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Kinematic Equations along Characteristics in Compressible Flows on the Facets of an Arbitrary Piecewise Linear Yield Criterion

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Abstract—The article deals with flows of the perfectly plastic compressible media for stress states corresponding to the facets of a piecewise linear yield criterion. Similar flows are observed, in particular, in loosely bonded Coulomb–Mohr media for plane strain states. It is assumed that the intermediate principle stress has no effect on the yielding or transition to the limit state. Under these conditions the system of kinematic differential equations belongs to the hyperbolic analytical type, the elements of the characteristic lines are instantly not elongating, the orthogonal projections of the displacement increment vector on characteristics are related by differential equations with the differentiations along the characteristic lines.

Keywords: piecewise linear yield criterion, Coulomb–Mohr media, compessibility, flow, principal stress, asymptotic directions, conjugate directors, hyperbolicity, characteristics

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1. General piecewise-linear yield criterion can be presented in the form of a linear equation relating the principal normal stresses σ_1, σ_2, σ_3 of symmetric stress tensor:

$$a_1\sigma_1 + a_2\sigma_2 - a_3\sigma_3 = b, \tag{1.1}$$

where a_1, a_2, a_3, b are the constitutive constants.

A significant simplification for equations of the theories of plasticity can be achieved by special numeration for the axes of the main trihedron of the stress tensor s: we enumerate the main axes in such a way that for the current stress state, the corresponding principle normal stresses σ_1, σ_2, σ_3 are arranged in decreasing order

$$\sigma_1 \geq \sigma_2 \geq \sigma_3. \tag{1.2}$$

Intermediate principal normal stress plays a special role in plasticity theories [1–6]. Its influence on the yield of metals and the deformation of loosely coupled media can often be neglected. Therefore, we assume that $a_2 = 0$. We divide equation (1.1) by a_1 and adopt the following notations

$$\bar{a} = \frac{a_3}{a_1}, \quad \bar{b} = \frac{b}{a_1},$$

after which the piecewise-linear yield condition used below is reduced to the form

$$\sigma_1 - \bar{a}\sigma_3 = \bar{b}. \tag{1.3}$$

The Coulomb–Mohr medium is one of the variants of the piecewise linear yield criterion, which perfectly simulates the mechanical behavior of dry sands, soils, granular media, i.e. loose materials with a granular, porous or granular structure [4]. In terms of principal stresses σ_1, σ_2, σ_3, the Mohr–Coulomb yield criterion for bulk media with internal friction and adhesion is formulated as follows [7, 8]:

$$\frac{\sigma_1 - \sigma_3}{2} = c\cos\gamma - \sin\gamma\frac{\sigma_1 + \sigma_3}{2}, \tag{1.4}$$

where, c, γ are the constitutive constants. The Coulomb–Mohr criterion (1.4) can also be reduced to the following form, which is equivalent to form of (1.3):

$$\sigma_1 - a\sigma_3 = 2k. \tag{1.5}$$

Here, the material constants a and k are related to the coefficient of adhesion and the angle of internal friction γ by the following relations:

$$a = \frac{1 - \sin\gamma}{1 + \sin\gamma}, \quad k = \frac{c\cos\gamma}{1 + \sin\gamma}.$$

2. In the kinematics of ideally plastic solids, it is convenient to operate with increments of the displacement vector du and strain tensor $d\boldsymbol{\varepsilon}$. We orient the unit basis vectors $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ along the principal axes of the stress tensor (and tensor $d\boldsymbol{\varepsilon}$).

The increment of the displacement vector du can be represented as an expansion in vectors of a local orthonormal basis $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$

$$d\mathbf{u} = \mathbf{l}du_{\langle 1\rangle} + \mathbf{m}du_{\langle 2\rangle} + \mathbf{n}du_{\langle 3\rangle}. \tag{2.1}$$

We set the spectral representation of the strain tensor increment de in the form

$$d\boldsymbol{\varepsilon} = \mathbf{l}\otimes\mathbf{l}(d\varepsilon_1) + \mathbf{m}\otimes\mathbf{m}(d\varepsilon_2) + \mathbf{n}\otimes\mathbf{n}(d\varepsilon_3), \tag{2.2}$$

where $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ is an orthonormal basis of eigenvectors common both for the stress tensor $\boldsymbol{\sigma}$ and for the strain tensor increment $d\boldsymbol{\varepsilon}$; $d\varepsilon_1$, $d\varepsilon_2$, $d\varepsilon_3$ are the principal increments of (plastic) deformation (eigenvalues of the tensor $d\boldsymbol{\varepsilon}$).

Let us further introduce a special numeration for the axes of the principal trihedron in such a way that the following inequalities as well as (1.2) are valid

$$d\varepsilon_1 \geq d\varepsilon_2 \geq d\varepsilon_3. \tag{2.3}$$

By virtue of the associated plastic flow rule

$$d\varepsilon_1 = d\lambda, \quad d\varepsilon_2 = 0, \quad d\varepsilon_3 = -\bar{a}d\lambda \quad (d\lambda \geq 0), \tag{2.4}$$

systems of ordered principal stresses and principal strain increments are consistent in the presence of a constitutive constraint

$$\bar{a} \geq 0.$$

The concept of asymptotic directors of the increment of the deformation tensor $d\boldsymbol{\varepsilon}$ and its representation in terms of asymptotic directors $\text{"}\mathbf{l}$, $\text{"}\mathbf{n}$ are considered in articles [7, 8], as well as in an earlier publication [9]. In particular, the dyadic representation of the strain tensor increment $d\boldsymbol{\varepsilon}$ has the form

$$d\boldsymbol{\varepsilon} = \mathbf{I}(d\varepsilon_2) + (d\varepsilon_1 - d\varepsilon_3)\operatorname{sym}(\text{"}\mathbf{l}\otimes\text{"}\mathbf{n}). \tag{2.5}$$

The angle between the asymptotic directors $\text{"}\mathbf{l}$, $\text{"}\mathbf{n}$ is calculated using the Lode kinematic parameter

$$\cos\text{"}\iota = -\nu, \tag{2.6}$$

where

$$\nu = \frac{2d\varepsilon_2 - d\varepsilon_1 - d\varepsilon_3}{d\varepsilon_1 - d\varepsilon_3}. \tag{2.7}$$

The flow on the facet of the piecewise-linear plasticity criterion obeys the kinematic constraint $d\varepsilon_2 = 0$ following from the associated flow rule.

In terms of notations of field theory, the system of differential equations of kinematics with respect to the physical components of the displacement vector increment $du_{\langle 1\rangle}$, $du_{\langle 3\rangle}$ has the following form:

$$\begin{gathered}(-\kappa_1 + \mathbf{n}\cdot\nabla)du_{\langle 1\rangle} + (-\kappa_3 + \mathbf{l}\cdot\nabla)du_{\langle 3\rangle} = 0,\\ \sin^2\frac{\text{"}\iota}{2}((\mathbf{l}\cdot\nabla)du_{\langle 1\rangle} + \kappa_1 du_{\langle 3\rangle}) + \cos^2\frac{\text{"}\iota}{2}((\mathbf{n}\cdot\nabla)du_{\langle 3\rangle} + \kappa_3 du_{\langle 1\rangle}) = 0;\end{gathered} \tag{2.8}$$

it belongs to the hyperbolic type; the characteristic directions coincide with the directions of the conjugate directors $\text{"}\mathbf{l}$, $\text{"}\mathbf{n}$.

Conjugate directors indicate directions in the plane orthogonal to the second principal axis of the tensor $d\boldsymbol{\varepsilon}$ that are orthogonal to the directions of asymptotic directors $''\mathbf{l}$, $''\mathbf{n}$. The director $''\mathbf{l}$ is orthogonal to the asymptotic director $''\mathbf{n}$, and the director $''\mathbf{n}$ is orthogonal to $''\mathbf{l}$.

3. For instantaneous elongation of the elements of the characteristic lines, we find

$$
\begin{gathered}
''\mathbf{l} \cdot (d\boldsymbol{\varepsilon}) \cdot ''\mathbf{l} = 0, \\
''\mathbf{n} \cdot (d\boldsymbol{\varepsilon}) \cdot ''\mathbf{n} = 0,
\end{gathered}
\tag{3.1}
$$

that is, during the flow, the linear elements perpendicular to the directions of the asymptotic directors $''\mathbf{l}$, $''\mathbf{n}$ do not undergo instantaneous elongation, i.e. material fibers oriented along the directors $''\mathbf{l}$, $''\mathbf{n}$ do not instantly elongate or shorten.

First of all, instead of physical components $du_{\langle 1\rangle}$, $du_{\langle 3\rangle}$, the displacement vector increment du relative to the basis l, n, we introduce its orthogonal projections onto the conjugate directions $''\mathbf{l}$, $''\mathbf{n}$: $du_{\langle\bar{1}\rangle}$, $du_{\langle\bar{3}\rangle}$.

Differentiations along isostatic d_1, d_3 and conjugate directions $\bar{d}_1$, $\bar{d}_3$ are related by the formulas [10]

$$
\begin{aligned}
d_1 &= \frac{1}{2\sin\frac{''\iota}{2}}(\bar{d}_1 + \bar{d}_3), \\
d_3 &= \frac{1}{2\cos\frac{''\iota}{2}}(\bar{d}_3 - \bar{d}_1).
\end{aligned}
\tag{3.2}
$$

Differential relations along characteristic lines (compare with [11])

$$
\begin{aligned}
&\bar{d}_1 du_{\langle\bar{1}\rangle} - \frac{\cos ''\iota\, du_{\langle\bar{1}\rangle} + du_{\langle\bar{3}\rangle}}{\sin ''\iota}\bar{d}_1\left(\theta - \frac{''\iota}{2}\right) = 0, \\
&\bar{d}_3 du_{\langle\bar{3}\rangle} + \frac{du_{\langle\bar{1}\rangle} + \cos ''\iota\, du_{\langle\bar{3}\rangle}}{\sin ''\iota}\bar{d}_3\left(\theta + \frac{''\iota}{2}\right) = 0,
\end{aligned}
\tag{3.3}
$$

where θ is the angle between some fixed direction in the flow plane and the eigenvector $\mathbf{l}$.

For a piecewise linear yield criterion, the angle $''\iota$ is constant. After a series of transformations, the flow kinematics relations (3.3) acquire the following final form:

$$
\begin{aligned}
&\bar{d}_1 du_{\langle\bar{1}\rangle} - (\underset{1}{\bar{a}}\, du_{\langle\bar{1}\rangle} + \underset{3}{\bar{a}}\, du_{\langle\bar{3}\rangle})\bar{d}_1\theta = 0, \\
&\bar{d}_3 du_{\langle\bar{3}\rangle} + (\underset{3}{\bar{a}}\, du_{\langle\bar{1}\rangle} + \underset{1}{\bar{a}}\, du_{\langle\bar{3}\rangle})\bar{d}_3\theta = 0,
\end{aligned}
\tag{3.4}
$$

where

$$
\underset{1}{\bar{a}} = \frac{1 - \bar{a}}{2\sqrt{\bar{a}}}, \quad \underset{3}{\bar{a}} = \frac{1 + \bar{a}}{2\sqrt{\bar{a}}}.
$$

It is curious to note that the constitutive constraint

$$
\bar{a} \geq 0
$$

and the inequality of irreversibility do not limit the sign of the dilation rate, i.e. during the flow, the medium can both loosen and contract irreversibly (in the most precise sense of the word).

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REFERENCES

1. D. D. Ivlev, *Theory of Ideal Plasticity* (Nauka, Moscow, 1966) [in Russian].
2. A. Nadai, *Plasticity: A Mechanics of the Plastic State of Matter* (McGraw-Hill, New York, 1931).
3. A. Nadai, *Theory of Flow and Fracture of Solids,* Vol. 1 (Wiley, New York, 1950).
4. A. Nadai, *Theory of Flow and Fracture of Solids,* Vol. 2 (Wiley, New York, 1950).

5. Yu. N. Radaev, *Spatial Problem of Mathematical Theory of Plasticity* (Samarsk. Gos. Univ., Samara, 2004) [in Russian].
6. Yu. N. Radaev, *Spatial Problem of Mathematical Theory of Plasticity*, 2nd ed. (Samarsk. Gos. Univ., Samara, 2006) [in Russian].
7. Yu. N. Radaev, "Instantly inextensible directors in the kinematics of three-dimensional flows of Coulomb media-Mora," Izv. Sarat. Univ. Nov. Ser. Ser. Mat. Mekh. Inf. **18** (4), 467–483 (2018).
8. Yu.N. Radaev, "On the theory of loosely coupled Coulomb media-Mora and generalized Prandtl plastic bodies," Vestn. Chuvash. Gos. Ped. Univ. Im. Yakovleva Ser.: Mekh. Pred. Sost., No 4 (38), 3–24 (2018).
9. Yu. N. Radaev, "Asymptotic axes of stress tensors and strain increment tensors in mechanics of compressible continua," Mech. Solids **48** (5), 546–552 (2013).
10. Yu. N. Radaev, "ON a hyperbolic model of plane irreversibly compressible flows of Coulomb-Mohr and Prandtl perfectly plastic media," Vestn. Chuvash. Gos. Ped. Univ. Im. Yakovleva Ser.: Mekh. Pred. Sost., No 4 (42), 56–68 (2019).
11. Yu. N. Radaev, "Kinematic differential equations along the instantaneously not elongating lines in compressible flows," Vestn. Chuvash. Gos. Ped. Univ. Im. Yakovleva Ser.: Mekh. Pred. Sost., No 4 (42), 84–91 (2019).

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