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On Analogs of the Bobylev–Steklov Case for a Gyrostat under the Action of a Moment of Gyroscopic Forces

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Abstract—Equations of motion of a gyrostat around a fixed point under the action of a moment of gyroscopic forces are studied. Analogs of the Bobylev–Steklov case are obtained; it is shown that, unlike the classical case of a rigid body, Bobylev's and Steklov's approaches are not equivalent and can provide complementary results. Conditions have been found under which parametric families of particular solutions expressed in terms of elliptic functions can be constructed. Six types of stationary solutions are singled out, and the conditions for their stability are obtained using the method of the Chetayev integral connections.

Keywords: gyrostat, Bobylev–Steklov case, parametric families of partial solutions, stationary solutions, stability

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1. INTRODUCTION

In the dynamics of a rigid body with a fixed point, of significant importance are both the classical cases of complete integrability (Euler, Lagrange, and Kovalevskaya cases) and cases of partial integrability in which parametric families of accurate solutions can be obtained. Such a partially integrable case with a three-parametric family of solutions was found in 1893 independently by Bobylev [1] and Steklov [2]. Although the approaches proposed in [1, 2] are formally different, they are essentially equivalent so the monographs on the dynamics of a rigid body use a unified term, the "Bobylev–Steklov case", and presently only one of these approaches: for example, monograph [3] describes Steklov's approach, monograph [4], Bobylev's approach, and in monograph [5], the Bobylev–Steklov case is presented on the basis of the Bobylev approach using the Hamiltonian.

The studies commenced in [1, 2] are now successfully developed in several directions. Study [6] examined asymptotic motions of a heavy rigid body the limiting motion of which is described by the Bobylev–Steklov solution. Work [7] studied the problem of orbital stability of periodic solutions for a heavy rigid body with a fixed point in the Bobylev–Steklov case. Kovacic algorithm [8] was applied in the problem, which made it possible to determine, under certain conditions, properties of the solutions of the periodic linear-approximation system and make on this basis conclusions about its stability.

Bobylev's approach was generalized by Kharlamov [9] for a gyrostat represented as a rigid body with cavities filled-in with an ideal liquid. Under certain conditions [9], it is possible to derive a system of solutions for the equations of motion of the gyrostat, which are expressed in terms of elliptical functions. An analog of the Bobylev–Kharlamov case was obtained [10] for the equations of motion of the gyrostat in a pseudo-Euclidian space.

Studied here are equations of motion of a gyrostat with a fixed point under the effect of the moment of forces (potential, gyroscopic, or circular gyroscopic). The primary goal was to obtain an analog of the Bobylev–Steklov case for the gyrostat under the action of the moment of gyroscopic forces. In this case Bobylev's and Steklov's approaches are not equivalent and yield the results that are complementary for the construction of parametric families of particular solutions. The construction of stationary solutions and the obtainment of the conditions of their stability using the method of Chetayev integral connections [11] are discussed.

2. EQUATIONS OF MOTION, FIRST INTEGRALS, AND DESCRIPTION OF THE PROBLEM

We consider the vector form of the equations of motion of the gyrostat with a fixed point under the action of moment of forces

$$I\dot{\omega} = (I\omega + \lambda)\times\omega + M, \tag{2.1}$$

$$\dot{\gamma} = \gamma\times\omega. \tag{2.2}$$

Here, $\omega = \mathrm{col}(p,q,r)$ is the angular speed vector; $\gamma = \mathrm{col}(\gamma_1,\gamma_2,\gamma_3)$ is the unit vector of the force field symmetry axis, which is specified by projections on the axes of the body-fixed reference frame; $I = I^T > 0$ is the symmetric positive-definite matrix of the tensor of inertia with respect to the fixed point; $\boldsymbol{\lambda} = \mathrm{col}(\lambda_1,\lambda_2,\lambda_3)$ is the gyrostatic moment vector; and $M = M(t,\gamma,\omega)$ is the vector of the moment of the force acting on the gyrostat. Following [12–14], we consider as the first integrals the following functions

$$J_1 = J_1(\gamma,\omega) = \omega^T I\omega + 2U(\gamma) = c_1 = \mathrm{const}, \tag{2.3}$$

$$J_2 = J_2(\gamma.\omega) = \gamma^T(I\omega+\lambda) + \frac{1}{2}\gamma^T S\gamma = c_2 = \mathrm{const}, \tag{2.4}$$

$$J_3 = J_3(\gamma) = \gamma^T\gamma = 1, \tag{2.5}$$

where $S = S^T$ is a symmetric matrix.

It should be noted that geometric integral (2.5) is available for any choice of moment $M = M(t,\gamma,\omega)$. However, for system (2.1) and (2.2) to have energy integral (2.3) and area integral (2.4), moment $M = M(t,\gamma,\omega)$ may not be arbitrary but should fulfill certain conditions. These necessary and sufficient conditions are specified by the following assertion proven in [15].

Assertion 1. For functions (2.3) and (2.4) to be the first integrals for system (2.1) and (2.2), it is necessary and sufficient that moment M is representable in form

$$M = \gamma\times\frac{\partial U}{\partial\gamma} - \omega\times S\gamma + L(t,\gamma,\omega)\,\omega\times\gamma, \tag{2.6}$$

where $L(t,\gamma,\omega)$ is an arbitrary function.

This assertion shows that first integrals (2.3) and (2.4) determine moment M in the right side of Eq. (2.1) in a unique way with an accuracy of a circular gyroscopic component $L(t,\gamma,\omega)\,\omega\times\gamma$. The first two components in the formula for moment (2.6) are, respectively, the moment of potential forces $\gamma\times\frac{\partial U}{\partial\gamma}$ with a potential $U(\gamma)$ and the moment of gyroscopic forces $-\omega\times S\gamma$, specified by matrix S.

Everywhere below we consider the matrix of inertia a diagonal one $I = \mathrm{diag}(A,B,C)$ and the potential, linear $U = a\gamma_1 + b\gamma_2 + c\gamma_3$ (this corresponds to a heavy rigid body), and consider the matrix that sets gyroscopic forces also diagonal $S = \mathrm{diag}(k_1,k_2,k_3)$. We represent system (2.1) and (2.2) in coordinate form

$$\begin{aligned} A\dot{p} &= (B-C)qr + \lambda_2 r - \lambda_3 q + c\gamma_2 - b\gamma_3 + k_2\gamma_2 r - k_3\gamma_3 q + L(q\gamma_3 - r\gamma_2),\\ B\dot{q} &= (C-A)pr + \lambda_3 p - \lambda_1 r + a\gamma_3 - c\gamma_1 + k_3\gamma_3 p - k_1\gamma_1 r + L(r\gamma_1 - p\gamma_3), \end{aligned} \tag{2.7}$$

$$\begin{gathered} C\dot{r} = (A-B)pq + \lambda_1 q - \lambda_2 p + b\gamma_1 - a\gamma_2 + k_1\gamma_1 q - k_2\gamma_2 p + L(p\gamma_2 - q\gamma_1),\\ \dot{\gamma}_1 = r\gamma_2 - q\gamma_3, \quad \dot{\gamma}_2 = p\gamma_3 - r\gamma_1, \quad \dot{\gamma}_3 = q\gamma_1 - p\gamma_2. \end{gathered} \tag{2.8}$$

Here, $L = L(t,\gamma,\omega)$ is a continuous function of t, γ, and ω.

The goal of this study was to:

(1) Find analogs of the Bobylev–Steklov case [1, 2] for system (2.7) and (2.8) and perform integration of the equations of motion for these cases.

(2) Identify stationary solutions specified by the constants that the right sides of equations of motion (2.7) and (2.8) are zero.

(3) Employ the first integrals to obtain, using the method of Chetayev integral connection [11], sufficient conditions for the stability of the identified stationary solutions.

The analysis has shown that the analogs of the Bobylev—Steklov case for system (2.7) and (2.8) can only be obtained under the following additional conditions: $\lambda_3 = 0$, $a = c = 0$, $k_2 = k_3 = 0$, $L = 0$. Equations of motion (2.7) can be represented then in form

$$A\dot{p} = (B - C)qr + \lambda_2 r - b\gamma_3,$$
$$B\dot{q} = (C - A)pr - \lambda_1 r - k_1\gamma_1 r, \tag{2.9}$$
$$C\dot{r} = (A - B)pq + \lambda_1 q - \lambda_2 p + b\gamma_1 + k_1\gamma_1 q.$$

If the gyrostatic moment is absent ($\lambda_1 = \lambda_2 = 0$), the moment of gyroscopic forces is not operative $(k_1 = 0)$, and the moments of inertia satisfy conditions $B = 2A$, system (2.8) and (2.9) corresponds to the classical Bobylev—Steklov case [1, 2].

Integrals (2.3) and (2.4) for system (2.8) and (2.9) are represented then in the following way

$$J_1 = Ap^2 + Bq^2 + Cr^2 + 2b\gamma_2 = c_1 = \text{const}, \tag{2.10}$$

$$J_2 = \gamma_1(Ap + \lambda_1) + \gamma_2(Bq + \lambda_2) + Cr\gamma_3 + \frac{1}{2}k_1\gamma_1^2 = c_2 = \text{const}. \tag{2.11}$$

3. CONSTRUCTING SOLUTIONS USING THE STEKLOV METHOD

In this section, to construct solutions of the gyrostat equations with the moment of gyroscopic forces (2.8) and (2.9), we apply the method proposed by Steklov [2] for equations of a heavy rigid body (see also [3]). Following [2], we seek the solution of (2.8) and (2.9) in form

$$p(t) = a_0 + a_1\gamma_1(t), \quad q(t) = q_0 = \text{const}, \quad r(t) = 0, \tag{3.1}$$

where a_0, a_1 are some real constants to be determined (it is assumed [2, 3] that $a_0 = 0$). Substituting (3.1) into system (2.9) we arrive at identities

$$Aa_1\dot{\gamma}_1 \equiv -b\gamma_3,$$
$$[(A - B)a_0 q_0 + \lambda_1 q_0 - \lambda_2 a_0] + [(A - B)a_1 q_0 - \lambda_2 a_1 + b + k_1 q_0]\gamma_1 \equiv 0.$$

Substituting (3.1) into system (2.8) we obtain a system of three differential equations from which γ_1, γ_2, and γ_3 should be found,

$$\dot{\gamma}_1 = -q_0\gamma_3, \quad \dot{\gamma}_2 = (a_0 + a_1\gamma_1)\gamma_3, \quad \dot{\gamma}_3 = q_0\gamma_1 - (a_0 + a_1\gamma_1)\gamma_2. \tag{3.2}$$

It follows then that $q_0 \neq 0, a_0, a_1$ should satisfy a system of three algebraic equations

$$[(A - B)q_0 - \lambda_2]a_0 + \lambda_1 q_0 = 0, \tag{3.3}$$
$$[(A - B)q_0 - \lambda_2]a_1 + b + k_1 q_0 = 0; \quad q_0 = b/(Aa_1).$$

Depending on the conditions set for parameters $A, B, b, \lambda_1, \lambda_2$, and k_1 system (3.3) has the following solutions q_0, a_0, and a_1:

If $\lambda_2 = k_1 = 0$, $B = 2A$, then $a_0 = \lambda_1/A$, $a_1 = b/(Aq_0)$, and $q_0 \neq 0$ is any real number.

If $\lambda_1 = 0, (A - B)b + k_1\lambda_2 = 0$, $\lambda_2 \neq 0$, $B \neq A$, $q_0 = \lambda_2/(A - B)$, $a_1 = b(A - B)/(A\lambda_2)$, a_0 is any real number.

If $k_1 \neq 0$, $B \neq A$, $D_1 = b^2(2A - B)^2 + 4Abk_1\lambda_2 \geq 0$, then q_0 may be any of the numbers $q_0 = (b(B - 2A) \pm \sqrt{D_1})/(2Ak_1)$ that are other than zero and $\lambda_2/(A - B)$, while a_0 and a_1 are calculated using formulas $a_0 = (\lambda_1 q_0)/((B - A)q_0 + \lambda_2)$, $a_1 = b/(Aq_0)$.

We now carry out integration of system (3.2). This system has integrals $J_3 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$ and $J_4 = a_0\gamma_1 + 0.5a_1\gamma_1^2 + q_0\gamma_2 = c_4 = \text{const}$. Using these integrals we represent γ_2 and γ_3 in terms of γ_1:

$$\gamma_2 = \frac{1}{q_0}\left(c_4 - a_0\gamma_1 - 0.5a_1\gamma_1^2\right),$$
$$\gamma_3 = F(\gamma_1) = \pm\sqrt{1 - \gamma_1^2 - \frac{1}{q_0^2}\left(c_4 - a_0\gamma_1 - 0.5\gamma_1^2\right)^2}.$$

The $\gamma_1(t)$ function can now be found from the first equation of system (3.2) by inverting elliptic integral

$$\int \frac{d\gamma_1}{F(\gamma_1)} = -q_0(t + c_5). \tag{3.5}$$

This procedure shows the validity of the assertions below.

Assertion 2. If $\lambda_2 = k_1 = 0$, $B = 2A$, system (2.8) and (2.9) has a family of solutions (3.1), (3.4), and (3.5), where $q_0 \neq 0$ is any real number and $a_0 = \lambda_1/A$, $a_1 = b/(Aq_0)a_1 = b/(Aq_0)$.

Assertion 3. If $\lambda_1 = 0, (A - B)b + k_1\lambda_2 = 0$, $\lambda_2 \neq 0$, $B \neq A$, system (2.8) and (2.9) has a family of solutions (3.1), (3.4), and (3.5), where $q_0 = \lambda_2/(A - B)$, $a_1 = b(A - B)/(A\lambda_2)$, and a_0 is any real number.

Assertion 4. If $k_1 \neq 0$, $B \neq A$, $D_1 = b^2(2A - B)^2 + 4Abk_1\lambda_2 \geq 0$, system (2.8) and (2.9) has a family of solutions (3.1), (3.4), and (3.5), where q_0 may be any of the two numbers $q_0^{\pm} = [b(B - 2A) \pm \sqrt{D_1}]/(2Ak_1)$ that are other than zero and $\lambda_2/(A - B)$, while numbers a_0 and a_1 are given by formulas $a_0 = \lambda_1 q_0/((B - A)q_0 + \lambda_2)$, $a_1 = b/(Aq_0)$.

We have found in this way that under conditions of assertions 2–4 the solutions of system (2.8) and (2.9) included in them are expressed in terms of elliptic functions of time. Assertion 2 yields a three-parametric family of solutions (parameters q_0, c_4, c_5); assertion 3, a three-parametric family of solutions (parameters a_0, c_4, c_5); and assertion 4 yields two two-parametric families of solutions (parameters c_4, c_5).

It is known [3] that an elliptic integral of form (3.5) can be represented in terms of elementary functions only provided that the fourth-power polynomial in the radicand has multiple roots. Sometimes this feature enables obtainment of an accurate solution of the system of equations for gyrostat (2.8) and (2.9) explicitly represented in terms of elementary functions.

Example 1. We consider a three-parametric family of the systems of form (2.8) and (2.9), where free parameters are A, C, and λ_2, which fulfill inequalities $0 < A < C < 3A$, $\lambda_2^2 \neq A^2$, while other coefficients, B, λ_1, k_1, and b, are expressed in terms of the parameters using formulas

$$B = 2A, \quad \lambda_1 = -\frac{\sqrt{3}(A + \lambda_2)}{9}, \quad k_1 = \frac{8\sqrt{3}\lambda_2}{9}, \quad b = \frac{8\sqrt{3}A}{9}.$$

It follows then from assertion 4 that each system in the family described above has accurate solution

$$p(t) = \frac{4\sqrt{3}}{9}\frac{t^2 - 6}{t^2 + 3} - \frac{\sqrt{3}}{9}, \quad q(t) = 1, \quad r(t) = 0,$$
$$\gamma_1(t) = \frac{1}{2}\frac{t^2 - 6}{t^2 + 3}, \quad \gamma_3(t) = -\frac{t}{t^2 + 3} + \frac{(t^2 - 6)t}{(t^2 + 3)^2}, \tag{3.6}$$
$$\gamma_2(t) = \frac{5\sqrt{3}}{9} + \frac{\sqrt{3}}{18}\frac{t^2 - 6}{t^2 + 3} - \frac{\sqrt{3}}{9}\frac{(t^2 - 6)^2}{(t^2 + 3)^2}.$$

It is apparent that for all components of solution (3.6) there are limits

$$\lim_{t\to\pm\infty} p(t) = p^* = \frac{1}{\sqrt{3}}, \quad \lim_{t\to\pm\infty} q(t) = q^* = 1, \quad \lim_{t\to\pm\infty} r(t) = r^* = 0,$$
$$\lim_{t\to\pm\infty} \gamma_1(t) = \gamma_1^* = \frac{1}{2}, \quad \lim_{t\to\pm\infty} \gamma_2(t) = \gamma_2^* = \frac{\sqrt{3}}{2}, \quad \lim_{t\to\pm\infty} \gamma_3(t) = \gamma_3^* = 0.$$

Summarizing, this solution describes the case of such a motion of the gyrostat when the remote past and the remote future are absolutely symmetric. The system slowly “quits” the Lyapunov-unstable stationary state $(p^*, q^*, r^*, \gamma_1^*, \gamma_2^*, \gamma_3^*)$, in which it stayed in the infinitely remote past (at $t \to -\infty$), performs an intense motion in the present (on a relatively short interval near $t = 0$), and slowly returns to the same unstable stationary state in the infinitely remote future (at $t \to +\infty$). The system is continuously under the action of both a moment of potential forces ($b \neq 0$) and a moment of gyroscopic forces ($k_1 \neq 0$); a constant gyrostatic moment is also in effect ($\lambda \neq 0$).

It should be noted that unlike the classical Bobylev–Steklov case [1, 2] for the heavy rigid body, for a gyrostat subject to the action of the moment of gyroscopic forces (i.e., for $k_1 \neq 0$), it is no longer possible to obtain in assertions 2 and 3 a family of solutions with any $q_0 \neq 0$. However, they do not require the conditions on the moments of inertia $B = 2A$ to be fulfilled.

4. CONSTRUCTING A SOLUTION USING THE BOBYLEV METHOD

In this section, to construct solutions of the gyrostat equations with a moment of gyroscopic forces (2.8) and (2.9), we apply the approach proposed by Bobylev [1] for equations for the heavy rigid body and extended to the gyrostat (without a moment of gyroscopic forces) by Kharlamov [9]). Following [1], we seek solution of system (2.8) and (2.9) in form

$$q(t) = q_0 = \text{const}, \quad r(t) = 0. \tag{4.1}$$

Substituting (4.1) into system (2.9), we arrive at identity

$$(A - B)\, p q_0 + \lambda_1 q_0 - \lambda_2 p + b\gamma_1 + k_1 q_0 \gamma_1 \equiv 0.$$

We then find

$$\gamma_1 = \frac{1}{b + k_1 q_0}\left[(\lambda_2 + (B - A)\, q_0 +)\, p - \lambda_1 q_0\right]. \tag{4.2}$$

Using equality (4.1) we obtain from integral (2.10)

$$\gamma_2 = \frac{1}{2b}\left[c_1 - Ap^2 - Bq_0^2\right]. \tag{4.3}$$

The first equation of system (2.9) is now represented in form

$$A\dot{p} = -b\gamma_3 = \mp b\sqrt{1 - \gamma_1^2 - \gamma_2^2} = \mp b\sqrt{P_4(p)}, \tag{4.4}$$

where the fourth-power polynomial $P_4(p)$ is represented with consideration for (4.2) and (4.3) as

$$P_4(p) = 1 - \frac{1}{4b^2}(c_1 - Ap^2 - Bq_0^2)^2 - \frac{1}{(b + k_1 q_0)^2}(((B - A)\, q_0 + \lambda_2)\, p - \lambda_1 q_0)^2.$$

However, Eq. (4.4), which is reduced to inverting an elliptic integral, can only be used under some additional conditions for the parameters of system (2.9). These conditions originate from the requirement to fulfil area integral (2.11). Substituting γ_1, γ_2 from (4.2) and (4.3) into integral (2.11) and taking into account (4.1), we represent this integral as a polynomial in power of p in the following form

$$J_2 = K_2 p^2 + K_1 p + K_0. \tag{4.5}$$

Here, coefficients K_j of polynomial (4.5) are given by formulas

$$K_2 = -\frac{A}{2b(b + k_1 q_0)}\left[b((2A - B)\, q_0 - \lambda_2) + k_1 q_0 (Bq_0 + \lambda_2)\right],$$

$$K_1 = -\frac{\lambda_1}{b + k_1 q_0}\left[((2A - B)\, q_0 - \lambda_2)\right],$$

$$K_0 = -\frac{G}{2b(b + k_1 q_0)},$$

$$G = B^2 k_1 q_0^4 + \left(B^2 b + Bk_1\lambda_2\right) q_0^3 + (Bb\lambda_2 - Bc_1 k_1)\, q_0^2 + \left(2b\lambda_1^2 - Bbc_1 - c_1 k_1 \lambda_2\right) q_0 - bc_1\lambda_2.$$

For polynomial (4.5) to be an integral, i.e., to preserve a constant value for any solution $p(t)$, it is necessary and sufficient that coefficients satisfy equalities $K_1 = 0,\ K_2 = 0$. It follows from these formulas that either

$$\begin{cases}(2A - B)q_0 - \lambda_2 = 0\\ k_1 q_0 (Bq_0 + \lambda_2) = 0,\end{cases} \tag{4.6}$$

or

$$\begin{cases}\lambda_1 = 0\\ b((2A - B)q_0 - \lambda_2) + k_1 q_0 (Bq_0 + \lambda_2) = 0.\end{cases} \tag{4.7}$$

Using three different solutions of system of equations (4.6) for parameters and Eq. (4.4), we conclude that the following three assertions are valid.

Assertion 5. If $\lambda_2 = k_1 = 0$, $B = 2A$, system (2.8) and (2.9) has a family of solutions for which $q(t) = q_0 = \text{const}$, $r(t) = 0$, and $p(t)$ is found by inverting the elliptic integral

$$\int \frac{dp}{\sqrt{P_4(p)}} = \mp \frac{b}{A}(t + c_5); \qquad P_4(p) = 1 - \frac{q_0^2}{b^2}(Ap - \lambda_1)^2 - \frac{1}{4b^2}\left(c_1 - Ap^2 - Bq_0^2\right)^2.$$

after which $\gamma_1(t),\ \gamma_2(t)$ are determined using Eqs. (4.2) and (4.3), and $\gamma_3(\mathrm{t})$ is found by differentiation $\gamma_3(t) = -\frac{A}{b}\dot{p}(t)$. Here, q_0 is any real number.

Assertion 6. If $k_1 = 0,\ B \neq 2A$, system (2.8) and (2.9) has a family of solutions for which $q(t) = q_0 = \lambda_2/(2A - B) = \text{const}$, $r(t) = 0$, and $p(t)$ is found by inverting the elliptic integral

$$\int \frac{dp}{\sqrt{P_4(p)}} = \mp \frac{b}{A}(t + c_5),$$

where $P_4(p) = 1 - \frac{\lambda_2^2}{b^2(2A - B)^2}(Ap - \lambda_1)^2 - \frac{1}{4b^2}\left(c_1 - Ap^2 - \frac{B\lambda_2^2}{(2A - B)^2}\right)^2$. Functions $\gamma_1(t), \gamma_2(t)$ are determined then using Eqs. (4.2) and (4.3), and $\gamma_3(t)$ is found by differentiation $\gamma_3(t) = -\frac{A}{b}\dot{p}(t)$. Here, q_0 is a fixed real number rather than any real number.

Assertion 7. If $\lambda_2 = 0,\ k_1 \neq 0$, system (2.8) and (2.9) has a family of solutions for which $q(t) = q_0 = 0 = \text{const}$, $r(t) = 0$, $\gamma_1(t) = 0$, and $p(t)$ is found by inverting elliptic integral

$$\int \frac{dp}{\sqrt{P_4(p)}} = \mp \frac{b}{A}(t + c_5),$$

where $P_4(p) = 1 - \frac{1}{4b^2}\left(c_1 - Ap^2\right)^2$. Function $\gamma_2(t)$ is determined then using Eq. (4.2), and $\gamma_3(t)$ is found by differentiation $\gamma_3(t) = -\frac{A}{b}\dot{p}(t)$.

Assertion 5 yields a three-parametric family of solutions (parameters q_0, c_1, c_5). Assertions 6 and 7 yield two-parametric families of solutions (parameters c_1, c_5).

Using the solution of system of equations (4.7) for parameters and Eq. (4.4), we conclude that the following assertion is valid.

Assertion 8. If $\lambda_1 = 0,\ k_1 \neq 0,\ D_2 = (b(2A - B) + k_1\lambda_2)^2 + 4Bbk_1\lambda_2 \geq 0$, system (2.8) and (2.9) has a family of solutions for which $q(t) = q_0 = \text{const}$, $r(t) = 0$, and $p(t)$ is found by inverting elliptic integral

$$\int \frac{dp}{\sqrt{P_4(p)}} = \mp \frac{b}{A}(t + c_5),$$

where $P_4(p) = 1 - \frac{1}{(b + k_1 q_0)^2}((B - A)q_0 + \lambda_2)p)^2 - \frac{1}{4b^2}\left(c_1 - Ap^2 - Bq_0^2\right)^2$. Functions $\gamma_1(t)$, $\gamma_2(t)$ are determined using Eqs. (4.2) and (4.3), and $\gamma_3(t)$ is found by differentiation $\gamma_3(t) = -\frac{A}{b}\dot{p}(t)$. Here, q_0 may only be that of the two numbers $q_0^{\pm} = [-(b(2A - B) + k_1\lambda_2) \mp \sqrt{D_2}]/(2Bk_1)$ which are different from $\widehat{q_0} = -b/k_1$.

Assertion 8 yields two two-parametric families of solutions (parameters c_1, c_5).

We have found in this way that under conditions of assertions 5–8 the solutions of system of equations (2.8) and (2.9) contained in them are expressed in terms of elliptic functions of time. It should be noted that the conditions of assertions 2 and 5 coincide, so they yield virtually the same family of solutions (except the case $q(t) = q_0 = 0$ not covered by assertion 2). In the case of the heavy rigid body, when in addition to the conditions of assertions 2 and 5 we have also $\lambda_1 = 0$, the Bobylev [1] and Steklov [2] methods are equivalent, so in monographs, only one of these methods is usually presented under the common name of the Bobylev–Steklov case. In a more general case of a gyrostat with the moment of gyroscopic forces when $k_1 \neq 0$, assertions 3, 4, 6, 7, and 8 show that the Steklov and Bobylev methods do not follow from each other and may yield complementary results. In particular, the solutions obtained above in example 1 on the basis of assertion 4 cannot be obtained from assertions 5–8, since in this example all parameters are not zero.

Example 2. We consider system (2.8) and (2.9) under the following values of parameters $B = A$, $\lambda_2 = 0$, $k_1 \neq 0$, $b > 0$ and seek a solution on the zero level of energy integral $c_1 = 0$ using assertion 7. We find then a solution in terms of Jacobi functions $q(t) = r(t) = \gamma_1(t) \equiv 0$, $p(t) = -\sqrt{\frac{2b}{A}}\,\mathrm{JacobiSN}\left(\frac{\sqrt{2}}{2\sqrt{Ab}}(t + c_5), \sqrt{-1}\right)$, $\gamma_2(t) = -\frac{A}{2b}p^2$, $\gamma_3(t) = -\frac{A}{2b}\dot{p}$. It should be noted that this solution cannot be found using the Steklov method since the parameters do not satisfy system (3.3).

5. STATIONARY SOLUTIONS

By stationary solutions we mean such constants that zero out the right-hand sides of system (2.8) and (2.9). Skipping the elementary analysis of the obtainment of such a solution, we represent six types of stationary solutions (ordered in the increasing number of non-zero elements) and the conditions on parameters of system (2.8) and (2.9) for which these solutions occur.

(a) For any values of parameters $A, B, C, \lambda_1, \lambda_2 k_1$ system (2.8) and (2.9) has a stationary solution

$$\overline{p} = \overline{q} = \overline{r} = \overline{\gamma_1} = \overline{\gamma_3} = 0, \quad \overline{\gamma_2} = \sigma = \pm 1.$$

(b) If $\lambda_1 = 0$, system (2.8) and (2.9) has a stationary solution

$$\overline{p} = \overline{r} = \overline{\gamma_1} = \overline{\gamma_3} = 0, \quad \overline{\gamma_2} = \sigma = \pm 1; \quad \overline{q} \in R - \text{is any number.}$$

(c) If $\lambda_2 \neq 0$, system (2.8) and (2.9) has stationary solution

$$\overline{q} = \overline{r} = \overline{\gamma_3} = \overline{\gamma_2} = 0, \quad \overline{\gamma_1} = \sigma = \pm 1, \quad \overline{p} = \frac{\sigma b}{\lambda_2}.$$

(d) If $A \neq B$, $b(A - B) + k_1\lambda_2 \neq 0$, $\left|\frac{\lambda_1\lambda_2}{b(A - B) + k_1\lambda_2}\right| < 1$, system (2.8) and (2.9) has stationary solution

$$\overline{r} = \overline{\gamma_3} = 0, \quad \overline{q} = \frac{\lambda_2}{A - B}, \quad \overline{\gamma_1} = \frac{\lambda_1\lambda_2}{b(A - B) + k_1\lambda_2}, \quad \overline{\gamma_2} = \sigma\sqrt{1 - \overline{\gamma}_1^2}, \quad \sigma = \pm 1, \quad \overline{p} = \frac{\lambda_2}{(A - B)}\frac{\overline{\gamma_1}}{\overline{\gamma_2}}.$$

(e) If $k_1 \neq 0$, $b(A - B) + k_1\lambda_2 \neq 0$, we set

$$\overline{r} = \overline{\gamma_3} = 0, \quad \overline{q} = \frac{\lambda_2}{A - B}, \quad \overline{\gamma_1} = \frac{\lambda_1\lambda_2}{b(A - B) + k_1\lambda_2},$$

$$\overline{\gamma_2} = \sigma\sqrt{1 - \overline{\gamma}_1^2}, \quad \sigma = \pm 1, \quad \overline{p} = \frac{\lambda_2}{(A - B)}\frac{\overline{\gamma_1}}{\overline{\gamma_2}}.$$

In cases (a)–(e) the stationary solutions are calculated in terms of the parameters using explicit formulas.

(f) If $\lambda_1 \neq 0$, the stationary condition is constructed in the following way. We set $\overline{r} = \overline{\gamma_3} = 0$, $\overline{q} \in R$ is number but $\overline{q} \neq \lambda_2/(A-B)$ and $\overline{q} \neq 0$. Next, we calculate $a_0 = \lambda_1\overline{q}/[(B-A)\overline{q} + \lambda_2]$, $a_1 = (b + k_1\overline{q})/[(B-A)\overline{q} + \lambda_2]$ and consider equation

$$z^2 + \frac{\overline{q}^2 z^2}{(a_0 + a_1 z)^2} = 1.$$

This equation always has either 2 or 4 real roots the absolute value of which does not exceed 1. We set $\overline{\gamma_1} = z$ for any of these roots and calculate $\overline{p} = a_0 + a_1\overline{\gamma_1}$, $\overline{\gamma_2} = \frac{\overline{q}}{\overline{p}}\overline{\gamma_1}$. Components $\overline{\gamma_1}$, $\overline{p}$, $\overline{\gamma_2}$ of the stationary solution turn out to be dependent on the selection of $\overline{q} \in R$. Thus, in case (f) system (2.8) and (2.9) has a continuum of stationary solutions.

6. ANALYSIS OF THE STABILITY OF STATIONARY SOLUTIONS

To obtain sufficient conditions of the stability of stationary solutions, we use the method of integral connections proposed by Chetayev [11]. We introduce notations for deviations from stationary solution $(\overline{p}, \overline{q}, \overline{r}, \overline{\gamma}_1, \overline{\gamma}_2, \overline{\gamma}_3)$

$$x_1 = p - \overline{p}, \quad x_2 = q - \overline{q}, \quad x_3 = r - \overline{r},$$

$$x_4 = \gamma_1 - \overline{\gamma}_1, \quad x_5 = \gamma_2 - \overline{\gamma}_2, \quad x_6 = \gamma_3 - \overline{\gamma}_3.$$

In terms of these variables, the integrals of the equations of perturbed motion are represented in the following way:

$$J_1 - \overline{J}_1 = 2A\overline{p}x_1 + 2B\overline{q}x_2 + 2C\overline{r}x_3 + 2bx_5 + Ax_1^2 + Bx_2^2 + Cx_3^2, \tag{6.1}$$

$$\begin{aligned} J_2 - \overline{J}_2 &= A\overline{\gamma}_1 x_1 + B\overline{\gamma}_2 x_2 + C\overline{\gamma}_3 x_3 \\ &+ (A\overline{p} + \lambda_1 + k_1\overline{\gamma}_1)x_4 + (B\overline{q} + \lambda_2)x_5 + (C\overline{r})x_6 \\ &+ Ax_1x_4 + Bx_2x_5 + Cx_3x_6 + \frac{1}{2}(k_1x_4^2), \end{aligned} \tag{6.2}$$

$$J_3 - \overline{J}_3 = 2\overline{\gamma}_1x_4 + 2\overline{\gamma}_2x_5 + 2\overline{\gamma}_3x_6 + x_4^2 + x_5^2 + x_6^2. \tag{6.3}$$

Here and below, $\overline{J}_i$ denotes the value of integral J_i on the stationary solution.

We consider first conditions of stability of stationary solutions for case (a). The Lyapunov function is constructed as a linear connection (linear combination) of integrals (6.1) and (6.3), which for the solutions of type (a) assumes form

$$\begin{aligned} V &= \alpha_1(J_1 - \overline{J}_1) + \alpha_3(J_3 - \overline{J}_3) \\ &= \alpha_1\left(2bx_5 + Ax_1^2 + Bx_2^2 + Cx_3^2\right) + \alpha_3\left(2\overline{\gamma}_2x_5 + x_4^2 + x_5^2 + x_6^2\right). \end{aligned}$$

To remove linear terms in the connection, we select coefficients $\alpha_i, i = 1,3$ in the following way $\alpha_1 = 1$, $\alpha_3 = -b\sigma$. We obtain $V = Ax_1^2 + Bx_2^2 + Cx_3^2 - b\sigma\left(x_4^2 + x_5^2 + x_6^2\right)$. For a stationary solution of type (a), which satisfies condition $\sigma = -\text{sign}(b)$, this function is positively defined. Thus, we have proven the following assertion.

Assertion 9. The stationary solution of type (a), which corresponds to $\sigma = -\text{sign}(b)$ is Lyapunov stable.

We now consider the issue of the necessary conditions for the stability of stationary solutions. Let Q be a 6×6 matrix of linear system $\dot{x} = Qx$ obtained by linearizing system (2.8) and (2.9) in the vicinity of stationary solution $(\overline{p}, \overline{q}, \overline{r}, \overline{\gamma}_1, \overline{\gamma}_2, \overline{\gamma}_3)$. Matrix Q is a Jacobi matrix composed of partial derivatives of right sides of (2.8) and (2.9) calculated on stationary solution $(\overline{p}, \overline{q}, \overline{r}, \overline{\gamma}_1, \overline{\gamma}_2, \overline{\gamma}_3)$. The explicit form of matrix Q is cumbersome and not presented here. The characteristic equation of matrix Q has form $s^2\left(s^4 + q_1s^2 + q_2\right) = 0$

where coefficients q_1 and q_2 depend on the parameters of system (2.8) and (2.9) and the selected stationary solution.

The necessary conditions of the stability of stationary solution $(\overline{p},\overline{q},\overline{r},\overline{\gamma}_1,\overline{\gamma}_2,\overline{\gamma}_3)$ are set by inequalities $q_1 \geq 0$, $q_2 \geq 0$, $q_1^2 - 4q_2 \geq 0$. If at least one of these three inequalities is violated, the characteristic equation has at least one root with positive real part, which, due to the Lyapunov theorem, results in the instability of the corresponding solution.

For the stationary solution of type (a), which corresponds to $\sigma = -\mathrm{sign}(b)$, the necessary conditions of stability are definitely fulfilled. For the stationary solution of type (a), which corresponds to $\sigma = \mathrm{sign}(b)$, we obtain the following coefficients of characteristic equation

$$q_1 = \frac{A\lambda_1^2 + B\lambda_2^2 - B|b|(A+C)}{ABC}, \quad q_2 = \frac{|b|\left(B|b| - \lambda_1^2\right)}{ABC}.$$

Therefore, this stationary solution will be unstable if at least one of the three inequalities below

$$B|b| - \lambda_1^2 < 0, \quad A\lambda_1^2 + B\lambda_2^2 - B|b|(A+C) < 0,$$

$$\left(A\lambda_1^2 + B\lambda_2^2\right)^2 - 2B|b|\left(A(A-C)\lambda_1^2 + B(A+C)\lambda_2^2\right) + B^2b^2(A-C)^2 < 0$$

is fulfilled.

For $\lambda_1^2 > 0$ and small $|b|$ the first inequality is fulfilled while for large $|b|$, the second inequality is satisfied. In the case when gyrostatic momentum is absent, $\lambda_1 = \lambda_2 = 0$, the second inequality is fulfilled. However, there are no reasons to assert that the stationary solution of type (a), which corresponds to $\sigma = \mathrm{sign}(b)$, is always (i.e., for any values of the parameters) unstable in linear approximation. For example, for the following values of parameters $A = 2$, $B = 3$, $C = 4$, $b = 1$, $\lambda_1 = 1$, $\lambda_2 = 4$ at $\sigma = 1 = \mathrm{sign}(b)$, coefficients of the characteristic equation are $q_1 = 4/3$ and $q_2 = 1/12$, and it does not have roots with a positive real part.

We now proceed to deriving conditions of stability of stationary equations $(\overline{p},\overline{q},\overline{r},\overline{\gamma}_1,\overline{\gamma}_2,\overline{\gamma}_3) = (0,\overline{q},0,0,\overline{\gamma}_2,0)$ of type (b) assuming that $\lambda_1 = 0$.

Following the Chetayev method [11], we construct the Lyapunov function as a connection of the integrals of the equations of perturbed motion (6.1)–(6.3)

$$V = (J_1 - \overline{J}_1) + \alpha_2(J_2 - \overline{J}_2) + \alpha_3(J_3 - \overline{J}_3) + \beta_2(J_2 - \overline{J}_2)^2 + \beta_3(J_3 - \overline{J}_3)^2.$$

To remove linear terms in the linear connection, we select coefficients $\alpha_i, i = 2,3$ in the following way

$$\alpha_2 = -\frac{2\overline{q}}{\overline{\gamma}_2} = -2\sigma\overline{q}, \quad \alpha_3 = \overline{q}(B\overline{q} + \lambda_2) - b\sigma.$$

We obtain

$$V = \left(Ax_1^2 + Bx_2^2 + Cx_3^2\right) - 2\sigma\overline{q}\left(Ax_1x_4 + Bx_2x_5 + Cx_3x_6 + \frac{1}{2}k_1x_4^2\right)$$
$$+ (\overline{q}(B\overline{q} + \lambda_2) - b\sigma)\left(x_4^2 + x_5^2 + x_6^2\right)$$
$$+ \beta_2(B\overline{\gamma}_2x_2 + (B\overline{q} + \lambda_2)x_5)^2 + \beta_3(2\overline{\gamma}_2x_5)^2 + o(\|x\|^2).$$

For quadratic part $V_2 = V - o(\|x\|^2)$ of integral V to be positively defined for sufficiently large $\beta_i > 0$ it is necessary and sufficient [16] that V_2 is positively defined on set $\Theta = \{B\overline{\gamma}_2x_2 + (B\overline{q} + \lambda_2)x_5 = 0, 2\overline{\gamma}_2x_5 = 0\}$. On set Θ we obtain

$$V_2 = Ax_1^2 - 2\sigma\overline{q}Ax_1x_4 + (-\sigma\overline{q}k_1 + \overline{q}(B\overline{q} + \lambda_2) - b\sigma)x_4^2$$
$$+ Cx_3^2 - 2\sigma\overline{q}Cx_3x_6 + (\overline{q}(B\overline{q} + \lambda_2) - b\sigma)x_6^2.$$

Applying the Silvestre criterion to two quadratic forms the sum of which constitutes V_2, we obtain the conditions for integral V to be positively defined:

$$\Delta_1 = (B-A)\bar{q}^2 + \bar{q}(\lambda_2 - k_1\sigma) - b\sigma > 0, \quad \Delta_2 = (B-C)\bar{q}^2 + \bar{q}\lambda_2 - b\sigma > 0. \tag{6.4}$$

It follows now from the Lyapunov theorem that the following assertion is valid.

Assertion 10. Each stationary solution $(0,\bar{q},0,0,\bar{\gamma}_2,0)$ of type (b), for which inequalities (6.4) are fulfilled, is Lyapunov stable.

It should be noted that a violation of conditions (6.4) does not imply yet that the corresponding solution is unstable, since conditions (6.4) are only sufficient ones. To compare the sufficient conditions of stability (6.4) with the necessary ones, we note that for a stationary solution of type (b inequality $q_2 \geq 0$ for the coefficient of the characteristic equation $s^2\left(s^4 + q_1s^2 + q_2\right) = 0$ is presented in the following form $q_2 = \Delta_1\Delta_2/AC \geq 0$.

Similar to the proof of assertion 10 based on the Chetayev method, the following assertions can be proven.

Assertion 11. Each stationary solution $(\bar{p},0,0,\bar{\gamma}_1,0,0)$ of type (c), for which inequalities

$$(A-C)\frac{b^2}{\lambda_2^2} + \frac{b(\sigma\lambda_1 + k_1)}{\lambda_2} > 0, \quad (A-B)\frac{b^2}{\lambda_2^2} + \frac{b(\sigma\lambda_1 + k_1)}{\lambda_2} + \frac{\lambda_2^2}{A} > 0$$

are fulfilled, is Lyapunov-stable.

Assertion 12. Each stationary solution $(\bar{p},\bar{q},0,\bar{\gamma}_1,\bar{\gamma}_2,0)$ of types (d)–(f), for which inequalities

$$(A-C)\bar{p}^2 + \bar{p}(\lambda_1 + k_1\bar{\gamma}_1) > 0,$$

$$\begin{aligned}
&3A^2\bar{p}^2\bar{\gamma}_1^2\bar{\gamma}_2^2 - AB\bar{p}\bar{\gamma}_1^4 - 4AB\bar{p}^2\bar{\gamma}_1^2\bar{\gamma}_2^2 - AB\bar{p}^2\bar{\gamma}_2^4 - 4AB\bar{p}\bar{q}\bar{\gamma}_1^3\bar{\gamma}_2 \\
&+ 3Ak_1\bar{\gamma}_1^3\bar{\gamma}_2^2 + 2B^2\bar{p}\bar{q}\bar{\gamma}_1^3\bar{\gamma}_2 + B^2\bar{q}^2\bar{\gamma}_1^4 - 2Bk_1\bar{p}\bar{\gamma}_1^3\bar{\gamma}_2^2 - Bk_1\bar{p}\bar{\gamma}_1\bar{\gamma}_2^4 \\
&- 2Bk_1\bar{q}\bar{\gamma}_1^4\bar{\gamma}_2 + k_1^2\bar{\gamma}_1^4\bar{\gamma}_2^2 - 4A\lambda_2\bar{p}\bar{\gamma}_1^3\bar{\gamma}_2 + 4A\lambda_1\bar{p}\bar{\gamma}_1^2\bar{\gamma}_2^2 + 2B\lambda_2\bar{p}\bar{\gamma}_1^3\bar{\gamma}_2 \\
&- 2B\lambda_1\bar{p}\bar{\gamma}_1^2\bar{\gamma}_2^2 + 2B\lambda_2\bar{q}\bar{\gamma}_1^4 - 2B\lambda_1\bar{q}\bar{\gamma}_1^3\bar{\gamma}_2 - 2k_1\lambda_2\bar{\gamma}_1^4\bar{\gamma}_2 + 2k_1\lambda_1\bar{\gamma}_1^3\bar{\gamma}_2^2 \\
&+ A\bar{p}\bar{\gamma}_1^4 + A\bar{p}\bar{\gamma}_1^2\bar{\gamma}_2^2 + B\bar{p}\bar{\gamma}_1^2\bar{\gamma}_2^2 + B\bar{p}\bar{\gamma}_2^4 + \lambda_2^2\bar{\gamma}_1^4 - 2\lambda_1\lambda_2\bar{\gamma}_1^3\bar{\gamma}_2 + \lambda_1^2\bar{\gamma}_1^2\bar{\gamma}_2^2 > 0
\end{aligned}$$

are fulfilled, is Lyapunov-stable.

Note. The second inequality in the conditions of assertion 12 is definitely fulfilled for those stationary solutions of types (d)–(f), for which inequalities $A\bar{p}\lambda_1 > 0$, $(A-B)\bar{p}^2 + \bar{p}(\lambda_1 + k_1\bar{\gamma}_1) > 0$ hold true.

We now consider a system of equations (2.7) and (2.8) more general compared to (2.8) and (2.9), which additionally contains the moment of circular gyroscopic forces, and parameters satisfy conditions

$$a = c = k_2 = k_3 = \lambda_3 = 0.$$

We denote the system constructed in this way as the system (2.7a) and (2.8). It follows from assertion 1 that system (2.7a) and (2.8) has the same first integrals (6.1)–(6.3) as system (2.8) and (2.9). It is also apparent that system (2.7a) and (2.8) has the same stationary solutions of types (a)–(f) under the conditions specified in Section 5. Therefore, assertions 9–12 are also valid for more general system (2.7a) and (2.8).

CONCLUSIONS

In conclusion we outline the directions in which the results obtained in this study could be extended. It is helpful to determine whether analogs of the Bobylev-Steklov case exist for a nonlinear potential $U(\gamma_2)$ specified by an analytic function. It is also reasonable to consider moment $L(t,\gamma,\omega)\,\omega\times\gamma$ as a control action that conserves the first integrals to clarify additional dynamic properties that can be maintained by selecting such a control. A set of stationary solutions has been found and their stability has been analyzed using the Chetayev method. It would be of interest to extend the list of stationary solutions and carry out a more detailed analysis of the solutions similar to that carried out in [17–19] for a gyrostat with only potential forces or using the Raus method [20] for a rigid body in the Hess case.

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