(x))' = -\frac{1}{2}\frac{L'(x)}{L''(x)}$.

If τ_a is the moment when the diffusion reaches the level a and $\tau_{a,b} = \min(\tau_a, \tau_b)$, then for $a \leq x \leq b$ we have

$$P_x(X_{\tau_a} = a) = \frac{L(b) - L(x)}{L(b) - L(a)}, \qquad P_x(X_{\tau_a} = b) = \frac{L(x) - L(a)}{L(b) - L(a)}.$$

Note that the second coordinate of the Markov process (X_t, M_t) changes only on the diagonal, i.e., at points where $X_t = M_t$.

Due to the superharmonic characterization of the function $V_*(x,s)$, the stop is not optimal at the points not lying on the diagonal (i.e., $x < s$) where we have

$$f''_{xx}(x,s) + 2\mu(x) f'_x(x,s) > 0. \tag{3}$$

For the points lying on the diagonal, the infinitesimal generator has a more complex form. Nevertheless, condition (3) is sufficient for a point (x,x) to belong to the domain of continuation of observations.

Lemma 1. *It is not optimal to stop at points (x,x) such that either $f'_s(x,x) > 0$, or $f'_s(x,x) = 0$ and (3) holds.*

The proof is based on the fact that in conditions of the lemma and for sufficiently large k we have $\lim_{\delta\to 0} \mathbf{E}_{xx} f(X_{\tau_{k,\delta}}, M_{\tau_{k,\delta}}) > 0$, where $\tau_{k,\delta} = \inf\{t : X_t = x + \delta$ or $X_t = x - k\delta\}$. The detailed proof of this inequality is relatively long and so we omit it.

The method of solution of problem (2) is typical for optimal stopping problems. We suppose that the condition of smooth gluing is valid and then prove the verification theorem. The Stefan problem has in this case the following form:

$$\mathbb{L}_X V(x,s) = 0 \text{ for } g(s) < x < s, \qquad \frac{\partial V}{\partial s}(x,s)\Big|_{x=s-} = 0,$$
$$V(x,s)|_{x=g(s)+} = f(x,s), \qquad \frac{\partial V}{\partial x}(x,s)\Big|_{x=g(s)+} = f'_x(x,s).$$

Denote $V_g(s) = V_g(s,s)$. Using that X is a strictly Markov process, we can write

$$V_g(s,x) = H(s)\frac{L(s) - L(x)}{L(s) - L(g(s))} + V_g(s)\frac{L(x) - L(g(s))}{L(s) - L(g(s))}. \tag{4}$$

This inequality holds for all $x \in (g(s), s)$. Taking the limit for $x \downarrow g(s)$, we get

$$\lim_{x\downarrow g(s)} \frac{V_g(x,s) - H(s)}{L(x) - L(g(s))} = \lim_{x\downarrow g(s)} \left[\frac{V_g(x,s) - H(s)}{x - g(s)} \Big/ \frac{L(x) - L(g(s))}{x - g(s)}\right] = \frac{\partial V_g(x,s)}{\partial x}\bigg|_{g(s)} \cdot \frac{1}{L'(g(s))} = \frac{F(s)}{L'(g(s))}.$$

Thus, inequality (4) implies

$$\frac{V_g(s) - H(s)}{L(s) - L(g(s))} = \frac{F(s)}{L'(g(s))}.$$

Expressing $V_g(s)$ from the latter equality, substituting it in into (4), and applying the principle of normal reflection, we get

$$V_g(x,s) = \left[\frac{F(s)}{L'(g(s))} \cdot (L(x) - L(g(s)))\right] + H(s). \tag{5}$$

Denote $\Lambda(g(s),s) = \frac{L(s) - L(g(s))}{L'(g(s))}$. At the point $(g(s),s)$ we have

$$g'(s) = -\frac{f'_s(g(s),s)\Lambda^{-1}(g(s),s) + f''_{xs}(g(s),s)}{f''_{xx}(g(s),s) + 2\mu(s) f'_x(g(s),s)}. \tag{6}$$

By I we denote the interval where the diffusion takes its values and by I^0 we denote the same interval with its lower border l_I.

Definition 1. A semicontinuous from below boundary $g : I \to I^0$, $g(s) \le s$, is called admissible if the points $(g(s), s)$ belong to the closure of the set $\{(x,s) : \mathbb{L}_X f(x,s) < 0\}$ and for $l_I < g(s) < s$ the function $g(s)$ satisfies (6).

Differentiating (5) with respect to $g(s)$ (as an independent variable), we obtain the following

Lemma 2. *If the inequality $g_1(s) \le g_2(s)$ holds for all s, then $V_{g_1}(x,s) \ge V_{g_2}(x,s)$.*

Further, since the solution to the differential equation continuously depends on the initial conditions, there exists the maximal admissible boundary $g_*(s)$. We say that the function f satisfies the single crossing condition if for any s there exists a boundary $\underline{x}(s)$ such that $\mathbb{L}_X(x,s) \le 0$ for $x < \underline{x}(s)$ and $\mathbb{L}_X(x,s) \ge 0$ for $s > x > \underline{x}(s)$.

Theorem 1. *Let f satisfy the single crossing condition. If the relation $\mathbf{E} f(X_{\tau_*}, M_{\tau_*}) < \infty$ holds for the stopping moment $\tau_* = \inf\{t > 0 : X_t \le g_*(M_t)\}$, then this moment is optimal in problem* (2).

Proof. Let $g(s)$ be an admissible boundary. Apply Ito's formula to the process $V_g(X_t, M_t)$. We get

$$V_g(X_t, M_t) = V_g(x,s) + \int_0^t \sigma(X_r) \frac{\partial V_g}{\partial x}(X_r, M_r) dB_r + \int_0^t \sigma(X_r) \frac{\partial V_g}{\partial s}(X_r, M_r) dM_r + \int_0^t (\mathbb{L}_X V_g)(X_r, M_r) dr. \quad (7)$$

The integral over M_r equals zero because of the principle of normal reflection.

The process $\int_0^t \sigma(X_r) \frac{\partial V_g}{\partial x}(X_r, M_r) dB_r$ is a continuous local martingale and the process $\int_0^t (\mathbb{L}_X V_g)(X_r, M_r) dr$ is not increasing due to the single crossing condition. This implies that $V_g(X_t, M_t)$ is a local supermartingale.

Let τ be an arbitrary stopping moment for X. Consider $\tau' = \inf\{t \ge \tau : \mathbb{L}_X(X_t, M_t) \le 0\}$. Applying Ito's formula to the process $f(X_t, M_t)$, similar to (7) we get

$$f(X_t, M_t) = f(x,s) + \widehat{Q}_t + \widehat{P}_t,$$

where $\widehat{Q}_t$ is also a local martingale and $\widehat{P}_t = \int_0^t (\mathbb{L}_X f)(X_r, M_r) dr$ increases for $\tau \le t \le \tau'$.

Take a sequence of stopping moment σ_n being localizing for Q and $\widehat{Q}$. In this case,

$$\mathbf{E}_{xs} f(X_{\tau\wedge\sigma_n}, M_{\tau\wedge\sigma_n}) \le \mathbf{E}_{xs} f(X_{\tau'\wedge\sigma_n}, M_{\tau'\wedge\sigma_n}) \le \mathbf{E}_{xs} V_g(X_{\tau\wedge\sigma_n}, M_{\tau\wedge\sigma_n}) \le V_g(x,s) + \mathbf{E}_{xs} Q_{\tau'\wedge\sigma_n} = V_g(x,s).$$

Tending n to infinity and using Fatou's lemma, we get $\mathbf{E}_{xs} f(X_\tau, M_\tau) \le V_g(x,s)$. Taking the supremum over all possible τ and the infimum over all possible g, we obtain

$$V_*(x,s) \le \inf_g V_g(x,s) = V_{g_*}(x,s),$$

and the equality holds because of $V_{g_*}(x,s) = \mathbf{E}_{xs} f(X_{\tau_*}, M_{\tau_*})$.

Now proceed to the case of finite horizon. Consider the problem

$$V_*(x,s,t) = \sup_{0\le\tau\le T} \mathbf{E}_{xst} f(X_\tau, M_\tau, \tau), \quad \text{where } (f'_s(x,s,t))|_{x=s} = 0 \text{ for any } s \text{ and } t \quad (8)$$

Its important particular case is

$$V_*(x,s,t) = \sup_{0\le\tau\le T} \mathbf{E}_{xst} h(X_\tau, M_T)$$

for the function $f(X_\tau, M_\tau, \tau) = \mathbf{E}(h(X_\tau, M_T) | X_\tau, M_\tau, \tau)$.

By the analogy with the previous case, for the Markov process (X_t, M_t, t) we have

Lemma 3. *If $f''_{xx}(x,s,t) + 2\mu(x) f'_x(x,s,t) + f'_t(x,s,t) > 0$ then the stop at the point (x,x,t) is not optimal.*

The proof utilizes the same idea as Lemma 1, and hence we omit it.

In order to derive an equation for the boundary of optimal stopping, we need one more auxiliary fact.

Lemma 4. *The derivative $V'_s(x, s+, t)$ exists and satisfies the following equation on the domain C of continuation of observation:*

$$\left(\frac{\partial}{\partial t} + \mathbb{L}_X\right) V'_s(x, s+, t) = 0. \quad (9)$$

Proof. Consider an arbitrary point (x_0, s_0, t_0) and prove the required assertion for it. Since the set C is open, we can take $\delta > 0$ so that $\{(x, s, t)\} \in C$ for $s_0 \le s \le s_0 + \delta$. For an arbitrary boundary $g = g(t)$ consider the stopping moment $\bar\tau_g = \inf\{\theta > t : X_\theta \ge s_0 \text{ or } X_\theta \le g(\theta)\}$. Define the payout function

$$W(x,s,t) = \begin{cases} V(s_0, s, t), & \text{for } x = s_0\ ; \\ f(x,s,t), & \text{for } x < s_0\ . \end{cases}$$

Finally, assume $w(x, s, t, g) = \mathbf{E}_{xt} W(X_{\bar\tau_g}, s, \bar\tau_g)$. Note that since $V_s'(s_0, s+, t)|_{s=s_0}$ exists for any t, then W is differentiable and hence w_s' exists too. Since the boundary g (and hence $\bar\tau_g$) does not depend on s, the following relation is valid:

$$w_s'(x,s,t,g) = \mathbf{E}_{xt} W_s'(X_{\bar\tau_g}, s, \bar\tau_g).$$

Therefore, w satisfies the equation $w_{st}'' + \mathbb{L}_X w_s = 0$.

Further, for $s_0 \le s \le s_0 + \delta$ we have $w(x_0, s, t, g) \le V_*(x_0, s, t_0)$, and the equality is attained for $g(\theta) = g_*(s, \theta)$.

By $\mathbf{X}$ we denote the closure of the family of distributions $(X_{\bar\tau_g}, M_{\bar\tau_g}, \bar\tau_g)$ with all possible $\bar\tau_g$. It is dense because for any $\varepsilon > 0$ and for sufficiently large number M the random variable belongs to the compact set $[x_0 - M, x_0 + M] \times [s_0, s_0 + M] \times [0, T]$ with the probability exceeding $1 - \varepsilon$. Therefore, according to Prokhorov's theorem, $\mathbf{X}$ is compact (in the weak convergence topology). Further, W, and hence w, are semicontinuous from above. This fact and the continuity of w_s' allow us to apply the envelope theorem in the formulation of Corollary 4 from [8]. This gives that the function $V_*(x, s, t) = w(x, s, t, g_*)$ is differentiable with respect to s from the right and the derivative satisfies (9). The lemma is proved.

Note that if $g(s, t)$ is smooth, then the condition of smooth gluing implies the equalities $V_s'(g(s, t), s, t) = f_s'(g(s, t), s, t)$ and $V_t'(g(s, t), s, t) = f_t'(g(s, t), s, t)$.

Derive the differential equation for the surface being the boundary of the optimal stopping domain for the process (X_t, M_t, t). Define the stopping moments $\tau_g = \inf\{t > 0 : X_t \le g(M_t, t)\}$, $\tau_0 = \inf\{t > 0 : X_t = M_t\}$, and $\tau_{g0} = \tau_g \wedge \tau_0$. Denote $V_g(x, s, t) = \mathbf{E}_{xst} f(X_{\tau_g}, M_{\tau_g}, \tau_g)$ and $V_g^0(s, t) = V_g(s, s, t)$.

Applying Ito's formula to the function $V_s'(x, s, t)$, we get

$$V_s'(X_{\tau_{g0}}, s, \tau_{g0}) = V_s(x,s,t) + \int_t^{\tau_{g0}} \left(\frac{\partial}{\partial t} + \mathbb{L}_X\right) V_s(X_r, s, r)dr + \int_t^{\tau_{g0}} \sigma(X_r) \frac{\partial V_s}{\partial x}(X_r, s, r) dB_r.$$

Taking a localizing sequence of stopping moments and using Fatou's lemma again, we conclude that $V_s'(x, s, t) = \mathbf{E} V_s'(X_{\tau_{g0}}, s, \tau_{g0})$, or

$$V_s'(x,s,t) = \int_t^T V_s'(s,s,\theta) d\Phi^h(x,s,\theta) + \int_t^T V_s'(g(s,\theta), s, \theta) d\Phi^l(x,s,\theta), \tag{10}$$

where $\Phi^h(x) = P(\tau_{g0} = \tau_0 \le x)$ and $\Phi^l(x) = P(\tau_{g0} = \tau_g \le x)$. The first summand in (10) equals zero due to the principle of normal reflection. Use the smoothness of gluing in s and differentiate with respect to x, We obtain

$$V_{xs}(g(s,t), s, t) = \int_t^T f_s(g(s,\theta), s, \theta) d(\Phi^l)_x'(g(s,t), s, \theta). \tag{11}$$

On the other hand, differentiating the original condition of smooth gluing with respect to s, we get

$$V_{xs}(g(s,t),s,t) + g_s(s,t) V_{xx}(g(s,t),s,t) = f_{xs}(g(s,t),s,t) + g_s(s,t) f_{xx}(g(s,t),s,t),$$

$$V_{xs}''(g(s,t),s,t) = \left(\frac{\partial}{\partial t} + \mathbb{L}_X\right) f(g(s,t),s,t) \cdot g_s'(s,t) + f_{xs}''(g(s,t),s,t).$$

Combining this with (11), we obtain the following equation:

$$g_s'(s,t) = \frac{\sigma^2(g(s,t))}{2} \cdot \frac{\int_t^T f_s'(g(s,\theta),s,\theta) d\psi(s,t,\theta) - f_{xs}''(g(s,t),s,t)}{(f_t' + \mathbb{L}_X f)(g(s,t),s,t)}, \tag{12}$$

where $\psi(\theta,s,t)=(\Phi^l)'_x(g(s,t),s,\theta)$ can be calculated as solutions to the second king Volterra equation

$$\Psi'_x(g(s,t),g(s,r),r)=\int_t^r \Psi(g(s,\theta),g(s,r),r-\theta)d\psi(s,t,\theta)$$

for the distribution function $\Psi(x,y,t)=\mathsf{P}(X_{t\vee\tau_0}\le y\mid X_0=x)$.

Proceed to the proof of the verification theorem.

Definition 2. A semicontinuous from below boundary $g: I\times[0,T)\to I^0$, $g(s,t)\le s$, is called admissible if at the points where $l_I<g(s,t)<s$ the function $g(s,t)$ is the solution to equation (12) and the inequality $\mathbb{L}_X f(g(s,t),s,t)\le 0$ holds for all (s,t), where $g(s,t)>l_I$.

We say that a function f satisfies the single crossing condition if for each s there exists a boundary $\underline{x}(s,t)$ in the interval I such that $\left(\frac{\partial}{\partial t}+\mathbb{L}_X\right)f(x,s,t)\ge 0$ for $x>\underline{x}(s,t)$ and $\left(\frac{\partial}{\partial t}+\mathbb{L}_X\right)f(x,s,t)\le 0$ for $x<\underline{x}(s,t)$.

Define $V_g(x,s,t)$ for $s\ge x\ge g(s,t)$ as the solution to Cauchy problem for the equation

$$\left(\frac{\partial}{\partial t}+\mathbb{L}_X\right)V_g(x,s,t)=0$$

with the instant stopping conditions and conditions of smooth gluing on the boundary g, and also with the condition $V_g(x,s,T)=f(x,s,T)$. In this case we can similarly prove that $V_g(X_t,M_t,t)$ is a local supermartingale.

Lemma 5. *If for all s and t the inequality $g_1(s,t)\le g_2(s,t)$ is valid, then $V_{g_1}(x,s,t)\ge V_{g_2}(x,s,t)$.*

Proof. For V_{g_1} and V_{g_2} we write representations similar to (7) with the use of the local martingales Q^1_r and Q^2_r and the nonincreasing processes P^1_r, P^2_r. Subtract one from the other and take a localizing sequence of stopping moments for $Q^2_r-Q^1_r$. We have

$$(V_{g_2}-V_{g_1})(X_{\tau_1\vee\sigma_n},M_{\tau_1\vee\sigma_n},\tau_1\vee\sigma_n)=V_{g_2}(x,s,t)-V_{g_1}(x,s,t)+(M^2-M^1)_{\tau_1\vee\sigma_n}+(P^2-P^1)_{\tau_1\vee\sigma_n},\quad(13)$$

where τ_1 is the moment when g_1 is attained. The latter summand is equal to $\int\limits_{\tau_2\vee\sigma_n}^{\tau_1\vee\sigma_n}\left(\frac{\partial}{\partial t}+\mathbb{L}_X\right)V_{g_2}(X_r,M_r,r)dr$ and is not less than zero,

For $n\to\infty$ the expression in the right-hand side of equality (13) tends to $V_{g_2}(X_{\tau_1},M_{\tau_1},\tau_1)-V_{g_1}(X_{\tau_1},M_{\tau_1},\tau_1)=f(X_{\tau_1},M_{\tau_1},\tau_1)-f(X_{\tau_1},M_{\tau_1},\tau_1)=0$. Using Fatou's lemma, we get $V_{g_2}(x,s,t)-V_{g_1}(x,s,t)\le 0$, which was required.

Finally, prove the last auxiliary result.

Lemma 6. *There exists the maximal admissible boundary $g_*(s,t)$.*

Proof. In contrast with the case of an infinite time horizon, the form of the equation for the boundary is too complex here to prove the existence of the maximal solution directly. Notice instead than it is sufficient to prove the existence of an admissible solution having the minimal function $(s,t)\mapsto V_g(s,s,t)$.

In fact, suppose $g_1(s,t)$ and $g_2(s,t)$ are two admissible boundaries and let the inequality $V_{g_1}(s,s,t)\le V_{g_2}(s,s,t)$ hold for all $s\in I$ and $t\in[0,T]$. In this case, $V_{g_1}(g_1(s,t),s,t)=f(g_1(s,t),s,t)\le V_{g_2}(g_1(s,t),s,t)$, by Lemma 5. Denote $C_1=\{(x,s,t): s>x>g_1(s,t)\}$, on the boundary of this set we have $V_{g_2}(x,s,t)-V_{g_1}(x,s,t)\ge 0$. Since in this case $\left(\frac{\partial}{\partial t}+\mathbb{L}_X\right)V_{g_1}=0$ inside of C_1, then due to the single crossing condition, inside of C_1 we have the equality $\left(\frac{\partial}{\partial t}+\mathbb{L}_X\right)(V_{g_2}-V_{g_1})\le 0$. Along with the inequality on the boundary of C_1 this means that $V_{g_2}-V_{g_1}\ge 0$ in C_1. Outside of C_1 we similarly get $V_{g_1}(x,s,t)=f(x,s,t)\le V_{g_2}(x,s,t)$.

Thus, if for a certain admissible function g_* and for all (s,t) the equality $V_{g_*}(s,s,t)=\inf\limits_{g\in\mathcal{G}}V_g(s,s,t)$ is valid (here we have denoted the set of all admissible boundaries by $\mathcal{G}$), then for all (x,s,t) we have

$$V_{g_*}(x,s,t)=\inf_{g\in\mathcal{G}}V_g(x,s,t).$$

Now suppose the inequality $g(s,t)>g_*(s,t)$ holds for some admissible function g and some s and t. In this case, $V_g(g(s,t),s,t)=f(g(s,t),s,t)<V_{g_*}(g(s,t),s,t)$, but this contradicts the minimality of $V_g(s,s,t)$.

It remains to show the existence of the function g_* minimizing $V_g(s,s,t)$. Consider the function $\widehat{V}(s,t)=\inf\limits_{g\in\mathcal{G}}V_g(s,s,t)$. The infimum is determined here because the values are bounded from below by the value $f(s,s,t)$.

Finally, define $\widehat{V}(x,s,t)$ as the solution to the Cauchy problem with conditions specified on the diagonal, i.e.,

$$\left(\frac{\partial}{\partial t}+\mathbb{L}_X\right)\widehat{V}(x,s,t)=0,\quad \widehat{V}(x,s,t)\Big|_{x=s}=\widehat{V}(s,t),\quad \frac{\partial\widehat{V}}{\partial x}(x,s,t)\Big|_{x=s}=\frac{\partial\widehat{V}}{\partial s}(x,s,t).$$

The surface $\widehat{V}(x,s,t)$ crosses $f(x,s,t)$ over some curve $g_*(s,t)$ which is admissible because the solution to the Cauchy problem continuously depends on initial conditions.

The final result is given by

Theorem 2. *Let $g_*(s,t)$ be the maximal admissible solution. If the stopping moment $\tau_*=\inf\{t>0: X_t\le g_*(M_t,t)\}$ satisfies the inequality $\mathbf{E}f(X_\tau,M_\tau,\tau)<\infty$, then it is optimal in problem* (8).

Proof. Let τ be an arbitrary stopping moment for the process X. In the representation of V_g we take a localizing sequence of stopping moments σ_n for Q. In this case,

$$\mathbf{E}_{xst}f(X_{\tau\wedge\sigma_n},M_{\tau\wedge\sigma_n},\tau\wedge\sigma_n)\le\mathbf{E}_{xst}V_g(X_{\tau\wedge\sigma_n},M_{\tau\wedge\sigma_n},\tau\wedge\sigma_n)\le V_g(x,s,t)+\mathbf{E}_{xst}Q_{\tau\wedge\sigma_n}=V_g(x,s).$$

Tend n to infinity and use Fatou's lemma. We obtain $\mathbf{E}_{xs}f(X_\tau,M_\tau)\le V_g(x,s)$.

Taking supremum over all possible τ and infimum over all admissible g, we get

$$V_*(x,s)\le\inf_g V_g(x,s)=V_{g_*}(x,s);$$

the equality takes place because of $V_{g_*}(x,s)=\mathbf{E}_{xst}f(X_{\tau_*},M_{\tau_*})$.

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