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Propagation of a Flat Shock Front in an Elastic Layer

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Abstract—The problem of a wave front in an anisotropic elastic layer is studied. It is shown that in the case of elastic isotropy, a uniform wave with a plane front in the layer is possible only in one particular case, at zero Poisson's ratio. In other cases, for the existence of a wave with a flat front, the wave must be inhomogeneous with respect to the transversal coordinate. An analytical solution providing the existence of a plane shock wave front has been obtained for the first time.

Keywords: anisotropy, wave front, acoustic tensor, elastic layer

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1. INTRODUCTION

In [1–17], the problems of propagation of harmonic dispersive and nondispersive waves in linearly elastic media were studied. A certain part of the research is carried out by numerical methods based on finite elements with spectral properties [18–21]. To solve wave problems, finite difference methods are also used [22–26], methods of boundary integral equations [27–29], as well as various variants of meshless methods are used, the most common of which are SPH and DEM methods [30–32]. The problem of determining the propagation velocities of elastic waves becomes especially difficult when dispersion occurs in the medium or structure and the wave profile begins to blur due to the difference in the propagation velocities of the frequency components of the wave profile. Apparently, this fact was first theoretically studied in [33], and further studies in this direction were continued in [34–39]. As applied to the dispersive waves in plates studied below, a large number of works are devoted to studying the long-wavelength limits of the Lamb, Rayleigh–Lamb, and Love waves [40–43], which are essentially dispersionless in the vicinity of zero frequency (for the symmetric fundamental mode) [44, 45].

It should be noted that there are a significant number of experimental studies devoted to studying the propagation of shock waves in rods [46–48], and there are studies on the formation and propagation of a shock front in one-dimensional waveguides made of bimodular materials [49, 50].

In this regard, it is of particular interest to study the conditions under which "plane" waves can propagate, the transverse profile of which remains flat during the motion. Below, in a linear formulation, we study the existence of waves in an elastic anisotropic layer with wave polarization independent of the transverse coordinate. Based on the potentials for the displacement field, analytical solutions are constructed that make it possible to describe the conditions for dispersive waves, in which, despite the dispersion, the wave front remains flat. The conditions for the existence of such waves, as a review of the literature shows, have been obtained for the first time.

2. WAVE FRONT IN AN INFINITE MEDIUM

The problem of the propagation of a plane shock wave front is considered in an elastic infinitesimal setting.

2.1. Anisotropic Environment.

The equations of motion in a linearly elastic anisotropic medium can be represented as

$$\ddot{\mathbf{u}}(\mathbf{x},t) - \frac{1}{\rho}\,\mathrm{div}_x \mathbf{C}\cdot\cdot\nabla_x \mathbf{u}(\mathbf{x},t) = 0, \tag{2.1}$$

where $\mathbf{u}$ is the displacement field, $\mathbf{x}$ is the spatial coordinate, t is time, ρ is the density of the medium, $\mathbf{C}$ is the quadrivalent symmetric elasticity tensor

$$\forall i, j, m, n \quad C_{imjn} = C_{imnj} = C_{mijn} = C_{jnim}. \tag{2.2}$$

Conditions (2.2) exclude micropolar media from consideration.

Considering the tensor $\mathbf{C}$ as an operator in the six-dimensional space of symmetric tensors of the second rank, let us write the condition for its strict ellipticity

$$\mathbf{a} \otimes \mathbf{b} \cdot\cdot \mathbf{C} \cdot\cdot \mathbf{b} \otimes \mathbf{a} > 0. \tag{2.3}$$

Condition (2.3) is satisfied for any nonzero decomposable tensors of the form $\mathbf{a} \otimes \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are arbitrary nonzero vectors.

Let us introduce the wave potential for a plane traveling wave

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{m}\,\psi(\mathbf{x} \cdot \mathbf{n} - ct). \tag{2.4}$$

In representation (2.4), $\mathbf{n}$ is the wave vector that determines the direction of wave propagation, c is the velocity, $\mathbf{m}$ is the normalized polarization of the wave ($\|\mathbf{m}\| = 1$), which determines the motion at the wave front, ψ is the scalar potential, which is used to set the wave profile. It is assumed below that the potential ψ is sufficiently smooth [48]

$$\psi \in C^{k-1}(\mathbb{R}) \quad \& \quad \partial\psi \notin C^{k-1}(\mathbb{R}), \quad k \geq 1. \tag{2.5}$$

Let us note that the following classification is applied to wave fronts: in the case when $k = 1$, that is, the potential ψ is continuous, and its first derivative is discontinuous, the wave front is considered strong; in this case, when the wave propagates, the stresses are discontinuous at the wave front, while displacements are continuous functions of the spatial coordinate. In the case when $k \geq 2$, the wave front is considered weak, for a weak wave front both stress and displacement are continuous functions of the spatial coordinate. In the case when $k < 1$, the wave front is considered superstrong, such a wave front is accompanied by discontinuities in displacements. In addition to condition (2.5), the following condition is usually introduced, which assumes the absence of displacements ahead of the wave front and the nonzero curvature of the wave potential behind the wave front

$$\psi(s) = \begin{cases} 0, & s > 0, \\ \partial^2\psi \neq 0, & s < 0. \end{cases} \tag{2.6}$$

Thus, it is assumed that the material ahead of the front is in a natural undeformed state. In addition, the motion of the plane wave is described by the condition

$$\mathbf{x} \cdot \mathbf{n} - ct = 0 \tag{2.7}$$

The equations of motion (2.1), together with representation (2.4) and condition (2.6), give the algebraic Christoffel equation for determining the vector amplitude $\mathbf{m}$

$$(\mathbf{A}(\mathbf{n}) - \rho c^2 \mathbf{I}) \cdot \mathbf{m} = 0 \tag{2.8}$$

In Eq. (2.8), $\mathbf{I}$ is the unit tensor (the unit diagonal matrix), $\mathbf{A}(\mathbf{n})$ is the acoustic tensor determined by the wave vector$\mathbf{n}$

$$\mathbf{A}(\mathbf{n}) = \mathbf{n} \cdot \mathbf{C} \cdot \mathbf{n} \tag{2.9}$$

Equation (2.8) shows that for any wave vector the acoustic tensor (2.9) is symmetric and strictly elliptic. This ensures the existence of three real and positive eigenvalues in the Jordan normal form of the tensor (2.9)

$$\mathbf{A}(\mathbf{n}) = \mathbf{Q}(\mathbf{n}) \cdot \mathbf{D}(\mathbf{n}) \cdot \mathbf{Q}^t(\mathbf{n}), \tag{2.10}$$

where $\mathbf{Q}(\mathbf{n})$ is an orthogonal tensor depending on the vector $\mathbf{n}$, and $\mathbf{D}(\mathbf{n})$ is a diagonal tensor consisting of the eigenvalues of the acoustic tensor, the superscript in (2.10) denotes the transposition of the corresponding tensor (matrix). Returning to the Christoffel equation, we note that the polarization (vector amplitude) is the eigenvector of the acoustic tensor and the root eigenvector of the tensor on the left side of Eq. (2.8). The symmetry of the acoustic tensor ensures the existence of three mutually orthogonal eigenvectors and, consequently, the polarizations corresponding to the eigenvalues of the acoustic tensor are mutually orthogonal. Moreover, even in the case when the acoustic tensor is not simple, for example, in the case of an isotropic medium, it is semisimple, nevertheless it has three mutually orthogonal eigenvectors.

Remarks 2.1. (a) Equations (2.4)–(2.8) ensure the constancy of propagation velocities, in the case of strong or weak shock wave fronts.

(b) The equations of motion (2.1) and the representation for the displacement field (2.4) show that whatever the function ψ, which, generally speaking, may not satisfy the conditions given by equations (2.5), (2.6), it determines a certain field of motion in a limitless environment. However, if the body has boundaries, then the function ψ is no longer arbitrary.

(c) Keeping Remark 2.1.b in mind, let us consider the function ψ as a function harmonic in time and space variables

$$\psi(\mathbf{x}\cdot\mathbf{n}-ct)=\exp\left(ir(\mathbf{x}\cdot\mathbf{n}-ct)\right), \tag{2.11}$$

where r is the wave number. Function (2.11) describes a plane harmonic wave of circular frequency $\omega = rc$ and length $l = 2\pi c/\omega$. Note that in the case of a harmonic wave, the wavefront is defined in space $\mathbb{R}^4$ as

$$\mathbf{x}\cdot\mathbf{n}-ct=\text{const}. \tag{2.12}$$

d) Of particular interest is the stress field at the wave front

$$\sigma_{\mathbf{n}}=\mathbf{n}\cdot\mathbf{C}\cdot\cdot\nabla_x\mathbf{u}. \tag{2.13}$$

Substituting into expression (2.13) the displacement field determined by representation (2.4), we obtain

$$\sigma_{\mathbf{n}}(\mathbf{x},t)=\mathbf{A}(\mathbf{n})\cdot\mathbf{m}\,\partial\psi(s)\big|_{s=\mathbf{x}\cdot\mathbf{n}-ct}. \tag{2.14}$$

2.2. Elastic Isotropy.

The elasticity tensor, considered as an operator in the space of symmetric tensors, can be written as a nondegenerate symmetric matrix of dimension 6 × 6

$$\mathbf{C}=\begin{pmatrix}\lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda & & & \\ \lambda & \lambda & \lambda+2\mu & & & \\ & & & 2\mu & & \\ & & & & 2\mu & \\ & & & & & 2\mu\end{pmatrix}. \tag{2.15}$$

In representation (2.15), λ and μ are the Lame constants

$$\lambda=\frac{E\nu}{(1+\nu)(1-2\nu)},\quad \mu=\frac{E}{2(1+\nu)}, \tag{2.16}$$

where E is the modulus of elasticity, ν is Poisson's ratio. The elasticity tensor (2.15) makes it possible to write Hooke's law in terms of the corresponding six-dimensional vectors $\vec{\sigma}_6=(\sigma_{11},\sigma_{22},\sigma_{33},\sigma_{13},\sigma_{23},\sigma_{12})$ and $\vec{\varepsilon}_6=(\varepsilon_{11},\varepsilon_{22},\varepsilon_{33},\varepsilon_{13},\varepsilon_{23},\varepsilon_{12})$.

The vector Christoffel equation (2.8) for the elasticity tensor of an isotropic medium (2.15) takes the form

$$\left(\frac{\lambda+2\mu}{\rho c^2}\mathbf{n}\otimes\mathbf{n}+\frac{\mu}{\rho c^2}(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})-\mathbf{I}\right)\cdot\mathbf{m}=0. \tag{2.17}$$

The Christoffel equation in the form (2.17) has eigenvalues that determine the velocities of the longitudinal wave $c_1 = c_L$ and two transverse waves $c_{2,3} = c_T$

$$c_L=\sqrt{\frac{\lambda+2\mu}{\rho}},\quad c_T=\sqrt{\frac{\mu}{\rho}}. \tag{2.18}$$

Remarks 2.2. (a) The matrix on the left side of Christoffel's equation (2.17) is not simple because its two eigenvalues are the same. However, this matrix remains semi-simple because there are no Jordan blocks in its structure.

(b) An analysis of equations (2.14), (2.17) shows that the stresses at the wave front can be represented as

$$\sigma_{\mathbf{n}} = (\lambda + 2\mu)\mathbf{n}\,\partial\psi(s)\big|_{s=\mathbf{x}\cdot\mathbf{n}-ct}. \tag{2.19}$$

Expression (2.19) shows that the stresses at the longitudinal wave front coincide in direction with the displacements. At the same time, on planes orthogonal to the wave front, i.e. on planes with a normal $\mathbf{p}\cdot\mathbf{n} = 0$, normal stresses $\sigma_{\mathbf{p}}$ are, generally speaking, also present

$$\sigma_{\mathbf{p}} = \lambda\mathbf{p}\,\partial\psi(s)\big|_{s=\mathbf{x}\cdot\mathbf{n}-ct}. \tag{2.20}$$

For $\lambda = 0$, there are no stresses on the planes $\Pi_{\mathbf{p}}$.

(c) In the case of a transverse wave, the stresses at the wave front are determined by the expression

$$\sigma_{\mathbf{n}} = \mu\mathbf{m}\,\partial\psi(s)\big|_{s=\mathbf{x}\cdot\mathbf{n}-ct}. \tag{2.21}$$

showing that the stresses $\sigma_{\mathbf{n}}$ are orthogonal to the wave vector $\mathbf{n}$ and coincide with the direction of the displacements. On orthogonal planes $\Pi_{\mathbf{m}}$ with normal $\mathbf{m}$ (let us recall that in the case under consideration $\mathbf{m}\cdot\mathbf{n} = 0$) the stresses $\sigma_{\mathbf{m}}$ can be represented as

$$\sigma_{\mathbf{m}} = \mu\mathbf{n}\,\partial\psi(s)\big|_{s=\mathbf{x}\cdot\mathbf{n}-ct}. \tag{2.22}$$

(d) If $\lambda = 0$ the elasticity tensor (2.15) is diagonal and the tensor $\mathbf{A}(\mathbf{n})$ takes the form

$$\mathbf{A}(\mathbf{n}) = \mu(\mathbf{n}\otimes\mathbf{n} + \mathbf{I}). \tag{2.23}$$

In addition, at $\lambda = 0$ on the planes orthogonal to the wave front, both the normal and tangential components of the surface stresses turn out to be zero.

3. WAVE FRONT IN THE LAYER

Let us consider a plate with free surfaces of thickness h. Let the origin of coordinates be located on the middle surface of the plate and the wave vector $\mathbf{n}$ be in the middle plane, and $\boldsymbol{\nu}$ be the vector of the unit normal to this plane.

3.1. Elastic Anisotropy.

The conditions on the side surfaces of the plate, which express the absence of the corresponding stresses, can be represented as

$$\sigma_{\nu} \equiv \boldsymbol{\nu}\cdot\mathbf{C}\cdot\cdot\nabla\mathbf{u}\big|_{x'=\pm h/2} = 0, \tag{3.1}$$

where it is denoted that $x' = \mathbf{x}\cdot\boldsymbol{\nu}$.

The displacement field for a shock wave propagating in the direction $\mathbf{n}$ and having a flat front is determined by the expression

$$\mathbf{u}(\mathbf{x},t) = \mathbf{m}\varphi(x')\psi(\mathbf{n}\cdot\mathbf{x} - ct), \tag{3.2}$$

where $\mathbf{u}$ is the displacement field, $\varphi(x')$ is an as yet unknown function characterizing the variation of the wave amplitude in the transversal direction. Boundary conditions (3.1), when (3.2) is taken into account, take the form

$$\sigma_{\nu} \equiv \begin{bmatrix} \mathbf{A}(\boldsymbol{\nu})(\partial\varphi|_{x'=\pm h/2})(\psi(s)|_{s=\mathbf{x}\cdot\mathbf{n}-ct}) + \\ + (\boldsymbol{\nu}\cdot\mathbf{C}\cdot\mathbf{n})(\varphi|_{x'=\pm h/2})\left(\partial\psi(s)|_{s=\mathbf{x}\cdot\mathbf{n}-ct}\right) \end{bmatrix}\cdot\mathbf{m} = 0. \tag{3.3}$$

Equations (2.1) and the representation of the displacement field (3.2) give the differential equation of the second one, which makes it possible to determine the polarization of the wave front $\mathbf{m}$, and this equation includes two, generally speaking, unknown functions φ and ψ

$$\begin{bmatrix} \mathbf{A}(\boldsymbol{\nu})(\partial^2\varphi)\left(\psi(s)|_{s=\mathbf{x}\cdot\mathbf{n}-ct}\right) + \mathbf{B}(\boldsymbol{\nu},\mathbf{n})(\partial\varphi)\left(\partial\psi(s)|_{s=\mathbf{x}\cdot\mathbf{n}-ct}\right) \\ + (\mathbf{A}(\mathbf{n}) - \rho c^2\mathbf{I})(\varphi)(\partial^2\psi(s)\big|_{s=\mathbf{x}\cdot\mathbf{n}-ct}) \end{bmatrix}\cdot\mathbf{m} = 0. \tag{3.4}$$

In Eq. (3.4)

$$\mathbf{A}(\boldsymbol{\nu}) = \boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu} \quad \mathbf{B}(\boldsymbol{\nu},\mathbf{n}) = \boldsymbol{\nu}\cdot\mathbf{C}\cdot\mathbf{n} + \mathbf{n}\cdot\mathbf{C}\cdot\boldsymbol{\nu}. \tag{3.5}$$

Remarks 3.1. (a) In the case when the wave profile ψ is known in advance (this is often the case when modeling shock waves), the differential equation (3.4) becomes a second-order ordinary differential equation with respect to the function φ.

(b) If the function ψ is harmonic, for example, defined by expression (2.11), Eq. (3.4) takes the form

$$[\mathbf{A}(\mathbf{v})(\partial^2\varphi) + ir\mathbf{B}(\mathbf{v},\mathbf{n})(\partial\varphi) - r^2(\mathbf{A}(\mathbf{n}) - \rho c^2\mathbf{I})(\varphi)] \cdot \mathbf{m} = 0. \tag{3.6}$$

3.2. Isotropy, $\lambda \neq 0$.

In the isotropic case, conditions (3.3) of zero stresses at the corresponding layer boundaries take the form

$$\sigma_\nu \equiv \begin{bmatrix} ((\lambda + 2\mu)(\mathbf{v}\otimes\mathbf{v}) + \mu(\mathbf{I} - \mathbf{v}\otimes\mathbf{v}))\left(\partial\varphi|_{x'=\pm h/2}\right)\left(\psi(s)|_{s=\mathbf{x}\cdot\mathbf{n}-ct}\right) \\ + (\lambda\mathbf{v}\otimes\mathbf{n} + \mu\mathbf{n}\otimes\mathbf{v})\left(\varphi|_{x э=\pm h/2}\right)\left(\partial\psi(s)|_{s=\mathbf{x}\cdot\mathbf{n}-ct}\right) \end{bmatrix} \cdot \mathbf{m} = 0. \tag{3.7}$$

Equation (3.4) is transformed in a similar way:

$$\begin{bmatrix} ((\lambda + 2\mu)\mathbf{v}\otimes\mathbf{v} + \mu(\mathbf{I} - \mathbf{v}\otimes\mathbf{v}))(\partial^2\varphi)\left(\psi(s)|_{s=\mathbf{x}\cdot\mathbf{n}-ct}\right) \\ + (2(\lambda+\mu)\mathrm{sym}(\mathbf{v}\otimes\mathbf{n}))(\partial\varphi)\left(\partial\psi(s)|_{s=\mathbf{x}\cdot\mathbf{n}-ct}\right) \\ + ((\lambda + 2\mu)\mathbf{n}\otimes\mathbf{n} + \mu(\mathbf{I} - \mathbf{n}\otimes\mathbf{n}) - \rho c^2\mathbf{I})(\varphi)(\partial^2\psi(s)|_{s=\mathbf{x}\cdot\mathbf{n}-ct}) \end{bmatrix} \cdot \mathbf{m} = 0. \tag{3.8}$$

In (3.8) it is denoted

$$\mathrm{sym}(\mathbf{v}\otimes\mathbf{n}) = \frac{1}{2}(\mathbf{v}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{v}). \tag{3.9}$$

Remark 3.2. For a longitudinal wave with polarization coinciding with the direction of propagation, equations (3.8) take the form

$$\begin{cases} \mu(\partial^2\varphi)(\psi(s)|_{s=\mathbf{x}\cdot\mathbf{n}-ct}) - ((\lambda + 2\mu) - \rho c^2)(\varphi)(\partial^2\psi(s)|_{s=\mathbf{x}\cdot\mathbf{n}-ct}) = 0, \\ (\lambda + \mu)(\partial\varphi)(\partial\psi(s)|_{s=\mathbf{x}\cdot\mathbf{n}-ct}) = 0. \end{cases} \tag{3.10}$$

Equations (3.10) show that for an arbitrary function ψ such that $\psi(s)$, $\partial\psi(s)$ and $\partial^2\psi(s)$ are not identically zero, they can be satisfied for $\varphi = \mathrm{const}$ only if

$$c = \sqrt{\frac{\lambda + 2\mu}{\rho}}. \tag{3.11}$$

However, it should be noted that conditions $\varphi = \mathrm{const}$ contradict conditions (3.7) at the boundary.

3.3. Isotropy, $\lambda = 0$.

Conditions (3.3) for $\lambda = 0$ take the form

$$\sigma_\nu \equiv \mu \begin{bmatrix} (\mathbf{v}\otimes\mathbf{v} + \mathbf{I})\left(\partial\varphi|_{x'=\pm h/2}\right)\left(\psi(s)|_{s=\mathbf{x}\cdot\mathbf{n}-ct}\right) \\ + \mathbf{n}\otimes\mathbf{v}\left(\varphi|_{x'=\pm h/2}\right)\left(\partial\psi(s)|_{s=\mathbf{x}\cdot\mathbf{n}-ct}\right) \end{bmatrix} \cdot \mathbf{m} = 0. \tag{3.12}$$

Equation (3.8) is transformed in a similar way:

$$\mu \begin{bmatrix} (\mathbf{v}\otimes\mathbf{v} + \mathbf{I})(\partial^2\varphi)\left(\psi(s)|_{s=\mathbf{x}\cdot\mathbf{n}-ct}\right) \\ + 2\mathrm{sym}(\mathbf{v}\otimes\mathbf{n})(\partial\varphi)\left(\partial\psi(s)|_{s=\mathbf{x}\cdot\mathbf{n}-ct}\right) \\ + \left(\mathbf{n}\otimes\mathbf{n} + \mathbf{I} - \frac{\rho c^2}{\mu}\mathbf{I}\right)(\varphi)(\partial^2\psi(s)|_{s=\mathbf{x}\cdot\mathbf{n}-ct}) \end{bmatrix} \cdot \mathbf{m} = 0. \tag{3.13}$$

Differential equations (3.13) for the wave described by Eq. (2.11) are transformed to the form

$$[(\mu(\mathbf{v}\otimes\mathbf{v}+\mathbf{I}))(\partial^2\varphi)+ir(2\mu\,\mathrm{sym}(\mathbf{v}\otimes\mathbf{n}))(\partial\varphi) \\ -r^2(\mu(\mathbf{n}\otimes\mathbf{n}+\mathbf{I})-\rho c^2\mathbf{I})(\varphi)]\cdot\mathbf{m}=0 \tag{3.14}$$

The last expression shows that conditions (3.12) at the boundary and differential equations (3.13) are satisfied for a wave with longitudinal polarization and a flat front with a constant cross-sectional function φ. The latter, taking Remark 3.2 into account, gives

Proposition (a) In the case of an isotropic layer, a *plane longitudinally polarized* wave exists only when the condition $\lambda = 0$ is satisfied.

(b) In the general case, when $\lambda \neq 0$, a longitudinally polarized shock or harmonic wave exists if (i) the wave front is non-planar, or (ii) the polarization of the wave is not constant in the layer cross section: $\varphi(x') \neq \mathrm{const}$.

4. CONCLUSIONS

The constructed solutions show that a plane wavefront of a longitudinal wave in an isotropic linear elastic layer with free boundary surfaces can propagate

(1) either under the condition of zero Poisson's ratio, which is equivalent to zero Lame constant $\lambda = 0$, and then the amplitude of the longitudinal wave is necessarily constant in the cross section;

(2) either with a non-zero Poisson's ratio and, accordingly, a non-zero Lame constant λ, but a variable amplitude in the cross section.

Thus, the initially flat shock front of a longitudinal wave in an isotropic elastic layer in the case of an arbitrary and non-zero Poisson's ratio must be transformed into a wave profile with an amplitude variable in the transversal direction. It seems interesting to generalize the obtained results to shock waves propagating in stratified and functionally graded plates.

In conclusion, it is necessary to note recent studies on the propagation of dispersive harmonic waves in rods [51–54], where the existence of plane shock fronts also plays an important role.

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