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On Solutions of the Equations of Motion of a Gyrostat with a Variable Gyrostatic Moment

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Abstract—The problem of the motion of a gyrostat with a variable gyrostatic moment under the action of potential and gyroscopic forces is considered. Three new solutions of the equations of motion are obtained, which are determined by three linear invariant relations for the components of the angular velocity vector of the gyrostat. In the case of a heavy gyrostat, a solution is found when the gyrostat mass distribution is characterized by the Kovalevskaya and Goryachev—Chaplygin generalized conditions. The next two solutions are valid for equations of the Kirchhoff—Poisson class. One of them exists in the case of dynamically symmetric gyrostats, and in the other solution the gyrostat mass distribution is arbitrary.

Keywords: gyrostatic moment, invariant relations, potential and gyroscopic forces, Kovalevskaya and Goryachev—Chaplygin generalized conditions

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INTRODUCTION

The problem of the motion of a gyrostat with a fixed point is a generalization of the classical problem described by the Euler—Poisson equations. Formulation of the problem of motion of a gyrostat and the first results were obtained in works [1—4] et al. Different determinations and types of gyrostats applied in the gyrostat dynamics are connected with consideration of various formulations.

The first formulation lies in the fact that a system of d bodies S consisting of a carrier body S_0 and weightless rotors S_1, ..., S_n rotating around their symmetry axes is studied. Specifically, gyrostats [1] are considered in this formulation. Monograph [5] provides a full definition of a gyrostat with a reference to article [4]. The main assumption in this definition is that the mass distribution in system S does not change with time. In addition, when studying gyrostats [1] it is assumed that they have constant relative components of total angular momentum, computed with regard to the carrier body. The motion of a gyrostat, which is statically and dynamically balanced [6] or is characterized by the property of dynamic symmetry of rotors rotating around their barycentric axes [7] is considered in [6, 7]. If the gyrostatic momentum of a gyrostat is constant, then, for example, the equations of motion of a heavy gyrostat have three first integrals.

The second formulation of the problem of motion of a gyrostat (Zhukovskii—Volterra gyrostats) is characterized by the fact that systems formed by the carrier body with internal cavities with liquid circulating in them are considered.

From the applied point of view, accounting for the variability of the gyrostatic moment [7, 8] is an important property of gyrostat motion. This circumstance is considered when studying gyro-satellites [9—11]. Mathematical simulation of the motion of a gyrostat with a fixed point under the impact of potential and gyroscopic forces is of particular significance, since it allows establishing the basic properties of dynamics of a gyrostat with a variable gyrostatic moment. Numerous studies have been published in this field, among which it is worth noting articles [11, 12] as well as monograph [13], which provide a survey of results obtained in the dynamics of a nonautonomous gyrostat. In the last monograph, most attention is paid to the analysis of results on studying the precessional motions of a gyrostat. This work is devoted to integration of the equations of motion of a gyrostat with variable gyrostatic moment under the impact of potential and gyroscopic forces.

In order to investigate the conditions for the existence of solutions of gyrostat equations of motion, we applied the method of invariant relations (IR). This method was developed in [5, 14] and generalized in article [15]. The method of IR was applied [16, 17] to other problems of dynamics. This approach is connected with the fact that in the general case of dynamics equations of a rigid body and a gyrostat are not integrable in Jacobi quadratures [18, 19]. In this article, we considered the problem of motion of a gyrostat under the impact of potential and gyroscopic forces on the equations of motion set by IR. Three new solutions in the dynamics of a nonautonomous gyrostat were obtained. For the case of a heavy gyrostat, the conditions of existence are characterized by the following conditions on distribution of masses of the gyrostat: the gyrostat is dynamically symmetrical, the center of mass is in the equatorial plane (generalized conditions of Kovalevskaya and Goryachev–Chaplygin). The two next solutions are valid in the problem of motion of a gyrostat under the impact of potential and gyroscopic forces; one of them is fulfilled for the same classes of gyrostats and the other corresponds to the case of an arbitrary distribution of the masses of the gyrostat.

1. PROBLEM FORMULATION

Most problems of dynamics of a rigid body and a gyrostat are described by a set of differential equations, which includes the Poisson equations [20–25]

$$\dot{\boldsymbol{\nu}} = \boldsymbol{\nu} \times \boldsymbol{\omega}, \tag{1.1}$$

where $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$ is a vector characterizing the direction of the symmetry axis of the force field; $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is a vector of the angular velocity of the carrier body of the gyrostat; the dot above the variable $\boldsymbol{\nu}$ means the derivative with respect to time t.

An important class of invariant relations (IR) was studied [25], which are denoted as [25, 26]

$$\omega_1 = \nu_1\varepsilon + \beta_1 g, \quad \omega_2 = \nu_2\varepsilon + \beta_2 g, \quad \omega_3 = h, \tag{1.2}$$

where β_1, β_2 are constant parameters, $\varepsilon = \varepsilon(\nu_3)$, $g = g(\nu_3)$, $h = h(\nu_3)$ are differentiated functions of the variable ν_3. The feature of IR (1.2) is that Eq. (1.1), which in scalar form is reduced to equation set [25]

$$\dot{\nu}_1 = \nu_2(h - \nu_3\varepsilon) - \beta_2\nu_3 g, \quad \dot{\nu}_2 = \nu_1(\nu_3\varepsilon - h) + \beta_1\nu_3 g, \quad \dot{\nu}_3 = (\beta_2\nu_1 - \beta_1\nu_2)g, \tag{1.3}$$

allows integral representation

$$\beta_1\nu_1 + \beta_2\nu_2 + \int \frac{[h - \nu_3\varepsilon]d\nu_3}{g} = c_0, \tag{1.4}$$

where c_0 is an arbitrary constant. The presence of relation (1.4) allowed building new classes of solutions of the equations of motion of a rigid body having, for example, a fixed point in a potential force field [26, 27], which were studied in [27]. However, IR (1.2) were not considered in the problem of motion of a gyrostat with a variable gyrostatic moment.

Let us denote the equations of motion of a gyrostat under the impact of potential and gyroscopic forces [23–25, 28]. As a movable coordinate system $Oxyz$ with unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ we will chose the main coordinate system of the carrier body:

$$\dot{\mathbf{x}} + \dot{\boldsymbol{\lambda}} = (\mathbf{x} + \boldsymbol{\lambda}(t)) \times a\mathbf{x} + a\mathbf{x} \times B\boldsymbol{\nu} + \mathbf{s} \times \boldsymbol{\nu} + \boldsymbol{\nu} \times C\boldsymbol{\nu}, \tag{1.5}$$

$$\dot{\boldsymbol{\nu}} = \boldsymbol{\nu} \times a\mathbf{x}, \tag{1.6}$$

where a is a gyratory tensor: $a = \mathrm{diag}(a_1, a_2, a_3)$, $B = \mathrm{diag}(B_1, B_2, B_3)$, $C = \mathrm{diag}(C_1, C_2, C_3)$; $\mathbf{s} = (s_1, s_2, s_3)$ is a constant vector. Let us write IR (1.2) in the components of vector $\mathbf{x}$. Using equalities $x_i = \omega_i / a_i$ $(i = \overline{1,3})$, from (1.2) we obtain [25]

$$x_1 = \frac{1}{a_1}(\nu_1\varepsilon + \beta_1 g), \quad x_2 = \frac{1}{a_2}(\nu_2\varepsilon + \beta_2 g), \quad x_3 = \frac{1}{a_3}h. \tag{1.7}$$

We will assume that the vector of the gyrostatic moment is denoted as [7]

$$\boldsymbol{\lambda}(t) = \sum_{i=1}^{3} D_i(a\mathbf{x} \cdot \mathbf{i}_i + \dot{\kappa}_i)\mathbf{i}_i, \tag{1.8}$$

where D_i ($i = \overline{1,3}$) is the moment of inertia of weightless bodies S_i ($i = \overline{1,3}$) with respect to the main axes of inertia; $\dot{\kappa}_i$ is the angular rotation velocities of these bodies around axes l_i ($i = \overline{1,3}$), directed along the main axes of inertia. The total angular momentum of the gyrostat (S_0, S_1, S_2, S_3) is expressed by the formula [7]

$$\mathbf{x}^* = \mathbf{x} + \boldsymbol{\lambda}(t). \tag{1.9}$$

In equality (1.9) $\mathbf{x} = A\boldsymbol{\omega}$, where $A = a^{-1}$ is the tensor of inertia, the function $\boldsymbol{\lambda}(t)$ is determined by equality (1.8). If we integrate Eqs. (1.5) and (1.6), then we have to additionally consider equations

$$D_i \dot{p}_i(t) = L_i(t) \quad (i = \overline{1,3}), \tag{1.10}$$

where due to (1.8), $p_i(t)$ are denoted as

$$p_i = a\mathbf{x} \cdot \mathbf{i}_i + \dot{\kappa}_i. \tag{1.11}$$

In Eqs. (1.10), $L_i(t)$ are the projections of the internal forces acting from the side of the carrier body S_i, on l_i.

Equations (1.5) and (1.6) admit two first integrals

$$\boldsymbol{\nu} \cdot \boldsymbol{\nu} = 1, \quad (\mathbf{x} + \boldsymbol{\lambda}(t)) \cdot \boldsymbol{\nu} - \frac{1}{2}(B\boldsymbol{\nu} \cdot \boldsymbol{\nu}) = k, \tag{1.12}$$

where k is an arbitrary constant.

2. THE CASE OF A HEAVY GYROSTAT

Let us assume in Eqs. (1.5) and (1.12) $B = 0$, $C = 0$, $\lambda_1(t) = 0$, $\lambda_2(t) = 0$ and write (1.6) in scalar form:

$$a_2 x_2 \lambda_3(t) = -\dot{x}_1 + (a_3 - a_2)x_2 x_3 + s_2\nu_3 - s_3\nu_2, \tag{2.1}$$

$$a_1 x_1 \lambda_3(t) = \dot{x}_2 - (a_1 - a_3)x_3 x_1 - s_3\nu_1 + s_1\nu_3, \tag{2.2}$$

$$\dot{\lambda}_3(t) = -\dot{x}_3 + (a_2 - a_1)x_1 x_2 + s_1\nu_2 - s_2\nu_1, \tag{2.3}$$

$$\dot{\nu}_1 = a_3 x_3 \nu_2 - a_2 x_2 \nu_3, \quad \dot{\nu}_2 = a_1 x_1 \nu_3 - a_3 x_3 \nu_1, \quad \dot{\nu}_3 = a_2 x_2 \nu_1 - a_1 x_1 \nu_2, \tag{2.4}$$

$$\nu_1^2 + \nu_2^2 + \nu_3^2 = 1, \quad (x_1 + \lambda_1(t))\nu_1 + (x_2\nu_2 + x_3\nu_3)\nu_3 = k. \tag{2.5}$$

The gyrostatic momentum $\boldsymbol{\lambda}(t)$ from (1.8) is simplified:

$$\boldsymbol{\lambda}(t) = \lambda_3(t)\mathbf{i}_3, \quad \lambda_3(t) = D_3[(a\mathbf{x} \cdot \mathbf{i}_3) + \dot{\kappa}_3]. \tag{2.6}$$

Let us consider Eqs. (2.1)–(2.6). We will exclude the variable $\lambda_3(t)$ from Eqs. (2.1) and (2.2) and present the result as

$$\left[\frac{1}{2}\gamma_+(a, x, x) - s_3\nu_3\right]^{\bullet} + a_3(a_2 - a_1)x_1 x_2 x_3 + \nu_3\gamma_-(s, ax) = 0, \tag{2.7}$$

where the dot indicates a derivative with respect to time of the function within the square brackets. Equation (2.7) was considered [12] in the case when the following equalities are true:

$$a_2 = a_1, \quad s_2 = 0, \quad s_1 = 0. \tag{2.8}$$

Due to conditions (2.8) from Eq. (2.7) we obtain the first integral

$$\frac{a_1}{2}(x_1^2 + x_2^2) - s_3\nu_3 = b_0, \tag{2.9}$$

where b_0 is an arbitrary constant. An additional integral [12] is found from Eq. (2.3)

$$x_3 + \lambda_3(t) = \text{const}. \tag{2.10}$$

Conditions (2.8) characterize the generalized Lagrangian integral (2.10) of the problem of the motion of a heavy rigid body. If we differentiate the second equation from (2.5) only by virtue of Eqs. (2.3) and (2.4), then we again obtain Eq. (2.7).

As the second resolving equation we will use a combination of Eqs. (2.1), (2.2), which is obtained as a result of exclusion of the parameter s_3 from these equations:

$$(x_3 + \lambda_3(t))\dot{\nu}_3 = -\gamma_+(1, \dot{x}, \nu) + a_3 x_3 \gamma_-(\nu, x) + \nu_3 \gamma_-(\nu, s). \tag{2.11}$$

Let us substitute the value of $x_3 + \lambda_3$, determined from the second equation of (2.5), into (2.11) and use the third equation from set (2.4):

$$\frac{k\dot{\nu}_3}{\nu_3} = \frac{\dot{\nu}_3}{\nu_3}\gamma_+(1, x, \nu) - \gamma_+(1, \dot{x}, \nu) + a_3 x_3 \gamma_-(\nu, x) + \nu_3 \gamma_-(\nu, s). \tag{2.12}$$

In this way, when considering the conditions for the existence of IR (1.7), it is necessary to study the reduced set, which consists of Eqs. (2.4), (2.7), (2.12).

Let us consider Eqs. (2.4), (2.7), (2.12) given the presence of IR (1.7) in them. First of all, let us study equation set (1.3) in the case where functions h and ε satisfy the equality

$$h = \nu_3 \varepsilon. \tag{2.13}$$

Then, from Eq. (1.3) and representation (1.4) we have

$$\dot{\nu}_1 = -\beta_2 \nu_3 g, \quad \dot{\nu}_2 = \beta_1 \nu_3 g, \quad \dot{\nu}_3 = \gamma_-(\nu, \beta) g, \tag{2.14}$$

$$\gamma_+(1, \beta, \nu) = c_0. \tag{2.15}$$

It was shown in [25, 27] that the components ν_1, ν_2 of vector $\boldsymbol{\nu}$ are the functions of an auxiliary variable ν_3 by virtue of the integral $\nu_1^2 + \nu_2^2 + \nu_3^2 = 1$ and IR (2.15):

$$\begin{gathered}\nu_1 = \frac{1}{\kappa_0^2}(c_0\beta_1 + \beta_2\sqrt{D(\nu_3)}), \quad \nu_2 = \frac{1}{\kappa_0^2}(c_0\beta_2 - \beta_1\sqrt{D(\nu_3)}), \\ D(\nu_3) = (\kappa_0^2 - c_0^2) - \kappa_0^2 \nu_3^2.\end{gathered} \tag{2.16}$$

and the dependence $v_3(t)$ is found by inversion of the integral

$$\int\limits_{\nu_3^{(0)}}^{\nu_3} \frac{d\nu_3}{g\sqrt{D(\nu_3)}} = t - t_0. \tag{2.17}$$

In formulas (2.16), (2.17) $\kappa_0^2 = \beta_1^2 + \beta_2^2$.

Let us write Eqs. (2.7), (2.12) taking equalities (1.7) and (2.13) into account:

$$\begin{gathered}\dot{\nu}_3\{\varepsilon'[\gamma_+(a, \nu, \nu)\varepsilon + \gamma_+(a, \nu, \beta) g] + g'[\gamma_+(a, \nu, \beta)\varepsilon + \gamma_+(a, \beta, \beta) g]\} \\ = a_1 a_2 [s_3 \dot{\nu}_3 - \nu_3 \gamma_-(s, \nu)\varepsilon - \nu_3 \gamma_-(s, \beta) g] - \nu_3 \varepsilon^2 [(a_2 - a_1)\nu_1\nu_2\varepsilon + \gamma_-(\beta, a\nu) g],\end{gathered} \tag{2.18}$$

$$\begin{gathered}\dot{\nu}_3[\gamma_+(a, \nu, \nu)\varepsilon' + \gamma_+(a, \nu, \beta) g'] = \frac{\dot{\nu}_3}{\nu_3}[\gamma_+(a, \nu, \nu)\varepsilon + \gamma_+(a, \nu, \beta) g] \\ + a_1 a_2\left[\nu_3 \gamma_-(\nu, s) - \frac{k\dot{\nu}_3}{\nu_3}\right] + \dot{\nu}_3 \nu_3 (a_1 + a_2)\varepsilon + (a_1 - a_2)\nu_1\nu_2\nu_3\varepsilon^2.\end{gathered} \tag{2.19}$$

For convenience of the use of Eqs. (2.18), (2.19) we did not substitute the value of

$$\dot{\nu}_3 = \gamma_-(\nu, \beta) g. \tag{2.20}$$

in them.

Thus the following statement is true.

Statement 1. The problem of integration of the equations of motion of a heavy gyrostat (2.1)−(2.4) on IR (1.7), where the function $h(\nu_3)$ is denoted as (2.13), is reduced to the integration of Eqs. (2.18), (2.19) and determination of the function $\nu_3(t)$ by inversion of integral (2.17).

Remark 1. The method of obtaining Eq. (2.18) and the form of Eq. (2.7) implies that on the considered IR with conditions

$$g = g_0, \quad \varepsilon = \varepsilon_0, \quad a_2 = a_1, \quad s_3 = 0, \quad s_1 = \sigma_0\beta_1, \quad s_2 = \sigma_0\beta_2, \tag{2.21}$$

where g_0, ε_0, σ_0 are constants, Eq. (2.7) has the first integral

$$\frac{a_1}{2}(x_1^2 + x_2^2) - s_3\nu_3 - \frac{\varepsilon_0}{2g_0}\sigma_0\nu_3^2 = c_*. \tag{2.22}$$

Here, c_* is an arbitrary constant. Based on conditions from (2.21) one may conclude that the distribution of masses of the gyrostat is determined by the generalized conditions of Kovalevskaya ($a_3 = 2a_1$) and Goryachev–Chaplygin ($a_3 = 4a_1$). Due to first two equalities of set (2.21) and equality (2.13), IR (1.2) are linear functions of variables ν_i $i = \overline{1,3}$.

For further consideration of Eqs. (2.18), (2.19), let us represent them in the following form:

$$\varepsilon' = H(\nu_3, \varepsilon, g), \quad g' = L(\nu_3, \varepsilon, g). \tag{2.23}$$

Apparently, this set is of only theoretical interest. Thus, let us consider an example of the integration of set (2.18), (2.19). We assume that functions ε and g take constant values

$$\varepsilon = \varepsilon_0, \quad g = g_0, \tag{2.24}$$

which are a part of conditions (2.21) (we do not use the rest of the equalities). Let us introduce new variable ψ instead of ν_3 according to the formula

$$\nu_3 = \frac{\mu_0}{\kappa_0}\sin\psi, \quad \mu_0 = \sqrt{\kappa_0^2 - c_0^2}. \tag{2.25}$$

Then, from the third equation of set (2.14), by virtue of (2.15) and (2.25), we obtain

$$\psi(t) = \kappa_0 g_0 t + \psi_0. \tag{2.26}$$

Based on (2.16) and (2.25) let us denote the components ν_1, ν_2:

$$\nu_i(\psi) = \frac{1}{\kappa_0^2}(c_0\beta_i - (-1)^i\mu_0\beta_{3-i}\cos\psi), \quad i = 1, 2. \tag{2.27}$$

Let us consider relations (1.7) in order to study the dependence of the other variables of the problem on the variable ψ. Based on formulas (2.13), (2.24), (2.25), (2.27) we find

$$\begin{gathered} x_i(\psi) = \frac{1}{a_i\kappa_0^2}[\beta_i(c_0\varepsilon_0 + g_0\kappa_0^2) - (-1)^i\beta_{3-i}\varepsilon_0\mu_0\cos\psi], \quad i = 1, 2, \\ x_3(\psi) = \frac{\varepsilon_0\mu_0}{a_3\kappa_0}\sin\psi. \end{gathered} \tag{2.28}$$

Using equalities (2.25), (2.27) and (2.28), let us require that Eqs. (2.18), (2.19) would be identities by variable ψ. Then, we obtain the following conditions:

$$a_2 = a_1, \quad s_3 = 0, \quad s_1 = \sigma_0\beta_1, \quad s_1 = \sigma_0\beta_1, \quad s_2 = \sigma_0\beta_2. \tag{2.29}$$

$$k = \frac{1}{a_1}(c_0 g_0 + \varepsilon_0), \quad \sigma_0 = -\frac{\varepsilon_0 g_0}{a_1}. \tag{2.30}$$

Conditions (2.29) coincide with conditions (2.21), under which the integral (2.22) is valid. Their characteristic is given above. It is necessary to note that in equalities (2.29) the parameter σ_0 takes a certain value from (2.30).

In order to study the properties of the forces acting on a weightless body from the side of the carrier body, let us consider the third equation from (1.10) at $i = 3$. Using the third equation from (1.11), due to

$$p_3(t) = \frac{\varepsilon_0\mu_0}{\kappa_0}\sin(\kappa_0 g_0 t + \psi_0) + \dot{\kappa}_3(t), \quad \lambda_3(t) = \frac{\varepsilon_0\mu_0}{\kappa_0}\frac{a_3 - a_1}{a_1 a_3}\sin(\kappa_0 g_0 t + \psi_0), \tag{2.31}$$

we find

$$L_3(t) = \varepsilon_0\mu_0 g_0\frac{a_3 - a_1}{a_1 a_3}\cos(\kappa_0 g_0 t + \psi_0). \tag{2.32}$$

Thus, projection of the inner forces on the rotation axis of the rotor S_3 equals the value of (2.32).

The following statement is true.

Statement 2. The necessary conditions for the existence of linear invariant relations in the main variables of problem (1.5) and (1.6) for system (2.18) and (2.19) are equalities (2.29) and (2.30), which characterize a dynamically symmetric gyrostat whose mass distribution is determined by the generalized conditions of Kovalevskaya and Goryachev–Chaplygin.

3. THE CASE OF MOTION OF GYROSTAT UNDER THE IMPACT OF POTENTIAL AND GYROSCOPIC FORCES

Let us write Eq. (1.5) in scalar form assuming that $\lambda_1(t) = 0$, $\lambda_2(t) = 0$:

$$a_2x_2\lambda_3(t) = -\dot{x}_1 + (a_3 - a_2)x_2x_3 + s_2\nu_3 - s_3\nu_2 + a_2B_3x_2\nu_3 - a_3B_2\nu_2x_3 + (C_3 - C_2)\nu_2\nu_3, \tag{3.1}$$

$$a_1x_1\lambda_3(t) = \dot{x}_2 - (a_1 - a_3)x_3x_1 - s_3\nu_1 + s_1\nu_3 - a_3B_1x_3\nu_1 + a_1B_3\nu_3x_1 - (C_1 - C_3)\nu_3\nu_1, \tag{3.2}$$

$$\dot{\lambda}_3(t) = -\dot{x}_3 + (a_2 - a_1)x_1x_2 + s_1\nu_2 - s_2\nu_1 + a_1B_2x_1\nu_2 - a_2B_1\nu_1x_2 + (C_2 - C_1)\nu_1\nu_2. \tag{3.3}$$

Equations (3.1)–(3.3) should be supplemented by Poisson equations (2.4) and first integrals (1.12):

$$\nu_1^2 + \nu_2^2 + \nu_3^2 = 1, \quad x_1\nu_1 + x_2\nu_2 + (x_3 + \lambda_3(t))\nu_3 - \frac{1}{2}(B_1\nu_1^2 + B_2\nu_2^2 + B_3\nu_3^2) = k, \tag{3.4}$$

since in chapter 2 it is shown that in this particular case complete analysis of the conditions of the existence of IR (1.2) can be carried out. Using the approach suggested in chapter 2, let us reduce equation set (3.1)–(3.3) to

$$\begin{gathered}\left[\frac{1}{2}\gamma_+(a,x,x) - s_3\nu_3\right]^{\bullet} + a_3(a_2 - a_1)x_1x_2x_3 + \nu_3\gamma_-(s,ax) \\ + a_3x_3\gamma_-(ax,B\nu) - \nu_3[\gamma_-(C\nu,x) - C_3\gamma_-(\nu,ax)] = 0,\end{gathered} \tag{3.5}$$

$$\begin{gathered}\frac{\dot{\nu}_3k}{\nu_3} = \frac{1}{\nu_3}\gamma_+(1,x,\nu)\dot{\nu}_3 - \gamma_+(1,\dot{x},\nu) + a_3x_3\gamma_-(\nu,x) + \nu_3\gamma_-(\nu,s) \\ - \frac{\dot{\nu}_3}{2\nu_3}(B_1\nu_1^2 + B_2\nu_2^2 - B_3\nu_3^2) + \nu_1\nu_2[a_3(B_1 - B_2)x_3 + (C_1 - C_2)\nu_3].\end{gathered} \tag{3.6}$$

In order to obtain a closed equation set it is necessary to consider set (3.5), (3.6) together with Poisson equations

$$\dot{\nu}_1 = a_3x_3\nu_2 - a_2x_2\nu_3, \quad \dot{\nu}_2 = a_1x_1\nu_3 - a_3x_3\nu_1, \quad \dot{\nu}_3 = a_2x_2\nu_1 - a_1x_1\nu_2. \tag{3.7}$$

After integrating set (3.5)–(3.7) we determine function $\lambda_3(t)$ from the second relation of set (3.4):

$$\lambda_3(t) = \frac{1}{\nu_3}\left[k - x_1\nu_1 - x_2\nu_2 - x_3\nu_3 + \frac{1}{2}(B_1\nu_1^2 + B_2\nu_2^2 + B_3\nu_3^2)\right]. \tag{3.8}$$

As mentioned above, in the assumption that $B = 0$, $C = 0$ the case corresponding to the following quantities in the notation of this chapter was considered: $s_1 = 0$, $s_2 = 0$, $a_2 = a_1$. Let us consider a more general option, putting in (3.3), (3.5), (3.6) and (3.8)

$$s_1 = 0, \quad s_2 = 0, \quad a_2 = a_1, \quad B_2 = B_1, \quad C_2 = C_1. \tag{3.9}$$

Then, (3.3) implies the first integral

$$\lambda(t) + x_3 + B_1\nu_3 = b_1, \tag{3.10}$$

where b_1 is an arbitrary constant. Under conditions (3.9) Eq. (3.5) is denoted as

$$\left[\frac{a_1}{2}(x_1^2 + x_2^2) - s_3\nu_3 + \frac{1}{2}(C_3 - C_1)\nu_3^2\right]^{\bullet} = -B_1a_3x_3\dot{\nu}_3. \tag{3.11}$$

Assuming $B_1 = 0$ in (3.11), we obtain an additional first integral

$$\frac{a_1}{2}(x_1^2 + x_2^2) - s_3\nu_3 + \frac{1}{2}(C_3 - C_1)\nu_3^2 = b_2, \tag{3.12}$$

where b_2 is an arbitrary constant.

Remark 2. The presence of the three first integrals (3.10)–(3.12) in equation set (3.3), (3.5), and (3.6) does not allow integrating the given set in quadratures. A similar problem occurs in the studies of article [12]. Thus it was additionally assumed in [12] that in relations (2.6) $\lambda_3 = \text{const}$. In the case under consideration this assumption by virtue of relation (3.10) leads to a solution [24], which is a generalization of the Kirchhoff solution [22].

This remark and results of chapter 2 show that it is reasonable to consider Eqs. (3.5)–(3.8) in the case of the existence of invariant relations in them [25]

$$x_1(\nu_3) = \frac{1}{a_1}(\nu_1(\nu_3)\varepsilon + \beta_1 g), \quad x_2(\nu_3) = \frac{1}{a_2}(\nu_2(\nu_3)\varepsilon + \beta_2 g), \quad x_3(\nu_3) = \frac{1}{a_3}\nu_3\varepsilon. \tag{3.13}$$

Let us substitute the values (3.13) into Eqs. (3.5) and (3.6) and use Poisson equations (3.7):

$$\begin{gathered}\gamma_-(\nu,\beta)\{[\varepsilon\gamma_+(a,\nu,\nu) + g\gamma_+(a,\nu,\beta)]g\varepsilon' + [\varepsilon\gamma_+(a,\nu,\beta) + g\gamma_+(a,\beta,\beta)]gg'\} \\ = -\nu_3\{(a_2 - a_1)\nu_1\nu_2\varepsilon + [\gamma_-(\beta, a\nu)g + a_1a_2(B_2 - B_1)\nu_1\nu_2]\}\varepsilon^2 \\ + a_1a_2\{\nu_3[(C_1 - C_2)\nu_1\nu_2 - \gamma_-(\beta, B\nu)g - \gamma_-(s,\nu)]\varepsilon \\ + [s_3\gamma_-(\nu,\beta) - \nu_3\gamma_-(s,\beta) + \beta_2(C_1 - C_3)\nu_1\nu_3 + \beta_1(C_3 - C_2)\nu_2\nu_3]g\},\end{gathered} \tag{3.14}$$

$$\begin{gathered}\frac{\dot\nu_3}{a_1a_2}[\gamma_+(a,\nu,\nu)\varepsilon' + \gamma_+(a,\nu,\beta)g'] = \frac{1}{a_1a_2}\left\{\frac{\dot\nu_3}{\nu_3}[\gamma_+(a,\nu,\nu)\varepsilon + \gamma_+(a,\nu,\beta)g]\right. \\ \left. + \nu_3[(a_1 - a_2)\varepsilon^2\nu_1\nu_2 + (a_1 + a_2)\varepsilon\dot\nu_3 + a_1a_2\gamma_-(\nu,s)]\right\} \\ - \frac{\dot\nu_3}{2\nu_3}(B_1\nu_1^2 + B_2\nu_2^2 - B_3\nu_3^2 + 2k) + \nu_1\nu_2[a_3(B_1 - B_2)x_3 + (C_1 - C_2)\nu_3],\end{gathered} \tag{3.15}$$

where $\dot\nu_3$ has the value (2.20). For convenience of studying Eq. (3.15), the expression of $\dot\nu_3$ is not included in (3.15). Due to the problem formulation, it is necessary to join Eqs. (3.14), (3.15) with Eqs. (2.14) and invariant relation (2.15). I.e., in Eqs. (3.14) and (3.15) functions $\nu_1(\nu_3)$, $\nu_2(\nu_3)$ are denoted as (2.16) and dependence $\nu_3(t)$ is established from (2.17). It is necessary to note that IR (2.15) describe precessional motions [28, 29] – motions, where the angle between vectors $\boldsymbol{\beta}(\beta_1, \beta_2, 0)$ and $\boldsymbol{\nu}$ is constant.

In the general case, integration of Eqs. (3.14) and (3.15) represents quite a complicated problem (see chapter 2 of this article). Thus, let us consider the option when IR (3.13) are linear functions, i.e., the components of angular velocity vector are denoted as

$$\omega_1 = \varepsilon_0\nu_1 + \beta_1 g_0, \quad \omega_2 = \varepsilon_0\nu_2 + \beta_2 g_0, \quad \omega_3 = \varepsilon_0\nu_3. \tag{3.16}$$

Due to equalities (3.16) precession of the carrier body refers to class of regular precessions [29–31].

Let us substitute the values (3.13) (where $\varepsilon(\nu_3) = \varepsilon_0$, $g(\nu_3) = g_0$) and the values (2.25) and (2.27) into Eqs. (3.14), (3.15) and require that the equalities obtained would be identities by ψ:

$$(a_2 - a_1)(2c_0\varepsilon_0 + g_0\kappa_0^2) + c_0a_1a_2(B_2 - B_1) = 0, \tag{3.17}$$

$$\sigma_0\frac{g_0}{\kappa_0^2}\left\{-\frac{\varepsilon_0(a_1\beta_2^2 + a_2\beta_1^2)}{a_1a_2} + \frac{1}{2}[\beta_1^2(B_3 - B_2) + \beta_2^2(B_3 - B_1)]\right\}, \tag{3.18}$$

$$\varepsilon_0^2(a_1 - a_2) + a_1a_2[\varepsilon_0(B_1 - B_2) + (C_1 - C_2)] = 0, \tag{3.19}$$

$$s_3 = 0, \quad s_1 = \sigma_0\beta_1, \quad s_2 = \sigma_0\beta_2, \tag{3.20}$$

$$\varepsilon_0^2 g_0(a_2\beta_1^2 + a_1\beta_2^2) + a_1a_2\{\varepsilon_0[\sigma_0\kappa_0^2 + g_0(B_2\beta_1^2 + B_1\beta_2^2)] + g_0[\beta_1^2(C_3 - C_2) + \beta_2^2(C_3 - C_1)]\} = 0, \tag{3.21}$$

where σ_0 is a parameter. Parameter k in equality (3.8) under conditions (3.17)–(3.21) has the value

$$\begin{gathered}k = \frac{1}{2a_1a_2\kappa_0^4}\{2\varepsilon_0[\kappa_0^4(a_1 + a_2) - c_0^2(a_1\beta_1^2 + a_2\beta_2^2)] \\ - a_1a_2[B_3\kappa_0^4 + c_0^2(\beta_1^2(B_1 - B_3) + \beta_2^2(B_2 - B_3))]\}.\end{gathered} \tag{3.22}$$

Function $\lambda_3(t)$ is determined based on equalities (2.25), (2.27, and (3.16)–(3.21):

$$\lambda_3(t) = \frac{\mu_0}{2a_1a_2a_3\kappa_0^3}\{2\varepsilon_0[a_3(a_1\beta_1^2 + a_2\beta_2^2) - \kappa_0^2 a_1a_2] - a_1a_2[\beta_1^2(B_2 - B_3) + \beta_2^2(B_1 - B_3)]\}\sin(\kappa_0 g_0 t + \psi_0). \tag{3.23}$$

Let us discuss conditions (3.17)–(3.22). Assuming that $B_i = 0$, $C_i = 0$ $(i = \overline{1,3})$ in these conditions, we obtain (2.29)–(2.32).

If we assume $a_2 \neq a_1$ in (3.17)–(3.21), then (3.17) implies

$$c_0 = \frac{g_0\kappa_0^2(a_1 - a_2)}{2\varepsilon_0(a_2 - a_1) + a_1a_2(B_2 - B_1)}.$$

Parameter ε_0 can be determined from Eq. (3.19), if

$$a_1a_2(B_1 - B_2)^2 + 4(a_2 - a_1)(C_1 - C_2) \geq 0.$$

Equality (3.21) can be considered as a condition on parameters C_i $(i = \overline{1,3})$.

In this way the statement is proved.

Statement 3. The new solution (2.25)–(2.28) of Eqs. (3.5) – (3.7), which describes the precessional motion of the carrier body in the problem of motion of a gyrostat under the impact of potential and gyroscopic forces, was obtained. The conditions of the existence of this solution are the equalities (3.17)–(3.22): in contrast to the case (2.29), they do not contain the requirement of dynamic symmetry of a gyrostat.

4. LINEAR IR OF THE EQUATIONS OF MOTION OF NONAUTONOMOUS GYROSTAT UNDER THE IMPACT OF POTENTIAL AND GYROSCOPIC FORCES IN THE CASE OF NONPRECESSIONAL MOTIONS

Earlier [25], a solution of the equations of motion of a dynamically symmetric rigid body ($a_2 = a_1$) in a potential force field was obtained:

$$x_1 = \frac{1}{a_1}\left(-\frac{\mu_1}{3}\nu_1 + \beta_1\mu_2\right), \quad x_2 = \frac{1}{a_1}\left(-\frac{\mu_1}{3}\nu_2 + \beta_2\mu_2\right), \quad x_3 = \frac{\mu_1}{a_3}\nu_3, \tag{4.1}$$

$$\begin{aligned}\nu_1(\nu_3) &= \frac{1}{3\mu_2\kappa_0^2}(\mu_1\beta_1(1 - 2\nu_3^2) - \beta_2\sqrt{F(\nu_3)}),\\ \nu_2(\nu_3) &= \frac{1}{3\mu_2\kappa_0^2}(\mu_1\beta_2(1 - 2\nu_3^2) + \beta_1\sqrt{F(\nu_3)}),\end{aligned} \tag{4.2}$$

where

$$F(\nu_3) = -\varepsilon_2^2\nu_3^4 + \varepsilon_1\nu_3^2 + \varepsilon_0, \quad \varepsilon_0 = 9\kappa_0^2\mu_2^2 - \mu_1^2, \quad \varepsilon_1 = 4\mu_1^2 - 9\kappa_0^2\mu_2^2, \quad \varepsilon_2 = 2\mu_1. \tag{4.3}$$

Let us highlight that in equalities (4.1) and (4.2) the property of dynamic symmetry of the gyrostat

$$a_2 = a_1 \tag{4.4}$$

is maintained. In formulas (4.1)–(4.3) μ_1, μ_2 are constant parameters, κ_0^2 is the parameter having the value $\beta_1^2 + \beta_2^2$. Dependence $\nu_3(t)$ is determined by inversion of the integral [27]:

$$\int_{\nu_3^{(0)}}^{\nu_3}\frac{d\nu_3}{\sqrt{F(\nu_3)}} = -\frac{1}{3}(t - t_0). \tag{4.5}$$

Quantities (4.2) are obtained from IR

$$\nu_1^2 + \nu_2^2 + \nu_3^2 = 1, \quad \beta_1\nu_1 + \beta_2\nu_2 = \frac{\mu_1}{3\mu_2}(1 - 2\nu_3^2), \tag{4.6}$$

which admit Poisson equation (1.1) on IR (4.1). Since the second IR from (4.6) differs from IR (2.15), then motion of the gyrostat is not precessional [28].

Let us assume additionally to condition (4.4) that the following equalities hold:

$$s_1 = \sigma_0\beta_1, \quad s_2 = \sigma_0\beta_2, \tag{4.7}$$

i.e., consider the conditions of the existence of IR (4.1), (4.2) ,and (4.6) of Eqs. (3.5) and (3.6) in the presence of limitations on parameters (4.4) and (4.7). The difference of IR (4.6) from IR of chapter 3 lies in the fact that the second IR from (4.6) is nonlinear.

In order to obtain vivid results, we will assume that the following equalities are true:

$$B_2 = B_1, \quad C_2 = C_1 \quad (i = \overline{1, 3}). \tag{4.8}$$

Let us substitute the values (4.1) and (4.2) into Eqs. (3.5) and (3.6). Taking equalities (4.7) and (4.8) in reduced equations of IR (4.6) into account, let us require that they would be identities in variable ν_3. Then, we obtain conditions

$$s_3 = 0, \quad k = -\frac{1}{2}B_{11}, \quad \sigma_0 = -\frac{\mu_2}{6a_1}[2\mu_1 + 3a_1(B_1 + B_3)], \tag{4.9}$$

$$4\mu_1^2 - 3a_1\mu_1(B_3 + 7B_1) + 18a_1(C_1 - C_3) = 0. \tag{4.10}$$

The first equality of set (4.9) implies that the center of mass of the gyrostat is in the plane of the circular cross-section of the inertia ellipsoid of the gyrostat. The third condition from (4.9) serves for determination of parameter σ_0, the value of parameter μ_1 can be obtained from equality (4.10), if inequality $C_3 > C_1$ is true.

The function of gyrostatic momentum $\lambda_3(\nu_3)$ in this solution can be determined using equality (3.8):

$$\lambda_3(\nu_3) = \frac{1}{6a_1a_3}[2\mu_1(a_3 - 3a_1) + 3a_1a_3(B_3 - B_1)]\nu_3, \tag{4.11}$$

where function $\nu_3(t)$ satisfies integral relation (4.5), which was studied earlier [27].

Let us consider an example of studying integral (4.5). According to [27], we assume that parameters from (4.3) obey the condition $\mu_1^2 = 9\kappa_0^2\mu_2^2$. The integral (4.5) becomes

$$\int\limits_{\nu_3^{(0)}}^{\nu_3} \frac{d\nu_3}{\nu_3\sqrt{\lambda_0^2 - \nu_3^2}} = -\frac{2\mu_1}{3}(t - t_0), \tag{4.12}$$

i.e., variable ν_3 changes on the segment

$$-\lambda_0 \le \nu_3 \le \lambda_0 \quad \left(\lambda_0 = \frac{\sqrt{3}}{2}\right), \tag{4.13}$$

where $\nu_3^{(0)} \neq 0$. Integral (4.12) is computed in elementary functions

$$\nu_3(t) = \frac{\sqrt{3}}{2\,\mathrm{ch}\,w(t)}, \quad w(t) = \frac{\mu_1(t - t_0)}{\sqrt{3}}. \tag{4.14}$$

Due to (4.1), (4.2), (4.13), and (4.14), motion of the carrier body tends to the state of rest at $t \to \infty$ [28]. The value of gyrostatic moment $\lambda_3(t)$ is found from (4.11). In this way, the following statement is true.

Statement 4. A class of solutions of motion equations of dynamically symmetric gyrostat under the action of potential and gyroscopic forces, which is *not precessional*, was constructed. For it, the motion of the carrier body possesses the property of asymptotic nature to the state of rest. The center of mass of the gyrostat is in plane of equal principal moments of inertia.

CONCLUSIONS

New solutions of the equations of motion of a gyrostat with variable gyrostatic moment in two problems of dynamics were built: in the problem of motion of a heavy gyrostat and in the problem of motion of a gyrostat under the action of potential and gyroscopic forces.

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