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On a Class of Semi-Regular Gyrostat Precessions with Variable Gyrostatic Moment

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Abstract—The article considers the problem of the motion of a gyrostat under the action of potential and gyroscopic forces in the case of a variable gyrostatic moment. The conditions for the existence of semi-regular precessions characterized by the constancy of their own rotation rate are studied. A new solution of the equations of the Kirchhoff–Poisson class is constructed, based on a special type of three invariant relations with respect to the main variables of these equations.

Keywords: potential and gyroscopic forces, variable gyrostatic moment, semi-regular precessions

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INTRODUCTION

The precessional motions of rigid bodies occupy a special place in the classical problem of the motion of a heavy solid and its generalizations. Applied problems related to the study of precessions of gyroscopic instruments are considered by Ishlinsky [1]. In the dynamics of a solid, precessions were studied by Grioli [2], Klein and Sommerfeld [3], the author of this article [4], and many other authors (see [5, 6]). Monograph [7] is devoted to the study of the conditions for the existence of precessional motions of a gyrostat with a variable gyrostatic moment. It provides an overview of the results obtained in this problem and formulates the main definitions of a gyrostat. The approaches adopted in the works of Wittenburg [8], Rumyantsev [9], Kharlamov [10] are of great importance in setting the problem of the motion of a gyrostat. Precessional motions are characterized by the property of constancy of the angle between two axes l_1, l_2 passing through a fixed point, one of which (l_1) is connected to the carrier body, and the other (l_2) is motionless in space. In the case when one of the axes of the moving coordinate system contains the l_1 axis, then it is advisable to call such a coordinate system a precessional coordinate system [11]. According to [2, 4], precession motions are divided into classes: if the speeds of precession and proper rotation are constant, then the precession is called regular; if the precession rate is constant, then the precession is called semi-regular of the first type; if only the speed of its own rotation is constant, then the precession is called semi-regular precession of the second type; in other cases, the precession is called a general type precession. The largest number of gyrostat precession movements was established for classes of regular and semi-regular precessions of the first type. It should be noted that the unique cases of precessions of a heavy rigid body described by the Euler–Poisson equations are the regular precessions obtained by Grioli [2] relative to an inclined axis and the case of Dokshevich [12], for which the product of the precession velocities and own rotation. Semi-regular precessions of the second type of a solid and a gyrostat are among the smaller number of precessions found. For example, it was proved in [4] that semiregular precessions of the second type are dynamically impossible in the classical problem. Despite this, in the problem of the motion of a gyrostat with a variable gyrostatic moment, some solutions have been obtained that have this property [7].

In this article, the conditions for the existence of semi-regular precessions of the second type of gyrostat with a variable gyrostatic moment under the action of potential and gyroscopic forces are investigated. The conditions on the parameters of the equations of motion and precession, under which the gyrostat performs a precession of the second type, are indicated.

1. STATEMENT OF THE PROBLEM

When studying the equations of motion of a gyrostat with a constant gyrostatic moment, we should take into account the well-known property of the analogy of the problem of the motion of a gyrostat under the action of potential and gyroscopic forces and the problem of the motion of a body in an ideal fluid, which was proved in a particular case by Steklov [13] and Kharlamov [14], and in the general case by Yahya [15]. For the case of a variable gyrostatic moment $\boldsymbol{\lambda}(t)$, there is no such analogy. Therefore, in this article we will use differential equations in the following form [6, 7, 15]:

$$A\dot{\boldsymbol{\omega}} + \dot{\boldsymbol{\lambda}}(t) = \left(A\boldsymbol{\omega} + \boldsymbol{\lambda}(t)\right) \cdot \boldsymbol{\omega} + \boldsymbol{\omega} \cdot B\boldsymbol{\nu} + \boldsymbol{\nu} \cdot (C\boldsymbol{\nu} - \mathbf{s}), \tag{1.1}$$

$$\dot{\boldsymbol{\nu}} = \boldsymbol{\nu} \cdot \boldsymbol{\omega}, \tag{1.2}$$

where the notation is introduced: $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is the angular velocity vector; $\boldsymbol{\lambda}(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t))$ is the gyrostatic moment vector; $A = \mathrm{diag}(A_1, A_2, A_3)$ is the gyrostat inertia tensor; $B = \mathrm{diag}(B_1, B_2, B_3)$ is a matrix characterizing gyroscopic forces; $C = \mathrm{diag}(C_1, C_2, C_3)$ is a matrix that determines terms that are quadratic in terms of the components of the vector $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$; $\mathbf{s} = (s_1, s_2, s_3)$ is the vector of the generalized center of mass of the gyrostat; the dot over the variables $\boldsymbol{\omega}(t)$, $\boldsymbol{\lambda}(t)$, $\boldsymbol{\nu}(t)$ denotes differentiation with respect to time t.

Equations (1.1), (1.2) have first integrals

$$\boldsymbol{\nu} \cdot \boldsymbol{\nu} = 1, \quad \left(A\boldsymbol{\omega} + \boldsymbol{\lambda}(t)\right) \cdot \boldsymbol{\nu} - \frac{1}{2}(B\boldsymbol{\nu} \cdot \boldsymbol{\nu}) = k, \tag{1.3}$$

where k is an arbitrary constant. All the above quantities are given in the principal moving coordinate system with unit vectors $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$.

System (1.1), (1.2) is a non-autonomous system of differential equations with respect to the variables $\boldsymbol{\omega}(t)$, $\boldsymbol{\lambda}(t)$, $\boldsymbol{\nu}(t)$. Its integration can be based on several approaches. In this article, we will assume that the rotor S_3, which carries the carrier body S_0, lies on the third coordinate axis, that is, $\boldsymbol{\lambda}(t) = (0, 0, \lambda_3(t))$. Then, by virtue of [10], we will consider system (1.1), (1.2) together with the equations

$$\dot{\lambda}_3(t) = L(t), \quad \lambda_3(t) = \mathrm{D}_3[\boldsymbol{\omega}(t) \cdot \mathbf{i}_3 + \dot{\kappa}(t)]. \tag{1.4}$$

here $\dot{\kappa}(t)$ is the speed of rotation of the rotor S_3; D_3 is the moment of inertia of the rotor S_3 relative to the axis of rotation Oz; $L(t)$ is the projection of moments and forces onto the Oz-axis from the side of the carrier body. Equations (1.4) can be studied using two approaches: if the function $L(t)$ is given, then first the function $\lambda_3(t)$ is found from the first equation of system (1.4) and equations (1.1), (1.2) are integrated, and then from the second equation (1.4) the function $\dot{\kappa}(t)$ is defined; if $\dot{\kappa}(t)$ is given and the function $\lambda_3(t)$ is known, then the function $L(t)$ is found from (1.4).

The problem of the motion of a gyrostat with a constant gyrostatic moment based on the Lagrange function was studied in [16].

We will study semiregular precessions using the method [17, 18]. According to this method, the gyrostat angular velocity vector can be represented as

$$\boldsymbol{\omega} = \varepsilon(\nu_3)\boldsymbol{\nu} + g_0\boldsymbol{\beta}, \tag{1.5}$$

where $\varepsilon(\nu_3)$ is a differentiable function; $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$ is a constant unit vector; g_0 is a constant parameter. The case $\varepsilon(\nu_3) = \varepsilon_0$, where ε_0 is a constant, was considered in [19].

We substitute the value (1.5) into equation (1.2):

$$\dot{\boldsymbol{\nu}} = g_0(\boldsymbol{\nu} \cdot \boldsymbol{\beta}). \tag{1.6}$$

From equation (1.6) the first integral follows

$$\beta_1\nu_1 + \beta_2\nu_2 + \beta_3\nu_3 = c_0, \tag{1.7}$$

which in vector form can be represented as follows: $\boldsymbol{\beta} \cdot \boldsymbol{\nu} = c_0$, where c_0 is a constant. That is, due to the equalities $|\boldsymbol{\beta}| = 1$, $|\boldsymbol{\nu}| = 1$, the parameter c_0 satisfies the condition $c_0 < 1$. It follows from (1.5) that the precession rate is equal to $\varepsilon(\nu_3)$, and the intrinsic rotation rate is g_0. That is, the precession of the gyrostat belongs to the type of semi-regular precession of the second type. We write (1.5), (1.6) in scalar form:

$$\omega_i = \varepsilon(\nu_3)\nu_i + g_0\beta_i, \quad (i = \overline{1,3}), \tag{1.8}$$

$$\dot{\nu}_1 = g_0(\beta_3\nu_2 - \beta_2\nu_3), \quad \dot{\nu}_2 = g_0(\beta_1\nu_3 - \beta_3\nu_1), \quad \dot{\nu}_3 = g_0(\beta_2\nu_1 - \beta_1\nu_2). \tag{1.9}$$

Using the equalities $\nu_1^2 + \nu_2^2 + \nu_3^2 = 1$ and (1.7), we find the functions $\nu_1(\nu_3)$, $\nu_2(\nu_3)$:

$$\begin{gathered} \nu_1(\nu_3) = \frac{1}{\kappa_0^2}[\beta_1(c_0 - \beta_3\nu_3) + \beta_2\sqrt{F(\nu_3)}], \\ \nu_2(\nu_3) = \frac{1}{\kappa_0^2}[\beta_2(c_0 - \beta_3\nu_3) - \beta_1\sqrt{F(\nu_3)}], \end{gathered} \tag{1.10}$$

where $\kappa_0^2 = \beta_1^2 + \beta_2^2$, and the function $F(\nu_3)$ is

$$F(\nu_3) = -\nu_3^2 + 2c_0\beta_3\nu_3 + (\kappa_0^2 - c_0^2). \tag{1.11}$$

Substituting $\nu_1(\nu_3)$, $\nu_2(\nu_3)$ from (1.10) into the third equation of system (1.9), we obtain that the function $\nu_3(t)$ can be obtained by inverting the integral [17]

$$\int_{\nu_3^{(0)}}^{\nu_3} \frac{d\nu_3}{\sqrt{F(\nu_3)}} = g_0(t - t_0). \tag{1.12}$$

Then the functions $\nu_i(t)$ $(i = 1,2)$, $\omega_i(t)$ $(i = \overline{1,3})$ are found from equations (1.10), (1.8). Since the function (1.11) satisfies the condition $F(\nu_3) < 0$ for $|\nu_3| > 1$, then for a real value of the parameter $\mu_0 = \sqrt{1 - c_0^2}$ the roots of the equation $F(\nu_3) = 0$ are real. That is, from (1.11), (1.12) we find

$$(\nu_3)_{1,2} = c_0\beta_3 \pm \mu_0\kappa_0\sin\psi, \tag{1.13}$$

where, by virtue of the third equation from (1.9), $\psi = g_0 t$. Choosing the plus sign in (1.13) for definiteness, from (1.13), (1.10) we obtain

$$\begin{gathered} \nu_1(\psi) = h_0 + h_1\cos\psi + h_2\sin\psi, \quad \nu_2(\psi) = r_0 + r_1\cos\psi + r_2\sin\psi, \\ \nu_3(\psi) = a_0 + a_2\sin\psi. \end{gathered} \tag{1.14}$$

Here we introduced the notation

$$\begin{gathered} h_0 = c_0\beta_1, \quad h_1 = \frac{\beta_2\mu_0}{\kappa_0}, \quad h_2 = -\frac{\beta_1\beta_3\mu_0}{\kappa_0}, \\ r_0 = c_0\beta_2, \quad r_1 = -\frac{\beta_1\mu_0}{\kappa_0}, \quad r_2 = -\frac{\beta_2\beta_3\mu_0}{\kappa_0}, \quad a_0 = c_0\beta_3, \quad a_2 = \kappa_0\mu_0. \end{gathered} \tag{1.15}$$

Let us note the form of solution (1.14) and notation (1.15) in the case of $\beta_3 = 0$, which was considered in [19] when studying regular gyrostat precessions $(\varepsilon(\nu_3) = \varepsilon_0)$:

$$\begin{gathered} \nu_1(\psi) = h_0 + h_1\cos\psi, \quad \nu_2(\psi) = r_0 + r_1\cos\psi, \quad \nu_3(\psi) = a_2\sin\psi, \\ h_0 = c\beta_1, \quad h_1 = \beta_2\mu_0, \quad h_2 = 0, \quad r_0 = c_0\beta_2, \\ r_1 = -\beta_1\mu_0, \quad r_2 = 0, \quad a_0 = 0, \quad a_2 = \mu_0. \end{gathered} \tag{1.16}$$

Let us pose the problem: to determine the conditions for the existence of a solution (1.8), (1.14) of equation (1.1).

2. STUDY OF EQUATION (1.1)

Let us write equation (1.1) in scalar form:

$$A_1\dot{\omega}_1 = (A_2 - A_3)\omega_2\omega_3 - \lambda_3(t)\omega_2 + \omega_2 B_3\nu_3 - \omega_3 B_2\nu_2 + s_2\nu_3 - s_3\nu_2 + (C_3 - C_2)\nu_2\nu_3, \tag{2.1}$$

$$A_2\dot{\omega}_2 = (A_3 - A_1)\omega_3\omega_1 + \lambda_3(t)\omega_1 + \omega_3 B_1\nu_1 - \omega_1 B_3\nu_3 + s_3\nu_1 - s_1\nu_3 + (C_1 - C_3)\nu_3\nu_1, \tag{2.2}$$

$$\dot{\lambda}_3(t) + A_3\dot{\omega}_3 = (A_1 - A_2)\omega_1\omega_2 + \omega_1 B_2\nu_2 - \omega_2 B_1\nu_1 + s_1\nu_2 - s_2\nu_1 + (C_2 - C_1)\nu_1\nu_2. \tag{2.3}$$

Let us substitute expressions ω_i from (1.8) into (2.1)–(2.3) and use equations (1.9). Then we get a system of three differential equations

$$\begin{aligned}\lambda_3(t)(v_2\varepsilon(\nu_3) + \beta_2 g_0) &= A_1 g_0 v_1\varepsilon'(\nu_3)(\beta_1 v_2 - \beta_2 v_1) + A_1 g_0\varepsilon(\nu_3)(\beta_2 v_3 - \beta_3 v_2)\\ &+ \nu_2\nu_3[\varepsilon^2(\nu_3)(A_2 - A_3) + \varepsilon(\nu_3)(B_3 - B_2) + C_3 - C_2]\\ &+ \nu_2\{\beta_3 g_0[\varepsilon(\nu_3)(A_2 - A_3) - B_2] - s_3\}\\ &+ \nu_3\{\beta_2 g_0[\varepsilon(\nu_3)(A_2 - A_3) + B_3] + s_2\} + \beta_2\beta_3 g_0^2(A_2 - A_3)\end{aligned} \tag{2.4}$$

$$\begin{aligned}-\lambda_3(t)(v_1\varepsilon(\nu_3) + \beta_1 g_0) &= A_2 g_0 v_2\varepsilon'(\nu_3)(\beta_1 v_2 - \beta_2 v_1) + A_2 g_0\varepsilon(\nu_3)(\beta_3 v_1 - \beta_1 v_3)\\ &+ \nu_1\nu_3[\varepsilon^2(\nu_3)(A_3 - A_1) + \varepsilon(\nu_3)(B_1 - B_3) + C_1 - C_3]\\ &+ \nu_1\{\beta_3 g_0[\varepsilon(\nu_3)(A_3 - A_1) + B_1] + s_3\}\\ &+ \nu_3\{\beta_1 g_0[\varepsilon(\nu_3)(A_3 - A_1) - B_3] - s_1\} + \beta_1\beta_3 g_0^2(A_3 - A_1),\end{aligned} \tag{2.5}$$

$$\begin{aligned}\dot{\lambda}_3(t) &= A_3 v_3\varepsilon'(\nu_3)(\beta_1 v_2 - \beta_2 v_1) + A_3 g_0\varepsilon(\nu_3)(\beta_1 v_2 - \beta_2 v_1)\\ &+ \nu_1\nu_2[\varepsilon^2(\nu_3)(A_1 - A_2) + \varepsilon(\nu_3)(B_2 - B_1) + C_2 - C_1]\\ &+ \nu_1\{\beta_2 g_0[\varepsilon(\nu_3)(A_1 - A_2) - B_1] - s_2\}\\ &+ \nu_2\{\beta_1 g_0[\varepsilon(\nu_3)(A_1 - A_2) + B_2] + s_1\} + \beta_1\beta_2 g_0^2(A_1 - A_2).\end{aligned} \tag{2.6}$$

The choice of the form of differential equations (2.4)–(2.6) is related to the applied technique for studying them in further transformations.

In some cases, it is advisable to use equations (2.1), (2.2), excluding the function $\lambda_3(t)$ from them:

$$\begin{aligned}\frac{1}{2}(A_1\omega_1^2 + A_2\omega_2^2)^{\bullet} &= (A_2 - A_1)\omega_1\omega_2\omega_3 + \omega_3(B_1\nu_1\omega_2 - B_2\nu_2\omega_1)\\ &+ s_3(\omega_2\nu_1 - \omega_1\nu_2) + \nu_3(\omega_1 s_2 - \omega_2 s_1) + \nu_3[\nu_2\omega_1(C_3 - C_2) - \nu_1\omega_2(C_3 - C_1)].\end{aligned} \tag{2.7}$$

In the general case, substituting functions (1.14) into equations (2.4), (2.5) and excluding the function $\lambda_3(t)$ from the obtained equations, we arrive at a Riccati type equation, the solution of which is not possible to establish. Therefore, in further transformations, we assume that following conditions hold:

$$A_2 = A_1, \quad B_2 = B_1, \quad C_2 = C_1. \tag{2.8}$$

In addition, we will consider two independent options for additional restrictions on parameters:

$$1.\ s_3 = 0, \quad s_1 = d_0\beta_1, \quad s_2 = d_0\beta_2 \tag{2.9}$$

$$2.\ s_1 = 0, \quad s_2 = 0, \quad s_3 \neq 0, \tag{2.10}$$

where d_0 is a parameter. In case (2.9), we transform equation (2.7) on the basis of (1.8), (1.9) to the form

$$\frac{1}{2}A_1(\omega_1^2 + \omega_2^2)'_{\nu_3} = \frac{d_0 + g_0 B_1}{g_0}\nu_3\varepsilon(\nu_3) + B_1 g_0\beta_3 + (C_1 - C_3)\nu_3. \tag{2.11}$$

By virtue of (2.8), (2.9), these equalities can be interpreted as generalized S.V. Kovalevskaya conditions.

When conditions (2.10) are satisfied, we represent equation (2.7) as follows:

$$\frac{1}{2}A_1(\omega_1^2 + \omega_2^2)'_{\nu_3} = B_1\nu_3\varepsilon(\nu_3) + (s_3 + \beta_3 g_0 B_1) + (C_1 - C_3)\nu_3. \tag{2.12}$$

Analogues of the first integrals following from (2.12) were considered in [20]. We write equation (2.3) in case (2.9):

$$A_3\omega_3(\nu_3)+\lambda_3(\nu_3)=-\frac{d_0+B_1g_0}{g_0}\nu_3+B_0, \tag{2.13}$$

where B_0 is an arbitrary constant. If we take into account conditions (2.10) in equation (2.3) and the third equation from (1.9), then we obtain

$$A_3\omega_3(\nu_3)+\lambda_3(\nu_3)=-B_1\nu_3+l_0. \tag{2.14}$$

Here, l_0 is an arbitrary constant. The analogy of equations (2.11) and (2.12), as well as (2.13), (2.14) is obvious.

Let us consider a linear combination of equations (2.4), (2.5), multiplying equation (2.4) by ν_1, equation (2.5) by ν_2, and adding the left and right parts of the resulting equations. Then, by virtue of the equation $\dot{\nu}_3=g_0(\beta_2\nu_1-\beta_1\nu_2)$, we find

$$\begin{gathered}\{A_1[(\nu_3^2-1)\varepsilon(\nu_3)]'-A_3\nu_3\varepsilon(\nu_3)+B_3\nu_3+g_0\beta_3(A_1-A_3)-\lambda_3(\nu_3)\}\dot{\nu}_3\\+s_2\nu_1-s_1\nu_2=0.\end{gathered} \tag{2.15}$$

In case (2.9), (2.13), it follows from (2.15)

$$\varepsilon(\nu_3)=\frac{1}{1-\nu_3^2}(E_2\nu_3^2+E_1\nu_3+E_0), \tag{2.16}$$

where E_0 is an arbitrary constant and E_2, E_1 are:

$$E_2=\frac{1}{2g_0A_1}[g_0(B_1+B_3)+2d_0],\quad E_1=\frac{1}{A_1}(\beta_3g_0A_1-B_0). \tag{2.17}$$

Let us write equation (2.15) under conditions (2.10), (2.14):

$$\varepsilon(\nu_3)=\frac{1}{1-\nu_3^2}(G_2\nu_3^2+G_1\nu_3+G_0), \tag{2.18}$$

where G_0 is an arbitrary constant and G_2, G_1 have the form

$$G_2=\frac{1}{2A_1}(B_1+B_3),\quad G_1=\frac{\beta_3g_0A_1-l_0}{A_1}. \tag{2.19}$$

It follows from (2.17) and (2.19) that the quantity E_2 at $d_0=0$ will take the value G_2, and to obtain the value G_1 from (2.17), we must assume $B_0=l_0$. However, equations (2.11), (2.12) do not have such an analogy.

3. CASE (2.9)

At the first stage, we will study this case under the condition

$$d_0+g_0B_1=0. \tag{3.1}$$

When equality (3.1) is satisfied, we write equation (2.11) in the form of its first integral

$$\begin{gathered}\frac{A_1}{2}[(1-\nu_3^2)\varepsilon^2(\nu_3)+2g_0\varepsilon(\nu_3)(c_0-\beta_3\nu_3-g_0B_1\beta_3\nu_3)\\-\beta_3g_0B_1\nu_3+\frac{1}{2}(C_3-C_1)\nu_3^2=D_0,\end{gathered} \tag{3.2}$$

where D_0 is an arbitrary constant. In equation (3.2), we take into account the invariant relation (1.8). Let us substitute the value $\varepsilon(\nu_3)$ from equation (2.16) into equality (3.2) and require that the resulting equality be an identity with respect to the variable ν_3. Then we obtain the following algebraic system for the parameters of the problem:

$$C_1-C_3+A_1E_2^2=0, \tag{3.3}$$

$$B_0 = \frac{g_0\beta_3 B_1}{E_2}, \quad E_1^2 + (E_0 + E_2)^2 + 2g_0c_0(E_0 + E_2) + E_1(E_1 - 2g_0\beta_3) = 0, \tag{3.4}$$

$$E_0(E_1 - \beta_3 g_0) + 2g_0c_0E_1 - \frac{\beta_3 g_0 B_1 - l_0}{A_1} = 0, \tag{3.5}$$

$$D_0 = \frac{A_1 E_0}{2}(E_0 + 2g_0c_0). \tag{3.6}$$

We shall considered equation (3.3) as a condition on the parameters C_1, C_3. The first equality from (3.4), due to (2.17) and assumption (3.1), for which the value of q has the form

$$E_2 = \frac{B_3 - B_1}{2A_1}. \tag{3.7}$$

let's write it like this

$$B_0 = \frac{2\beta_3 g_0 A_1 B_1}{B_3 - B_1}. \tag{3.8}$$

Based on the value (3.8), we express the parameter E_1 from (2.17) in terms of the parameters of the problem:

$$E_1 = \frac{\beta_3 g_0(B_3 - 3B_1)}{B_3 - B_1}. \tag{3.9}$$

From the second equation of system (3.4) and equation (3.6), we find the values of c_0, E_0

$$c_0 = \delta_0 \frac{2\beta_3 B_1}{B_3 - B_1}, \quad E_0 = \delta_0 E_1 - \frac{B_3 - B_1}{2A_1}, \tag{3.10}$$

where $\delta_0 = \pm 1$. Equation (3.6) on the basis of (3.10) serves to determine the constant D_0. Let us study the function (2.16) taking into account (3.7), (3.10). For definiteness, we set $\delta_0 = 1$. Then

$$\varepsilon(\nu_3) = \frac{1}{1 - \nu_3}(E_2\nu_3 + E_2^*), \quad E_2^* = \frac{2\beta_3 g_0(B_3 - 3B_1) - (B_3 - B_1)^2}{2A_1(B_3 - B_1)}. \tag{3.11}$$

In order for the function $\varepsilon(\nu_3)$ not to take a constant value, we assume that the conditions $\beta_3 \neq 0$, $B_3 - 3B_1 \neq 0$ are satisfied. Thus, the function $\varepsilon(\nu_3)$ from (3.11) is a linear-fractional function of $\nu_3 = a_0 + a_2 \sin g_0 t$. Let us indicate the value of $\lambda_3(\nu_3)$ in the case under consideration. From equations (2.13), (3.11) we have

$$\lambda_3(\nu_3) = \frac{\beta_3 g_0[2A_1B_1 - A_3(B_3 - B_1)]}{B_3 - B_1} - \frac{A_3\nu_3(E_2\nu_3 + E_2^*)}{1 - \nu_3}. \tag{3.12}$$

For the final solution of the problem of the conditions for the existence of semi-regular gyrostat precessions, it is necessary to consider equations (1.4) together with the value of function (3.12), setting $\nu_3(t) = a_0 + a_2 \sin g_0 t$. Due to the obviousness of these transformations, we confine ourselves to their explanations.

4. CASE $d_0 + g_0 B_1 \neq 0$

Let us consider equation (2.11) under the condition $d_0 + g_0 B_1 \neq 0$. The function $\varepsilon(\nu_3)$ has the form (2.16) with the notation (2.17). Let us substitute ω_1, ω_2 from (1.8) into it and take equations (1.9) into account. Then we get a differential equation for the function $\varepsilon(\nu_3)$:

$$\begin{gathered}A_1\varepsilon'(\nu_3)[(1-\nu_3^2)\varepsilon(\nu_3)+g_0(c_0-\beta_3)\nu_3]-A_1\nu_3\varepsilon^2(\nu_3)\\ -\frac{1}{g_0}\varepsilon(\nu_3)[(d_0+B_1g_0)\nu_3+g_0^2\beta_3A_1]+g_0(C_3-C_1)\nu_3-\beta_3g_0B_1=0.\end{gathered}\tag{4.1}$$

We introduce the function (2.16) into equation (4.1). It is convenient to present the result as follows:

$$\begin{gathered}A_1[E_1\nu_3^2+2(E_0+E_2)\nu_3+E_1](E_2\nu_3^2+\tilde{E}_1\nu_3+\tilde{E}_0)\\ -A_1\nu_3(E_2\nu_3^2+E_1\nu_3+E_0)^2-(1-\nu_3^2)(E_2\nu_3^2+E_1\nu_3+E_0)(R_0+R_1\nu_3)\\ +(1-\nu_3^2)^2[(C_3-C_1)\nu_3-\beta_3B_1g_0]=0,\end{gathered}\tag{4.2}$$

where

$$\tilde{E}_0=E_0+c_0g_0,\quad \tilde{E}_1=E_1-\beta_3g_0,\quad R_1=\frac{d_0+g_0B_1}{g_0},\quad R_0=\beta_3g_0A_1.\tag{4.3}$$

Let us require that equation (4.2) be an identity in the variable ν_3. The zero coefficient of ν_3^5 has the form

$$E_2R_1+C_3-C_1-A_1E_2^2=0.\tag{4.4}$$

To simplify other conditions, we express the quantity C_3-C_1 from equality (4.4) and substitute it into (4.2). Let us write down the equality to zero of the coefficient at ν_3^3. Using equations (4.3), we obtain

$$R_1(E_0+E_2)=0.\tag{4.5}$$

Let us show that the equality $R_1=0$ must hold. Suppose the contrary; then from (4.5) we have

$$E_0+E_2=0.\tag{4.6}$$

Condition (4.6) allows us to consider equation (4.2) for $\nu_3=\pm1$:

$$E_1(2c_0g_0\pm E_1\mp 2g_0\beta_3)=0.\tag{4.7}$$

The case $E_1=0$, due to (4.6) for the function $\varepsilon(\nu_3)$ from (2.16), leads to a constant value $\varepsilon(\nu_3)$. Therefore, in (4.7) it is necessary to put

$$c_0=0,\quad E_1=2g_0\beta_3.\tag{4.8}$$

From the equalities to zero in equation (4.2) of the coefficients at ν_3^4 and the free term, it follows

$$E_1R_1=E_2R_0+g_0B_1\beta_3,\quad R_0E_0-g_0B_1\beta_3=0.\tag{4.9}$$

Since by assumption (4.6) $E_0=-E_2$, then by virtue of $E_1\neq 0$ from (4.9) we obtain $R_1=0$, which was to be proved. So, the option when the parameters satisfy the condition $d_0+g_0B_1\neq 0$ is impossible.

5. CASE (2.10)

Let us consider equations (2.11), (2.12) and functions (2.13), (2.14), (2.16), (2.18). In order to study this option, we can formally go from equation (2.11) to equation (2.12), from equation (2.13) go to equation (2.14), from function (2.16) to function (2.18), then case (2.10) is obtained from case (2.9) by setting $d_0=0$ $B_0=l_0$ in it and replacing the expression $B_1g_0\beta_3$ with the expression $s_3+B_1g_0\beta_3$. Therefore, equation (4.1) will correspond to an equation in which $R_1=B_1$, and instead of the last term in (4.1) it is necessary to consider the term $-(s_3+B_1g_0\beta_3)$. Taking into account the result of section 4, in this variant we obtain $R_1=B_1=0$. That is, the quadratic form $B_1\nu_1^2+B_2\nu_2^2+B_3\nu_3^2$ becomes degenerate $B_3\nu_3^2$. This result is of no interest for the dynamics of a gyrostat under the action of potential and gyroscopic forces.

6. CONCLUSIONS

When considering the conditions for the existence of semi-regular precessions of the second type of gyrostat under the action of potential and gyroscopic forces, a new solution of the equations of motion is constructed, in which the property of variability of the gyrostatic moment is taken into account. This solution is characterized by three invariant relations: (1.8) and formulas (1.14). The key conditions for the existence of this solution are equalities (2.8), (2.9), which characterize the mass distribution, which can be attributed to the generalized S.V. Kovalevskaya conditions. The solution is described by elementary functions of time, and the carrier body precession rate is a linear-fractional function of the trigonometric function $\sin g_0 t$. In the general case, the solution of the problem posed is quite difficult. It can be described as follows: at the first stage of studying the existence conditions, based on the second integral from (1.3), using the invariant relation (1.8) and solving (1.14), (1.15) (in a particular case, instead of (1.14), (1.15) we can draw (1.16)) the function $\lambda_3(\psi)$ is defined; at the second stage, this function is substituted into equations (2.4)–(2.6) and three differential equations for the function $\varepsilon(\psi)$ are found (obviously, they will be dependent); at the third stage, the problem of conditions for integrating the obtained equations in quadratures is studied.

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