

Thermoelastic Behavior of an Incompressible Elastomer at Finite Strains

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Abstract—Parts made of elastomers usually work in a temperature range in which they exhibit highly elastic deformation. Thermal expansion is not a highly elastic deformation and is present in the entire temperature range in which the elastomer exists. These types of deformation react differently to temperature changes. The aim of this article is to elucidate, within the framework of the phenomenological approach, the influence of the temperature dependences of these deformations on the temperature dependence of the stress tensor components describing the stress state of an incompressible elastomer at finite deformation using a specific problem as an example. A variant of the formulation of the problem of statics of a homogeneous isotropic incompressible material at finite deformations is used, which makes it possible to take into account thermal expansion as a self-sufficient process, independent of highly elastic deformation. Within the framework of this formulation, a new exact solution of the coupled problem of thermoelasticity of a finite longitudinal shear of a long cylindrical bushing with the Gent-Thomas strain energy potential in a non-uniform temperature field is obtained. It is shown that the temperature dependence of the shear stress causing longitudinal shear is determined only by thermal expansion, while the temperature dependence of normal stresses is significantly determined by thermal expansion, and to a lesser extent by the temperature change of highly elastic deformation. The temperature dependence of the shear stiffness is mainly determined by the temperature change of the highly elastic deformation.

Keywords: thermomechanics, thermal expansion, highly elastic finite deformation, inhomogeneous temperature field, hyperelasticity, incompressibility, Gent-Thomas potential

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1. INTRODUCTION

For engineering applications of elastomers operating in a wide temperature range, methods for calculating the temperature dependence of the stress-strain state are important. Typically, elastomers are used in the temperature range at which highly elastic deformation occurs. Thermal expansion is not a highly elastic deformation and is present in the entire temperature range in which the elastomer exists. These types of deformation react differently to temperature changes. The aim of this article is to elucidate, within the framework of the phenomenological approach, the influence of the temperature dependences of these deformations on the temperature dependence of the stress tensor components describing the stress state of an incompressible elastomer at finite deformation using the example of a specific problem.

In the last century, a statistical theory of the high elasticity of polymers was developed. The history of the issue can be found in [1–4]. An important result of this theory, from the point of view of engineering applications, was the existence of a single strain energy potential (Treloar potential, neo-Hookean potential)

$$W = \frac{\mu}{2}(I_1 - 3), \quad \mu = \frac{\rho \tilde{R} T}{M_c}. \quad (1.1)$$

Here μ is the shear modulus, ρ is the mass density, \tilde{R} is the universal gas constant, M_c is the average molecular mass, T is the absolute temperature, I_1 is the first of the three principal strain invariants

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_3^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \lambda_1^2 \lambda_2^2, \quad I_3 = \lambda_1^2 \lambda_2^2 \lambda_3^2, \quad (1.2)$$

where λ_i are the multiplicities of stretching along the main strain directions. Further experiments [5] showed that $\partial W / \partial I_2 \neq 0$ and it decrease with increasing I_2 . A problem arose, consisting in describing the general form of the strain energy function that explains this fact. This problem has not been solved so far, which manifests itself in a variety of forms of the strain energy function proposed by different authors [6]. A realistic formula that reasonably approximates the general data on rubber deformations given in [5] was put forward by Gent and Thomas [7]. This expression has the form

$$W = c_1(I_1 - 3) + c_2 \ln\left(\frac{I_2}{3}\right), \quad c_1, c_2 - \text{const.} \quad (1.3)$$

The form (1.3) was chosen mainly because of the mathematical simplicity [1] and the transition for $c_2 = 0$ and $c_1 = \mu/2$ to expression (1.1). Differentiation gives the relations

$$\frac{\partial W}{\partial I_1} = c_1, \quad \frac{\partial W}{\partial I_2} = \frac{c_2}{I_2}$$

corresponding to a nonlinear dependence of $\partial W / \partial I_2$ on I_2 . In Eq. (1.1), the temperature dependence of the strain energy potential manifests itself through the dependence of the shear modulus on temperature. We emphasize that in (1.1) thermal expansion is not taken into account, that is, this equation describes the temperature dependence of highly elastic deformation. This approach will be used below to describe the temperature dependence of potential (1.3), which is more complex than (1.1).

The proposed work belongs to the phenomenological. A variant of the formulation of the problem of statics of a homogeneous isotropic incompressible material under finite deformations is used, which makes it possible to take into account thermal expansion as a self-sufficient process, independent of highly elastic deformation. It differs from other works on the thermoelasticity of rubber by specifying the method for taking into account thermal expansion. The paper considers the standard problem of the finite longitudinal shear of a hollow cylinder, but within the framework of the proposed formulation, a new exact solution of the coupled problem of thermoelasticity is obtained. Solutions are given both in an inhomogeneous axisymmetric temperature field and in its absence, both with and without thermal expansion. A comparison is made with the solution within the potential (1.1).

2. GOVERNING EQUATIONS

When considering problems of thermoelasticity, it is usually assumed that stress, increments of free energy and entropy caused by deformation disappear in the reference configuration [8]. When thermal expansion is taken into account, the difference temperature $\theta = T - T_0$ is introduced, where T is the absolute temperature of the deformed configuration, and T_0 is the homogeneous absolute temperature of the reference configuration. In this case, it is assumed that there is no thermal expansion at $\theta = 0$ [8–10]. Taking these provisions into account, the following version of the formulation of problems of static thermoelasticity of a homogeneous isotropic incompressible material under finite deformations is proposed.

We will describe the final deformation of the elastomer using the strain gradient tensor \mathbf{F} given by the expression

$$d\mathbf{r} = \mathbf{F} \cdot d\mathbf{R}. \quad (2.1)$$

Here, \mathbf{r} and \mathbf{R} are the position vectors of the points of the body in the deformed (current) and undeformed (reference) configurations, respectively. The dot means dot product (convolution over one tensor index).

According to rational continuum mechanics [12, 13], for a simple homogeneous material with a mass density of free energy $\Psi = \Psi(\mathbf{F}, T)$, the equations of state (governing equations) take the form

$$\mathbf{P} = \rho_0 \frac{\partial \Psi}{\partial \mathbf{F}^T}, \quad \hat{\eta} = -\frac{\partial \Psi}{\partial T}. \quad (2.2)$$

Here, ρ_0 is the mass density in the reference configuration, \mathbf{P} is the Piola stress tensor [12] (according to [13] this is the transposed first Piola–Kirchhoff tensor), $\hat{\eta}$ is the entropy mass density. If the material is initially isotropic, then $\Psi = \Psi(I_1, I_2, I_3, T)$. Here $I_1 = \mathbf{1} \cdot \mathbf{B}$, $I_2 = 1/2(I_1^2 - \mathbf{1} \cdot \mathbf{B}^2)$, $I_3 = J^2 = \det \mathbf{B}$ are the principal algebraic invariants of the tensor $\mathbf{B} = \mathbf{F} \cdot \mathbf{F}^T$, which is the left Cauchy–Green deformation tensor [13] (in [12] this is the Finger deformation measure), $\mathbf{1}$ is the unit tensor. The sign T denotes trans-

position, and a pair of dots denotes a double scalar product (meaning a variant of sequential multiplication, when nearest vectors are scalarly multiplied). These invariants coincide with (1.2).

Assuming that the change in volume occurs only due to thermal expansion, we choose the equation for the change in volume in the form [8–10]

$$J = \det \mathbf{F} = g(\theta). \tag{2.3}$$

Here, $g(\theta)$ is some monotonic function of the temperature increment θ , with $g(0) = 1$. The expressions $g(\theta) = e^{3\alpha\theta}$ are commonly used, where α is the linear coefficient of thermal expansion. It was shown in [10] that from (2.3) and (2.2) follows the expression for the Cauchy stress tensor \mathbf{S}

$$\mathbf{S} = 2 \left(\frac{1}{g(\theta)} \frac{\partial W}{\partial I_1} \mathbf{B} - g(\theta) \frac{\partial W}{\partial I_2} \mathbf{B}^{-1} \right) + \gamma \mathbf{1}, \tag{2.4}$$

where the specific energy of elastic deformation (strain energy potential) W , per unit volume of the reference configuration, is related to the mass density of free energy by the relation $W = \rho_0 \Psi$, γ is a function of hydrostatic pressure. Expressions (2.3) and (2.4) take into account thermal expansion.

The temperature dependence of highly elastic deformation is taken into account through the temperature dependence of the elastic moduli [1, 11]. Thus, the influence of the temperature field on the strain energy potential manifests itself in two ways, through the dependence of the elastic moduli and, implicitly, through the mechanism of thermal expansion, which affects the tensor invariants. The Gent-Thomas potential (1.3) is taken as the strain energy potential. In [14], an expression for this potential is given in a form that transforms into Hooke’s law at small strains

$$W = \mu \left\{ \frac{3 - 2\beta}{6} (I_1 - 3) + \beta \ln \left(\frac{I_2}{3} \right) \right\}, \quad 0 \leq \beta < \frac{3}{2}.$$

Here, the shear modulus μ in accordance with [1] has the form

$$\mu = \frac{\rho \tilde{R} T}{M_c} = \frac{\rho_0 \tilde{R} T_0}{M_c} \frac{T}{T_0 J} = \frac{\mu_0}{g(\theta)} \left(1 + \frac{\theta}{T_0} \right), \quad \mu_0 = \frac{T_0 \rho_0 \tilde{R}}{M_c}. \tag{2.5}$$

In other words, the shear modulus depends on temperature both explicitly and implicitly through thermal expansion. Thus, we get

$$W = \frac{\mu_0}{g(\theta)} \left(1 + \frac{\theta}{T_0} \right) \left\{ \frac{3 - 2\beta}{6} (I_1 - 3) + \beta \ln \left(\frac{I_2}{3} \right) \right\}, \quad 0 \leq \beta < \frac{3}{2}. \tag{2.6}$$

At $\beta = 0$, expression (2.6) goes over into the Treloar potential

$$W = \frac{\mu_0}{2g(\theta)} \left(1 + \frac{\theta}{T_0} \right) (I_1 - 3) = \frac{\mu}{2} (I_1 - 3).$$

In the absence of body forces, the equilibrium equation will be written in the form [12]

$$\overset{0}{\nabla} \cdot \mathbf{P} = 0.$$

Here, $\overset{0}{\nabla} = \mathbf{R}^k \partial / \partial q^k$ is the Hamilton operator in the reference configuration, \mathbf{R}^k is the reciprocal basis with the basis \mathbf{R}_k , that is, $\mathbf{R}_k = \partial \mathbf{R} / \partial q^k$, $\mathbf{R}^k \cdot \mathbf{R}_p = \delta_p^k$, where q^1, q^2, q^3 are the material coordinates of the points of the body, δ_p^k is the Kronecker symbol. This equation can also be expressed in terms of the Cauchy stress tensor [12] as

$$\overset{0}{\nabla} \cdot \mathbf{P} = \nabla \cdot \mathbf{S} = 0. \tag{2.7}$$

Here, $\nabla = \mathbf{r}^k \partial / \partial q^k$ is the Hamiltonian operator in the deformed configuration, \mathbf{r}^k is the reciprocal basis with the basis \mathbf{r}_k , i.e.

$$\mathbf{r}_k = \mathbf{F} \cdot \mathbf{R}_k, \quad \mathbf{r}^k = (\mathbf{F}^{-1})^T \cdot \mathbf{R}^k. \tag{2.8}$$

According to (2.7), from (2.4) we obtain

$$\nabla\gamma = -2\nabla \cdot \left[\frac{W_1}{g(\theta)} \mathbf{B} - g(\theta) W_2 \mathbf{B}^{-1} \right]. \quad (2.9)$$

Here, the designations $W_1 = \partial W / \partial I_1$, $W_2 = \partial W / \partial I_2$ are introduced. Eliminating the function γ in (2.9), we will have

$$\nabla \times \nabla \cdot \left[\frac{W_1}{g(\theta)} \mathbf{B} - g(\theta) W_2 \mathbf{B}^{-1} \right] = 0. \quad (2.10)$$

Among simple, homogeneous, and initially isotropic materials, we will consider materials that obey the Fourier law

$$\mathbf{q} = -\lambda \nabla T = -\lambda \nabla \theta. \quad (2.11)$$

Here, \mathbf{q} is the specific heat flux in the current configuration, λ is a constant thermal conductivity coefficient. In the absence of internal heat sources for a stationary heat flux, the heat balance equation has the form [12]

$$\nabla \cdot \mathbf{q} = 0. \quad (2.12)$$

Equation (2.12) and expression (2.11) imply the stationary heat conduction equation

$$\nabla \cdot \nabla \theta = 0, \quad (2.13)$$

which is supplemented by boundary conditions. Equations (2.3) and (2.13) form a connected system of equations describing the temperature field and thermal expansion, regardless of the highly elastic deformation.

3. SOLUTION OF THE PROBLEM OF THERMOELASTICITY OF A FINITE LONGITUDINAL SHEAR OF A LONG CYLINDRICAL BUSHING WITH THE POTENTIAL OF THE GENT-THOMAS DEFORMATION ENERGY IN A NON-UNIFORM TEMPERATURE FIELD

On the inner side surface, the bushing is glued to a fixed rigid cage. A longitudinal shear force with modulus Q and zero normal and circumferential forces act on the outer side surface. It is assumed that the load is "dead", that is, the vector $\mathbf{Q} = Q\mathbf{k}$ does not change its magnitude and direction during deformation ($\mathbf{e}_R, \mathbf{e}_\Phi, \mathbf{k}$ is the unit basis of the cylindrical coordinate system). Temperature T_1 is maintained on the inner side surface, and T_2 on the outer. R_1 and R_2 denote the inner and outer radii of the elastomeric sleeve, and H denotes its length. This problem for various strain energy potentials, both in linear and non-linear formulations, was solved by many authors. In the framework of coupled thermoelasticity, this problem was investigated in [8] with an original thermoelastic analog of the one-constant Gent-Thomas strain energy potential. For the proposed statement with two-constant Gent-Thomas potential, the exact solution is obtained for the first time.

3.1. General Formulation of Problems within the Framework of the Semi-Inverse Method

The material coordinate system is chosen to coincide in the reference configuration with the cylindrical system (R, Φ, Z) , and the OZ axis coincides with the axis of symmetry of the sleeve. The sleeve is assumed to be long enough to ignore end effects and consider the stress-strain state to be independent of Z .

Under these conditions, it is possible to use the kinematic hypothesis of coaxial sections, that is, the cylindrical and coaxial sections before deformation remain the same after deformation (the same axis for all such sections coincides with the axis of symmetry of the bushing). In a cylindrical coordinate system, due to axial symmetry, the place vector in the deformed (actual) configuration is given by the relation

$$\mathbf{r} = f(R) \mathbf{e}_R + [Z + w(R)] \mathbf{k}. \quad (3.1)$$

Here, $r = f(R)$, $\varphi = \Phi$, $z = Z + w(R)$ are the coordinates of the points in the actual configuration in a cylindrical coordinate system, and $w(R)$ and $f(R)$ are the functions to be determined. The boundary conditions for these functions take the form $f(R_1) = R_1$, $w(R_1) = 0$. The temperature field is also assumed to be axisymmetric $\theta = \theta(R)$. The Hamilton operator of the material coordinate system in the reference configuration is written in the form

$$\overset{0}{\nabla} = \mathbf{e}_R \frac{\partial}{\partial R} + \frac{\mathbf{e}_\Phi}{R} \frac{\partial}{\partial \Phi} + \mathbf{k} \frac{\partial}{\partial Z}. \tag{3.2}$$

In the current configuration, it can be written as

$$\nabla = \mathbf{e}_R \frac{\partial}{\partial r} + \frac{\mathbf{e}_\Phi}{r} \frac{\partial}{\partial \varphi} + \mathbf{k} \frac{\partial}{\partial z}. \tag{3.3}$$

Equation (2.13) in a cylindrical coordinate system has the form

$$\nabla \cdot \nabla \theta = \frac{\partial}{\partial r} \left(r \frac{\partial \theta}{\partial r} \right) = 0. \tag{3.4}$$

The transposed strain gradient tensor by (3.1) and (3.2) is obtained in the form

$$\mathbf{F}^T = \overset{0}{\nabla} \mathbf{r} = \frac{df(R)}{dR} \mathbf{e}_R \mathbf{e}_R + \frac{f(R)}{R} \mathbf{e}_\Phi \mathbf{e}_\Phi + \frac{dw(R)}{dR} \mathbf{e}_R \mathbf{k} + \mathbf{k} \mathbf{k} \tag{3.5}$$

So that the incompressibility condition ($J = \det \overset{0}{\nabla} \mathbf{r} = B(R)$ где $B(R) = g[\theta(R)] = e^{3\alpha\theta(R)}$) takes the form of a differential equation

$$\frac{f(R)f'(R)}{R} = B(R) \quad \text{or} \quad f'(R) = \frac{RB(R)}{f(R)}, \tag{3.6}$$

where the prime denotes the derivative with respect to R . Equations (3.4) and (3.6) form a connected system of equations. This system describes the thermal expansion and the temperature field, regardless of the strain energy potential. We will look for a solution to the system in a parametric form. The solution of Eq. (3.4) supplemented by the boundary conditions $r(R_1) = f(R_1) = R_1, \theta(R_1) = \theta[f(R_1)] = \theta_1$ has the form

$$\theta(R) = \eta \ln \frac{f(R)}{R_1} + \theta_1. \tag{3.7}$$

Here, $\eta[^\circ\text{C}]$ is the dimensional integration constant, which plays the role of a parameter.

From (3.7) we calculate $B(R) = e^{3\alpha\theta(R)} = a(f(R)/R_1)^{3\alpha\eta}, a = e^{3\alpha\theta_1}$, and substitute in (3.6). We get $f^{1-3\alpha\eta}(R) f'(R) = aRR_1^{-3\alpha\eta}$. The solution of this equation with the boundary condition $f(R_1) = R_1$ has the expression

$$f(R) = R_1 \left\{ 1 + \frac{a(2-3\alpha\eta)}{2} \left[\left(\frac{R}{R_1} \right)^2 - 1 \right] \right\}^{\frac{1}{2-3\alpha\eta}} \tag{3.8}$$

and (3.7) can be rewritten

$$\theta(R) = \frac{\eta}{2-3\alpha\eta} \ln \left\{ 1 + \frac{a(2-3\alpha\eta)}{2} \left[\left(\frac{R}{R_1} \right)^2 - 1 \right] \right\} + \theta_1. \tag{3.9}$$

Functions (3.8) and (3.9) describe the process of thermal expansion and the temperature field in a parametric form. You can also rewrite the expression for $B(R)$

$$B(R) = a \left\{ 1 + \frac{a(2-3\alpha\eta)}{2} \left[\left(\frac{R}{R_1} \right)^2 - 1 \right] \right\}^{\frac{3\alpha\eta}{2-3\alpha\eta}}.$$

Using (3.6), from (3.5) we obtain the expressions

$$\begin{aligned} \mathbf{F}^T &= \frac{RB(R)}{f(R)} \mathbf{e}_R \mathbf{e}_R + \frac{f(R)}{R} \mathbf{e}_\Phi \mathbf{e}_\Phi + w'(R) \mathbf{e}_R \mathbf{k} + \mathbf{k} \mathbf{k}, \\ \mathbf{F} &= \frac{RB(R)}{f(R)} \mathbf{e}_R \mathbf{e}_R + \frac{f(R)}{R} \mathbf{e}_\Phi \mathbf{e}_\Phi + w'(R) \mathbf{k} \mathbf{e}_R + \mathbf{k} \mathbf{k}, \end{aligned} \tag{3.10}$$

The inverse tensor to \mathbf{F} is calculated using the corollary of the Hamilton–Cayley theorem

$$\begin{aligned}\mathbf{F}^{-1} &= [\mathbf{F}^2 - I_1(\mathbf{F})\mathbf{F} + I_2(\mathbf{F})\mathbf{1}]J^{-1}, \\ \mathbf{F}^{-1} &= \frac{f(R)}{RB(R)}\mathbf{e}_R\mathbf{e}_R + \frac{R}{f(R)}\mathbf{e}_\Phi\mathbf{e}_\Phi - w'(R)\frac{f(R)}{RB(R)}\mathbf{ke}_R + \mathbf{kk}, \\ (\mathbf{F}^T)^{-1} &= \frac{f(R)}{RB(R)}\mathbf{e}_R\mathbf{e}_R + \frac{R}{f(R)}\mathbf{e}_\Phi\mathbf{e}_\Phi - w'(R)\frac{f(R)}{RB(R)}\mathbf{e}_R\mathbf{k} + \mathbf{kk}.\end{aligned}\quad (3.11)$$

These expressions are used to calculate

$$\mathbf{B} = \frac{R^2 B^2(R)}{f^2(R)}\mathbf{e}_R\mathbf{e}_R + \frac{f^2(R)}{R^2}\mathbf{e}_\Phi\mathbf{e}_\Phi + w'(R)\frac{RB(R)}{f(R)}(\mathbf{e}_R\mathbf{k} + \mathbf{ke}_R) + (1 + w'^2(R))\mathbf{kk}, \quad (3.12)$$

$$\mathbf{B}^{-1} = \frac{f^2(R)(1 + w'^2(R))}{R^2 B^2(R)}\mathbf{e}_R\mathbf{e}_R + \frac{R^2}{f^2(R)}\mathbf{e}_\Phi\mathbf{e}_\Phi - w'(R)\frac{f(R)}{RB(R)}(\mathbf{e}_R\mathbf{k} + \mathbf{ke}_R) + \mathbf{kk}, \quad (3.13)$$

$$I_1(\mathbf{B}) = \frac{R^2 B^2(R)}{f^2(R)} + \frac{f^2(R)}{R^2} + 1 + w'^2(R),$$

$$I_2(\mathbf{B}) = B^2(R)\left(1 + \frac{R^2}{f^2(R)}\right) + \frac{f^2(R)(1 + w'^2(R))}{R^2}. \quad (3.14)$$

Using (2.4), (3.12), and (3.13) (the hydrostatic pressure function is considered to depend only on R), we obtain

$$\mathbf{S} = S_{RR}\mathbf{e}_R\mathbf{e}_R + S_{\Phi\Phi}\mathbf{e}_\Phi\mathbf{e}_\Phi + S_{RZ}(\mathbf{e}_R\mathbf{k} + \mathbf{ke}_R) + S_{ZZ}\mathbf{kk}, \quad (3.15)$$

where

$$\begin{aligned}S_{RR} &= s_{RR}(R) + \gamma(R), & S_{\Phi\Phi} &= s_{\Phi\Phi}(R) + \gamma(R), & S_{ZZ} &= s_{ZZ}(R) + \gamma(R), & S_{RZ} &= s_{RZ}(R), \\ s_{RR}(R) &= 2\left[W_1\frac{R^2 B(R)}{f^2(R)} - W_2\frac{f^2(R)(1 + w'^2(R))}{R^2 B(R)}\right], & s_{\Phi\Phi}(R) &= 2\left[W_1\frac{f^2(R)}{R^2 B(R)} - W_2\frac{R^2 B(R)}{f^2(R)}\right], \\ s_{ZZ}(R) &= 2\left[W_1\frac{1 + w'^2(R)}{B(R)} - W_2 B(R)\right], & s_{RZ}(R) &= 2\left[W_1\frac{R}{f(R)} + W_2\frac{f(R)}{R}\right]w'(R).\end{aligned}$$

Using (3.3) and (3.15), we rewrite (2.7) in the components

$$\frac{\partial}{\partial r}[rS_{RZ}(R)] = 0, \quad (3.16)$$

$$r\frac{\partial}{\partial r}S_{RR}(R) + S_{RR}(R) - S_{\Phi\Phi}(R) = r\frac{\partial}{\partial r}[s_{RR}(R) + \gamma(R)] + s_{RR}(R) - s_{\Phi\Phi}(R) = 0. \quad (3.17)$$

The first integral of equation (3.16) is written in the form

$$rS_{RZ}(R) = c \Leftrightarrow S_{RZ}(R) = \frac{c}{f(R)}. \quad (3.18)$$

The integration constant c is found from the force boundary conditions on the cylindrical side surface. The unit normal vector to the lateral surface does not change during deformation and coincides with \mathbf{e}_R . Using (3.15) and (3.18), we obtain the main vector on the lateral surface in the form

$$\frac{Q}{H}\mathbf{k} = \int_0^{2\pi} \mathbf{e}_R \cdot \mathbf{S} r d\Phi = \int_0^{2\pi} \{S_{RR}(R)\mathbf{e}_R + S_{RZ}(R)\mathbf{k}\} f(R) d\Phi = 2\pi S_{RZ}(R) f(R)\mathbf{k} = 2\pi c\mathbf{k},$$

That is $c = Q/(2\pi H)$. Now we have an expression for the longitudinal shear stress

$$S_{RZ} = \frac{Q}{2\pi H} \frac{1}{f(R)}. \tag{3.19}$$

And from (3.15) and (3.18) we obtain the equation for finding the longitudinal displacement

$$w'(R) = \frac{Q}{4\pi H} \frac{1}{f(R) \left[W_1 \frac{R}{f(R)} + W_2 \frac{f(R)}{R} \right]}. \tag{3.20}$$

Based on Eq. (3.19), shear stresses do not depend on the form of the strain energy potential, and the dependence on temperature is determined only by thermal expansion. According to Eq. (3.20), the longitudinal displacement corresponding to the shear stress depends on the type of strain energy potential, and its temperature dependence is determined not only by thermal expansion, but also by the temperature dependence of highly elastic deformation.

Let us turn to Eq. (3.17). By (2.8), using (3.10) and (3.11), we write

$$\mathbf{r}_1 = \frac{RB(R)}{f(R)} \mathbf{e}_R + w'(R) \mathbf{k}, \quad \mathbf{r}_2 = f(R) \mathbf{e}_2, \quad \mathbf{r}_3 = \mathbf{k},$$

$$\mathbf{r}^1 = \frac{f(R)}{RB(R)} \mathbf{e}_R, \quad \mathbf{r}^2 = \frac{1}{f(R)} \mathbf{e}_\Phi, \quad \mathbf{r}^3 = -\frac{f(R)w'(R)}{RB(R)} \mathbf{e}_R + \mathbf{k}$$

and obtain the Hamilton operator in the material coordinate system (R, Φ, Z) in the actual configuration

$$\nabla = \frac{f(R)}{RB(R)} \mathbf{e}_R \frac{\partial}{\partial R} + \frac{1}{f(R)} \mathbf{e}_\Phi \frac{\partial}{\partial \Phi} + \left[-\frac{f(R)w'(R)}{RB(R)} \mathbf{e}_R + \mathbf{k} \right] \frac{\partial}{\partial Z}. \tag{3.21}$$

Comparing (3.21) and (3.3), we obtain

$$r \frac{\partial}{\partial r} = \frac{f^2(R)}{RB(R)} \frac{\partial}{\partial R} - \frac{w'(R)f^2(R)}{RB(R)} \frac{\partial}{\partial Z}$$

using which we rewrite Eq. (3.17) in the form

$$\frac{\partial}{\partial R} [\gamma(R)] = -\frac{\partial}{\partial R} [s_{RR}(R)] - \frac{RB(R)[s_{RR}(R) - s_{\Phi\Phi}(R)]}{f^2(R)},$$

where

$$\gamma(R) = -s_{RR}(R) - \int_{R_1}^R \frac{tB(t)[s_{RR}(t) - s_{\Phi\Phi}(t)]}{f^2(t)} dt + \gamma_0.$$

Here, γ_0 is the integration constant, which is found from the boundary condition $S_{RR}(R_2) = 0$ or

$$\gamma_0 = \int_{R_1}^{R_2} \frac{tB(t)[s_{RR}(t) - s_{\Phi\Phi}(t)]}{f^2(t)} dt \quad \text{and} \quad \gamma(R) = -s_{RR}(R) + \int_R^{R_2} \frac{tB(t)[s_{RR}(t) - s_{\Phi\Phi}(t)]}{f^2(t)} dt.$$

Substituting (3.15) and (3.15), we will get

$$\begin{aligned} \gamma(R) = & -2 \left[W_1 \frac{R^2 B(R)}{f^2(R)} - W_2 \frac{f^2(R)(1 + w'^2(R))}{R^2 B(R)} \right] \\ & + 2 \int_R^{R_2} W_1 \left(\frac{t^3 B^2(t)}{f^4(t)} - \frac{1}{t} \right) + W_2 \left(\frac{t^3 B^2(t)}{f^4(t)} - \frac{(1 + w'^2(t))}{t} \right) dt. \end{aligned}$$

Let's move on to dimensionless quantities, variables and parameters.

$$\bar{\rho} = \frac{R}{R_1}, \quad \kappa = \frac{R_2}{R_1}, \quad \delta = \frac{\Delta}{R_1}, \quad \omega = \frac{w}{R_1}, \quad \upsilon = \frac{f}{R_1}, \quad q = \frac{Q}{2\pi R_1 H \mu_0}, \quad \tilde{\gamma}(\bar{\rho}) = \frac{\gamma(\bar{\rho})}{\mu_0},$$

$$\sigma_{RZ} = \frac{S_{RZ}}{\mu_0}, \quad \sigma_{RR} = \frac{S_{RR}}{\mu_0}, \quad \sigma_{\Phi\Phi} = \frac{S_{\Phi\Phi}}{\mu_0}, \quad \dot{\omega}(\bar{\rho}) = \frac{\partial \omega(\bar{\rho})}{\partial \bar{\rho}} = \frac{\partial w(R)}{\partial R} = w'(R),$$

$$\Sigma_{RR} = \frac{S_{RR}}{\mu_0}, \quad \Sigma_{\Phi\Phi} = \frac{S_{\Phi\Phi}}{\mu_0}, \quad \Sigma_{ZZ} = \frac{S_{ZZ}}{\mu_0}, \quad \Sigma_{RZ} = \frac{S_{RZ}}{\mu_0}.$$

Here, Δ is the longitudinal displacement of the outer side surface of the sleeve relative to the inner one. Using these relations, we get ($\bar{\rho} \in [1, \kappa]$)

- expression characterizing the temperature field

$$\theta(\bar{\rho}) = \frac{\eta}{2-3\alpha\eta} \ln \left\{ 1 + \frac{a(2-3\alpha\eta)}{2} [\bar{\rho}^2 - 1] \right\} + \theta_1; \quad (3.22)$$

- expressions describing the stress-strain state

$$\nu(\bar{\rho}) = \left\{ 1 + \frac{a(2-3\alpha\eta)}{2} [\bar{\rho}^2 - 1] \right\}^{\frac{1}{2-3\alpha\eta}}, \quad (3.23)$$

$$B(\bar{\rho}) = a \left\{ 1 + \frac{a(2-3\alpha\eta)}{2} [\bar{\rho}^2 - 1] \right\}^{\frac{3\alpha\eta}{2-3\alpha\eta}}, \quad a = e^{3\alpha\theta_1},$$

$$\dot{\omega}(\bar{\rho}) = \frac{\mu_0 q}{2} \left[W_1 \bar{\rho} + W_2 \frac{\nu^2(\bar{\rho})}{\bar{\rho}} \right]^{-1}, \quad (3.24)$$

$$\begin{aligned} \Sigma_{RR}(\bar{\rho}) &= \sigma_{RR}(\bar{\rho}) + \tilde{\gamma}(\bar{\rho}), & \Sigma_{\Phi\Phi}(\bar{\rho}) &= \sigma_{\Phi\Phi}(\bar{\rho}) + \tilde{\gamma}(\bar{\rho}), \\ \Sigma_{ZZ}(\bar{\rho}) &= \sigma_{ZZ}(\bar{\rho}) + \tilde{\gamma}(\bar{\rho}), & \Sigma_{RZ}(\bar{\rho}) &= \sigma_{RZ}(\bar{\rho}), \end{aligned} \quad (3.25)$$

where

$$\begin{aligned} \sigma_{RR}(\bar{\rho}) &= \frac{2}{\mu_0} \left[W_1 \frac{\bar{\rho}^2 B(\bar{\rho})}{\nu^2(\bar{\rho})} - W_2 \frac{\nu^2(\bar{\rho})(1 + \dot{\omega}^2(\bar{\rho}))}{\bar{\rho}^2 B(\bar{\rho})} \right], & \sigma_{RZ}(\bar{\rho}) &= \frac{q}{\nu(\bar{\rho})}, \\ \sigma_{\Phi\Phi}(\bar{\rho}) &= \frac{2}{\mu_0} \left[W_1 \frac{\nu^2(\bar{\rho})}{\bar{\rho}^2 B(\bar{\rho})} - W_2 \frac{\bar{\rho}^2 B(\bar{\rho})}{\nu^2(\bar{\rho})} \right], & \sigma_{ZZ}(\bar{\rho}) &= \frac{2}{\mu_0} \left[W_1 \frac{1 + \dot{\omega}^2(\bar{\rho})}{B(\bar{\rho})} - W_2 B(\bar{\rho}) \right], \\ \tilde{\gamma}(\bar{\rho}) &= -\sigma_{RR}(\bar{\rho}) + \frac{2}{\mu_0} \int_{\bar{\rho}}^{\kappa} W_1 \left(\frac{t^3 B^2(t)}{\nu^4(t)} - \frac{1}{t} \right) + W_2 \left(\frac{t^3 B^2(t)}{\nu^4(t)} - \frac{(1 + \dot{\omega}^2(t))}{t} \right) dt. \end{aligned}$$

For the Gent-Thomas potential

$$W_1 = \frac{\mu_0}{B(R)} \left[1 + \frac{\theta(\bar{\rho})}{T_0} \right] \left(\frac{1}{2} - \frac{\beta}{3} \right), \quad W_2 = \frac{\mu_0}{B(R)} \left[1 + \frac{\theta(\bar{\rho})}{T_0} \right] \frac{\beta}{I_2}, \quad (3.26)$$

$$I_2 = B^2(\bar{\rho}) \left(1 + \frac{\bar{\rho}^2}{\nu^2(\bar{\rho})} \right) + \frac{\nu^2(\bar{\rho})(1 + \dot{\omega}^2(\bar{\rho}))}{\bar{\rho}^2} \quad (3.27)$$

and excluding I_2 in expressions (3.24), (3.26), and (3.27), we obtain the cubic equation for $\dot{\omega}$

$$\begin{aligned} a_3 \dot{\omega}^3(\bar{\rho}) + a_2 \dot{\omega}^2(\bar{\rho}) + a_1 \dot{\omega}(\bar{\rho}) + a_0 &= 0, \\ a_3 &= \frac{\bar{\rho} \nu^4(\bar{\rho})(3-2\beta)}{B(R)} \left[1 + \frac{\theta(\bar{\rho})}{T_0} \right], & a_2 &= -3q \nu^4(\bar{\rho}), \\ a_1 &= \{ \bar{\rho}^3 B^2(\bar{\rho}) [\bar{\rho}^2 + \nu^2(\bar{\rho})] (3-2\beta) + \bar{\rho} (4\beta+3) \nu^4(\bar{\rho}) \} B^{-1}(R) \left[1 + \frac{\theta(\bar{\rho})}{T_0} \right], \\ a_0 &= -3q \{ \bar{\rho}^2 [\bar{\rho}^2 + \nu^2(\bar{\rho})] B^2(\bar{\rho}) + \nu^4(\bar{\rho}) \}. \end{aligned} \quad (3.28)$$

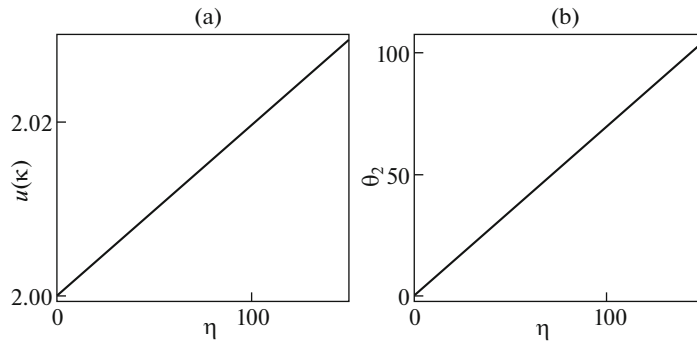


Fig. 1. Dependences $u(\kappa)$ (a) and $\theta_2 = \theta(\kappa)$ (b) on η .

3.2. Calculations Results

In this article, the linear coefficient of thermal expansion $\alpha = 0.0002^\circ\text{C}^{-1}$, which is characteristic of elastomers, is adopted. The undeformed configuration corresponds to a uniform temperature $T_0 = 293$ K. The value of the difference temperature on the inner surface $\theta_1 = T_1 - T_0$ is assumed to be 0°C . The value of the difference temperature on the outer surface $\theta_2 = T_2 - T_0$ is assumed to be 100°C . The case of no heating corresponds to the difference temperature on the outer surface $\theta_2 = 0^\circ\text{C}$.

Equation (3.28) has real coefficients and for the accepted values q , w it has one real and two complex conjugate roots. Only the real root has physical meaning.

All calculations were carried out for $\kappa = 2$. The dependences of the dimensionless stresses on the dimensionless radius were calculated for $q = 0.8$.

Figures 1a and 1b show the dependences of $u(\kappa)$ and $\theta_2 = \theta(\kappa)$ on η . Equality $\theta_2 = 100^\circ\text{C}$ for $\kappa = 2$ is achieved at $\eta = 141.448^\circ\text{C}$, and $\theta_2 = 0^\circ\text{C}$ at $\eta = 0$.

Having the analytical solution (3.22)–(3.25) in the form of functions depending on α and θ_2 as parameters, it is easy to find out the degree of influence of thermal expansion and the temperature dependence of the highly elastic component on the properties of this solution.

Below are the results of calculations in the form of graphs. Each figure shows three curves. Curve 1 corresponds to $\theta_2 = 0$ (no temperature field). Curve 2 corresponds to $\theta_2 = 100^\circ\text{C}$, $\alpha = 0^\circ\text{C}^{-1}$ (temperature field is present, but thermal expansion is absent). Curve 3 corresponds to $\theta_2 = 100^\circ\text{C}$, $\alpha = 0.0002^\circ\text{C}^{-1}$ (temperature field and thermal expansion are present).

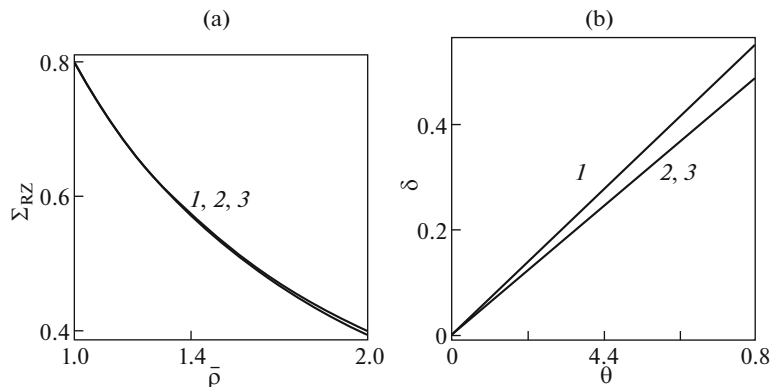


Fig. 2. Dependences Σ_{RZ} on \bar{p} . (a) Dependences δ on q at $\beta = 0.6$. (b) (1) – $\theta_2 = 0$, (2) – $\theta_2 = 100^\circ\text{C}$, $\alpha = 0$, (3) – $\theta_2 = 100^\circ\text{C}$, $\alpha = 2 \times 10^{-4} \text{ }^\circ\text{C}^{-1}$.

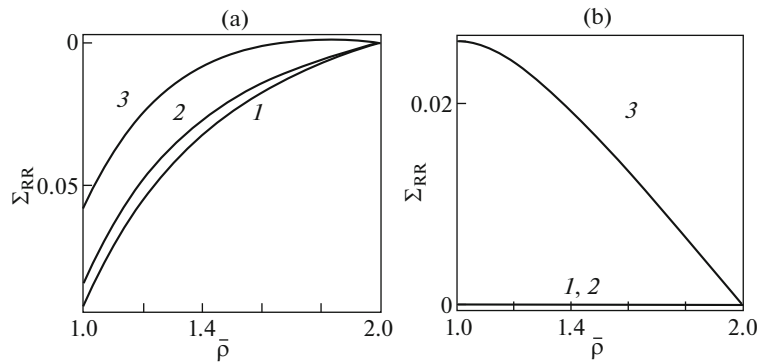


Fig. 3. Dependences Σ_{RR} on $\bar{\rho}$ at $\beta = 0.6$. (a) Dependences Σ_{RR} on $\bar{\rho}$ at $\beta = 0$. (b) (1) $\theta_2 = 0$, (2) $\theta_2 = 100^\circ\text{C}$, $\alpha = 0$, (3) $\theta_2 = 100^\circ\text{C}$, $\alpha = 2 \times 10^{-4} \text{ }^\circ\text{C}^{-1}$.

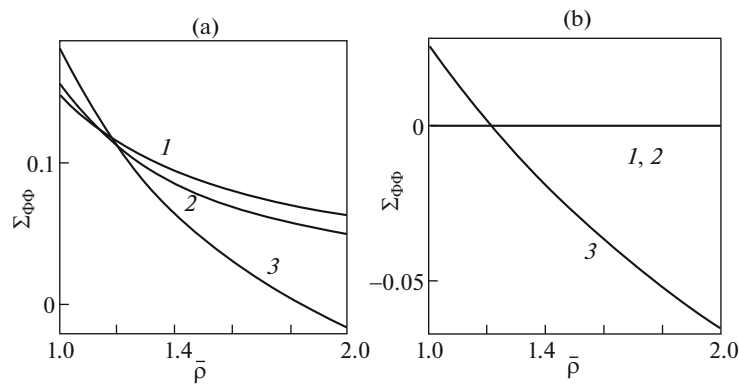


Fig. 4. Dependences $\Sigma_{\Phi\Phi}$ on $\bar{\rho}$ at $\beta = 0.6$. (a) Dependences $\Sigma_{\Phi\Phi}$ on $\bar{\rho}$ at $\beta = 0$. (b) (1) $\theta_2 = 0$, (2) $\theta_2 = 100^\circ\text{C}$, $\alpha = 0$, (3) $\theta_2 = 100^\circ\text{C}$, $\alpha = 2 \times 10^{-4} \text{ }^\circ\text{C}^{-1}$.

Figure 2a shows the dependences $\Sigma_{RZ} = \Sigma_{RZ}(\bar{\rho})$. All three curves visually coincide, since the temperature dependence of this stress is determined only by a very slight temperature change in the area of the cylindrical surface (3.24) due to thermal expansion. Figure 2b for $\beta = 0.6$ shows the dependencies $\delta = \delta(q)$ (stiffness characteristics). An increase in temperature increases the shear stiffness, and the coincidence of curves 2 and 3 indicates that the process is more strongly influenced by the temperature dependence of the highly elastic component of deformation.

Figure 3a shows the dependence curves of $\Sigma_{RR} = \Sigma_{RR}(\bar{\rho})$ at $\beta = 0.6$. The closeness of curves 1 and 2 indicates that the temperature dependence of the highly elastic deformation is weak. The significant difference between curve 3 and curves 1 and 2 shows that the temperature dependence of the radial stress is mainly determined by thermal expansion. Figure 3b shows the curves $\Sigma_{RR} = \Sigma_{RR}(\rho)$ for $\beta = 0$ (Tréloar potential). Here, the highly elastic deformation does not contribute to Σ_{RR} at all, and these stresses arise exclusively due to thermal expansion. A similar situation is also true for the temperature behavior of the $\Sigma_{\Phi\Phi} = \Sigma_{\Phi\Phi}(\bar{\rho})$ dependence, which is illustrated in Fig. 3a with $\beta = 0.6$ and Fig. 3b with $\beta = 0$.

CONCLUSION

Within the framework of the phenomenological approach, a variant of setting the problems of static thermoelasticity of a homogeneous isotropic incompressible material at finite deformations is proposed, which allows taking into account both thermal expansion and temperature changes in stresses caused by highly elastic deformation in an inhomogeneous temperature field. The problem of a finite longitudinal shear of a hollow cylinder in an axisymmetric temperature field is considered. Within the framework of

the proposed formulation with the Jent-Thomas potential, a new analytical solution of the coupled problem of thermoelasticity is obtained. The general result of thermoelastic analysis is that the temperature dependence of different characteristics of the stress-strain state can be controlled by different mechanisms. Thus, the temperature dependence of shear stress, which causes longitudinal shear, is determined only by thermal expansion, while the temperature dependence of shear stiffness is determined mainly by the temperature change in highly elastic deformation. The temperature dependence of normal stresses is significantly determined by thermal expansion, and to a lesser extent by the temperature change of highly elastic deformation at $W_2 \neq 0$. At $W_2 = 0$, the highly elastic deformation does not contribute to the normal stresses at all, and they arise exclusively due to thermal expansion. Such information is of particular interest in the design of rubber-metal parts, since the thermal expansion is affected by the presence of free surfaces that can be specified constructively.

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