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To the memory of K. I. Babenko (on the 100th anniversary of his birth)

Konstantin Ivanovich Babenko (July 21, 1919–June 10, 1987), an outstanding scientist in the field of fluid mechanics, numerical analysis, harmonic analysis, theory of functions, approximation theory, and theory of partial differential equations, Corresponding Member of Academy of Sciences of the USSR (1976); Doctor of Physics and Mathematics (1952), Professor (1958), recipient of the State Award of the USSR, member of the editorial board of the Journal of Applied Mathematics and Mechanics (1978–1987).

About the Works of K. I. Babenko in the Field of Mechanics and Applied Mathematics (on the 100th Anniversary of His Birth)

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Abstract—The paper provides an overview of the scientific achievements of K.I. Babenko, Corresponding Member of Academy of Sciences of the USSR, in the field of mechanics and applied mathematics, including numerical studies of ideal gas flows around a body, investigation of viscous fluid flows, and research in the field of computational mathematics. A characteristic feature of these works is the combination of deep analytical methods and tools and numerical solutions of specific problems.

Keywords: fluid and gas dynamics, numerical solution, computational algorithms

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INTRODUCTION

K.I. Babenko's scientific activity is notable for an extraordinary large creative range: from abstract problems of function theory and functional analysis to particular applied problems of mechanics. It is impossible to highlight any single area of studies where his main findings belong. His papers in "abstract mathematics" were often initiated by real applied problems, and solving some applied problems became possible due to his deep insight into their mathematical essence.

The first scientific papers were written by K.I. Babenko in 1945–1948, when he was a postgraduate student at the Engineering School, Zhukovsky Air Force Engineering Academy; the papers were presented to the Doklady Akademii Nauk SSSR Journal by Academicians M.V. Keldysh and S.N. Bershtein. Later, one of these papers, "On Conjugate Functions" [1], played a significant role in his work with the doctoral dissertation.

K.I. Babenko's candidate dissertation "Determining forces and moments acting on the oscillating swept wing in supersonic gas flow" can be truly named a classical paper in mechanics. In this paper, his already developed analytic talent was applied to solve the differential equation describing the wing motion. In 1972, Konstantin Ivanovich wrote in his autobiography: "...I have solved the problem on determining forces and moments in the linearized formulation and gave the explicit formulas to find them. For this purpose I had to consider effectively the influence of lateral edges of the wing, which I managed to do. When I defended my dissertation, it was classified and consequently was not published timely. Later some of my results were obtained abroad." K.I. Babenko received the Zhukovski award and medal in 1949 for his candidate dissertation (see https://www.keldysh.ru/memory/babenko/cand.pdf).

Over those years, communications with Academician M.V. Keldysh greatly influenced the destiny of Konstantin Ivanovich. Apparently, his interest in the boundary value problems for the mixed-type equations was formed under the influence of papers written by M.V. Keldysh and F.I. Frankl. K.I. Babenko defended his doctoral dissertation "On mixed-type equations" (see https://www.keldysh.ru/memory/babenko/doktor.pdf) in 1952. A year earlier, at the initiative of M.V. Keldysh, Konstantin Ivanovich was transferred to the Steklov Mathematical Institute, Academy of Sciences of the USSR. It is known that since 1946, the calculations for development of the nuclear-missile shield of the USSR were carried out at the Steklov Mathematical Institute, Academy of Sciences of the USSR (MIAS). The Decree of the Council of Ministers of the USSR dated May 9, 1951, which was mentioned in the order about K.I. Babenko's transfer to the MIAS, referred to the period of development of the thermonuclear weapon and concerned the creation of structures responsible for organizing the calculations within the Soviet atomic project. In particular, the applied mathematics department headed by Academician M.V. Keldysh was organized at the MIAS. Konstantin Ivanovich actively joined the work of this department aimed at solving the problems on production issues, as the classified works accomplished by orders of the decision-making authorities were called then. A new science, computational mathematics, was created, and K.I. Babenko was actively involved in its formation. Part of this activity is now called mathematical simulation since full substantiations of discretization for nonlinear applied problems are more often the exception than the rule. Studies of gas dynamics and later, hydrodynamics took the key place in this research.

1. WORKING ON GAS DYNAMICS

The results of the first years of K.I. Babenko's work on production issues at the Applied Mathematics Department, MIAS, and then at the Applied Mathematics Division created at the institute in 1953 are partially presented in the known report of 1954 "Solution of the problem on axisymmetric gas motion with shock wave" devoted to solving the 2D gas dynamic problem on a strong explosion in an inhomogeneous atmosphere. (Today the declassified report is kept at the M.V. Keldysh memorial study room (M.V. Keldysh, Mathematics, Collected Works)). In this paper, the first technique for calculating 2D problems was proposed. I.M. Gelfand, N.A. Dmitriev, M.V. Keldysh, O.M. Lokutsievskii, and N.N. Chentsov were K.I. Babenko's coauthors.

In 1974, Academician M.V. Keldysh wrote about the scientific work carried out by K.I. Babenko: "Due to poor computing facilities available at that time we had to overcome serious difficulties. This required engaging many tools of mathematical analysis, including various asymptotic methods. K.I. Babenko put his inherent analytical skill into this work." In the report, Konstantin Ivanovich presented a number of important theoretical results. In particular, the method of solving 2D difference equations proposed by Keldysh was investigated there. At the same time, K.I. Babenko and I.M. Gelfand proposed a method to study the stability of difference equations with respect to the variation of boundary conditions, which application helped to handle the arising instability of the scheme. This fact is mentioned in the report of the Applied Mathematics Division, MIAS of 1955, "Point explosion in the atmosphere," published in M.V. Keldysh, *Mechanics, Collected Works.* Later, in 1972, K.I. Babenko wrote in his autobiography: "First at the MIAS, and then at the Institute of Applied Mathematics (IAM), I joined the work in computational mathematics. I dare to hope that the high scientific level attained by the computational mathematics at the IAM and the outstanding achievements gained in this field are to a certain extent based on my contribution."

The year of 1956 was the next milestone in his activity. Then the necessity of detailed geometrical description of processes occurring in charge interactions emerged. The team headed by K.I. Babenko developed the methods and algorithms, where the part concerning the calculations of supersonic flow around blunted bodies was unclassified in [2].

In K.I. Babenko's file, in the list of his papers dated 1974, five reports on the production issues accomplished with coauthors from 1954 to 1965 are mentioned; although these papers are not included in the known list of his papers. Konstantin Ivanovich was awarded two Orders of the Red Banner of Labor (in 1955 and 1956); he was appointed the Head of Department (Department no. 4 at the IAM), and awarded the title of Professor in 1959, which confirms his significant contribution to this scientific work. Konstantin Ivanovich's unclassified papers on the numerical methods of gas dynamics were published only in the 1960s.

K.I. Babenko with his team wrote a large cycle of papers on numerical methods of solving the problems of spatial ideal gas flow around bodies. The papers rendered great influence on the numerical methods of gas dynamics and were further developed in many Russian and foreign papers. "Spatial flow around the head of a blunted body by an ideal gas" monograph [3] published in 1967, gained wide popularity, and received the State Award. In the monograph, the method of supersonic gas spatial flow around pointed bodies was stated in detail and the theoretical investigation of systems of finite-difference equations was carried out. The first finite-difference method for solving the problems of supersonic spatial flow around bodies with the second-order accuracy developed by K.I. Babenko and G.P. Voskresenskii was presented there. The tables of nonaxisymmetric flow around bodies in a broad range of Mach numbers given in this book provided exhaustive information on the gas flow.

"The first, theoretical part of the book is especially typical for Konstantin Ivanovich's style. There is a clear and detailed statement of the method used, and a number of issues of general purpose referred to the difference boundary problems and the methods of their solution are considered," noted M.V. Keldysh in his review of this paper.

During the next few years, in the papers devoted to spatial flows of fluids and gases and to supersonic flow around bodies, the developed algorithms were updated and extended according to practical needs.

2. VISCOUS FLUID FLOW AROUND BODIES

Since late 1960s, K.I. Babenko addressed the problems of the viscous fluid flow around bodies [4–21]. We consider in detail an example of an applied problem that produced remarkable theoretical results.

The exterior problem for the Navier–Stokes equations is considered:

$$(u\cdot\nabla)u + \operatorname{grad} p = \frac{1}{2\lambda}\Delta u$$
$$\operatorname{div} u = 0,$$

where u and p are the dimensionless velocity vector and pressure, density $\rho = 1$, and Reynolds number is 2λ. A flow around a finite body T, with the boundary S that satisfies the Hölder condition is considered. The length unit is assumed $l = \operatorname{diam} T$, and the axes are directed so that $u_\infty = (1, 0, 0)$. Then the boundary conditions on the body and at an infinitely remote point are set as follows:

$$u\big|_S = u_0, \quad \lim_{|x|\to\infty} u(x) = u_\infty.$$

The issue of setting the boundary conditions for numerical solution of the problem of the flow away from the body [4] inclined Konstantin Ivanovich to thorough theoretical study of the flow problems. Many outstanding scientists, whose names are perpetuated in world science, devoted their papers to clarifying the structure of solutions of the problem of viscous incompressible fluid flow around a body. While the issue of existence of the generalized stationary solutions at any Reynolds numbers was clarified in different versions by J. Leray, O.A. Ladyzhenskaya, and others, the question, in what sense the conditions were satisfied at infinity for these solutions remained open for many years. Considerable progress in this field was achieved in R. Finn's papers, who introduced the class of "physically reasonable" (*PR*) solutions of flow problems. For these problems, he postulated that, at infinity, they satisfied the condition:

$$|u(x) - u_\infty| \le C|x|^{-\alpha}$$

for particular $\alpha > 1/2$ in the case of spatial flow and $\alpha > 1/4$ in the case of planar flow. In these conditions, he obtained the leading term of the asymptotic expansion $u(x)$ at $|x| \to \infty$ and estimated the residual. Konstantin Ivanovich also proved the existence of these solutions at sufficiently small Reynolds numbers.

For *PR* solutions, K.I. Babenko [5] first obtained several first terms of asymptotic expansions of the stationary flow problem solution in the plane case with the residual $O(|r|^{-3/2}\ln|r|)$, $r = (x^2 + y^2)^{1/2}$, and the asymptotic formula for the vortex

$$\omega(x, y) = A\frac{y\lambda}{r^{3/2}}e^{\lambda(x-r)} + e^{\mu(x-r)}O(r^{-3/2}\ln|r|), \quad \mu = \lambda - \varepsilon.$$

In particular, the exponential decrease of the vortex outside any angle containing the semiaxis $x > 0$, $y = 0$ follows from this formula.

In addition, in this paper, an analog of the Zhukovski formula for a viscous fluid formally derived by Filon [22] was rigorously formulated for *PR* solutions, and the formula for drag was obtained.

When solving the spatial problem [14], the method [5] was applied to find the formula for a vortex, which was similar to the plane case, and two terms of the asymptotic expansion for velocity were obtained for the case when the force acting on the body coincided with the drag force.

Later these expansions were repeatedly refined, in particular, in M.M.Vasilev's and N.I. Yavorskii's papers.

In all papers published before 1972, the problem on existence of a *PR* solution for arbitrary Reynolds numbers remained open. In the key paper in this field of science [13], it was demonstrated that any solution of the spatial flow problem with the finite Dirichlet integral $D[u] < \infty$ satisfied the inequality

$$|u(x) - u_\infty| < C|x|^{-1},$$

i.e., any solution of the spatial flow problem with the finite Dirichlet integral satisfied the *PR* condition with $\alpha = 1$. This comprehensive fact connected the studies of J. Leray, R. Finn, O.A. Ladyzhenskaya, and other mathematicians and described completely the structure of the stationary solution of the spatial flow problem at any Reynolds number.

The next result obtained by K.I. Babenko in the qualitative theory of the Navier–Stokes equations was the mathematically rigorous perturbation theory for stationary solutions of the flow problem published in [17, 19]. This large work begins with modifying the classical theory of potentials to avoid degeneration at zero Reynolds number. Using the following representation of the terms of the series

$$u(x,\lambda) = \sum_{n=0}^{\infty} u^{(n)}(x,\lambda)\lambda^n,$$

where $u^{(n)}(x,\lambda)$ is the solution of the uniform ($n = 0$) and the nonuniform ($n > 0$) Oseen equations. Using the modified potentials K.I. Babenko not only proved the convergence of this series and studied the asymptotic behavior of the first terms, but also proved the solution uniqueness at sufficiently small λ. Thus, he obtained a rigorous substantiation of using external and internal expansions in this problem, and the asymptotic formula for the forces acting on the body moving in a fluid was an application to the constructed theory. In particular, the well-known Stokes formula for the resistance of a sphere was rigorously substantiated

$$F_1 = F_1^0(1 + 0.9\lambda^2\ln(2\lambda)) + B\lambda + O(\lambda^2\ln^2(2\lambda)),$$

where F_1^0 corresponds to the Oseen approximation and B is the effectively calculated constant. Then he moved to constructing the perturbation theory at finite λ and proved the theorems from which it follows:

1. If $u(x,\lambda)$ is a solution of the flow problem, and the homogenous linearized problem has only a trivial solution at $\lambda = \lambda_0$, then at $\lambda = \lambda_0 + \varepsilon$, the flow problem has a solution that can be presented by the series

$$u(x,\lambda) = \sum_{n=0}^{\infty} u^{(n)}(x)\varepsilon^n.$$

2. If at $\lambda = \lambda_0$ there is a single solution, and at $\lambda > \lambda_0$ there is more than one solution of the flow problem, then at $\lambda = \lambda_0$, the linearized problem has a nontrivial solution.

Later, in the 1980s, Konstantin Ivanovich explored the properties of the continuous and point spectra of the linearized flow problem [20] and, in particular, demonstrated that the eigenvalue $\lambda = 0$ belongs to the spectrum at all Reynolds numbers. He showed [21] that, if a pair of complex conjugate eigenvalues crossed the imaginary axis, then at certain nondegeneracy conditions, a set of time-periodic solutions bifurcate from the stationary solution. In the same paper, he also stated that, at every fixed $t = t_0$, this solution had a finite Dirichlet integral, but did not belong to $L_2(\mathbb{R}^3 \setminus T)$.

Concerning this result, it is interesting that neither classical Lyapunov–Schmidt reduction, nor the standard method of reduction of the bifurcation problem to the central manifold are applicable to this problem.

These results were partially restated and their proofs simplified in L.I. Sazonov's papers and his doctoral dissertation, and in the papers of J.G. Heywood, G.P. Galdi, and others (see [23]).

In the 1980s, a series of studies of transition of flows of viscous incompressible fluid from the laminar to the turbulent state were accomplished under the supervision of K.I. Babenko. In these studies, alongside the general statements, the solutions of problems on loss of stability by particular viscous fluid flows were given: Poiseuille and Couette flows, Taylor vortices; the transition to turbulence was considered, and composite attractors of the Navier-Stokes system were investigated. These studies were summed up in [24–26].

3. SURFACE WAVES

Engaged in numerical solution of the applied problems of mechanics, K.I. Babenko had to find balances between the memory capacity, computer performance, and fundamental requirements, and paid great attention to studying the optimality of the computational methods. The computational algorithms he created automatically responded to the solution smoothness (algorithms without saturation) and appeared indispensable for solving many problems, in particular, problems of the numerical study of the hydrodynamic instability. Based on these algorithms, Konstantin Ivanovich developed the method of computer-assisted proofs—the method of controlled computer-assisted calculations that, being combined with the analytical studies, led to a rigorous proof of new statements. In particular, the computer-assisted proofs were applied to study the equation describing surface waves of small amplitude. In the algorithm for the numerical solution of the Rayleigh–Taylor instability problem [27–30] developed by K.I. Babenko, a simple variation of gravity direction produced an algorithm for numerical solution of the surface wave problem. One of the algorithm elements was a numerical solution of the linearized problem. Konstantin Ivanovich noticed that, at small amplitudes, the matrix of the indicated algorithm appeared to be similar to symmetric one. This prompted in him an idea to search for a new equation, for which the linearization would lead to a self-adjoint problem. K.I. Babenko derived a new equation equivalent to the known Stokes and Nekrasov equations, which described the waves on the surface of ideal fluid:

$$\frac{1}{2}J(y^2) + (y - c^2)J(y) + y - kR^{-1}(1 + J(y)) - kHp[R^{-1}D(y)] = 0,$$

where $J = DH$, D is the operator of differentiation, H is the Hilbert operator, k is the surface tension coefficient, and R is the radius of curvature, for which there is a known expression. The fundamental data on the structure of many solutions, i.e., on the domain of existence of surface waves with finite amplitudes, were obtained for this equation using the method of computer-assisted proofs [31–35]. These studies were continued by J.F. Toland, P.I. Plotnikov, P.M. Lushnikov, V.E. Zakharov, A.I. Dyachenko, and others in [36–40] where, in particular, the problem of the existence and structure of the Stokes wave was solved.

By the moment of joining the MIAS, Konstantin Ivanovich had already showed his analytic skill and obtained significant results in the theory of functions and the theory of differential equations. Later, he gained impressive achievements in the field of applied mathematics. At the same time, the world of "pure mathematics" attracted him, he aspired to solve beautiful and fundamental analytical problems, and he kept working in this field of mathematics. These studies formed the basis for his achievements in solving the problems of applied mathematics and mechanics.

Konstantin Ivanovich Babenko was an excellent example of a person with an amazing capacity for work, commitment to science, adherence to principles, and profound integrity for all who knew him.

REFERENCES

1. K. I. Babenko, "On conjugate functions," Dokl. Akad. Nauk SSSR **62** (2), 157–160 (1948).
2. K. I. Babenko, V. N. Ivanova, E. P. Kazandzhan, M. A. Kukarkina, and Yu. B. Radvogin, "Unsteady Flow around the head of a blunted body by an ideal gas," Preprint No. 72 (Keldysh Institute of Applied Mathematics, 1969).
3. K. I. Babenko, G. P. Voskresenskii, V. V. Rusanov, and A. N. Lyubimov, *Spatial Flow around Smooth Bodies by Ideal Gas* (Nauka, Moscow, 1964) [in Russian].
4. K. I. Babenko, N. D. Vvedenskaya, and M. M. Orlova, "On stationary viscous fluid flow past a circular cylinder," Preprint No. 4 (Keldysh Institute of Applied Mathematics, Moscow, 1969).
5. K. I. Babenko, "On the asymptotic behavior of a vortex in a plane viscous fluid flow away from the body," Prikl. Mat. Mekh. **34** (5), 911–925 (1970).
6. K. I. Babenko, N. D. Vvedenskaya, and M. G. Orlova, "On stationary flow of viscous fluid past a circular cylinder fluid dynamics," Fluid Dyn. Trans. **5** (11), 37–49 (1970).
7. K. I. Babenko, "On the asymptotic behavior of a vortex in a plane viscous fluid flow away from the body," Preprint No. 2 (Keldysh Institute of Applied Mathematics, Moscow, 1970).
8. K. I. Babenko and M. M. Vasil'ev, "Asymptotic behavior of a solution of a viscous fluid flow problem past a finite body," Preprint No. 84 (Keldysh Institute of Applied Mathematics, Moscow, 1971).
9. K. I. Babenko, N. D. Vvedenskaya, and M. M. Orlova, "The results of calculation for flow around an infinite circular cylinder by a viscous fluid," Preprint No. 38 (Keldysh Institute of Applied Mathematics, Moscow, 1971).
10. K. I. Babenko, N. D. Vvedenskaya, and M. M. Orlova, "Boundary value problem for Navier-Stokes equations in a plane flow problem," Preprint No. 39 (Keldysh Institute of Applied Mathematics, Moscow, 1971).
11. K. I. Babenko and N. D. Vvedenskaya, "Numerical solution of the boundary value problem for the Navier-Stokes equation," USSR Comput. Math. Math. Phys. **12** (5), 329–339 (1972).
12. K. I. Babenko, M. M. Vasil'ev, E. P. Kazandzhan, and M. M. Orlova, "Numerical solution of the problem on viscous incompressible fluid unsteady motion within a given region," Preprint No. 54 (Keldysh Institute of Applied Mathematics, Moscow, 1972).
13. K. I. Babenko, "On stationary solutions of the problem on flow past a body of a viscous incompressible fluid," Mat. Sb. **91 (133)** (1), 3–26 (1973).
14. K. I. Babenko and M. M. Vasil'ev, "On asymptotic behavior of stationary flow of viscous incompressible fluid far from the body," Prikl. Mat. Mekh. **37** (4), 690–705 (1973).
15. K. I. Babenko, N. D. Vvedenskaya, and M. M. Orlova, "Calculation of the steady flow of a viscous fluid past a circular cylinder," USSR Comput. Math. Math. Phys. **15** (1), 176–190 (1975).
16. K. I. Babenko and N. D. Vvedenskaya, "Calculation of equations and vortex at the boundary in numerical solution of the boundary value problem for Navier-Stokes equations," in *Proc. 4th All-Union Seminar on Numerical Methods of Viscous Fluid Mechanics* (Novosibirsk, 1975) [in Russian].
17. K. I. Babenko, "Perturbation theory of stationary viscous incompressible fluid flows at small Reynolds numbers," Preprint No. 79 (Keldysh Institute of Applied Mathematics, Moscow, 1975). https://keldysh.ru/papers/1975/prep1975_79.pdf
18. K. I. Babenko, "Perturbation theory of stationary flows of viscous incompressible fluid for small Reynolds numbers," Sov. Phys. Dokl. **21**, 143–145 (1976).
19. K. I. Babenko, "On properties of steady viscous incompressible fluid flows," in *Approximation Methods for Navier-Stokes Problems* (Springer, Berlin, 1980).
20. K. I. Babenko, "Spectrum of the linearized problem of flow of a viscous incompressible liquid around a body," Sov. Phys. Dokl. **262** (1), 25–27 (1982).
21. K. I. Babenko, "Periodic solutions of the flow of a viscous fluid past a body," Sov. Math. Dokl. **25**, 211–216 (1982).
22. L. N. G. Filon, "The forces on a cylinder in a stream of viscous fluid," Proc. R. Soc. London A **113**, 7–27 (1926).

23. G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Springer Monographs in Mathematics* (Springer, New York, 2011).
24. K. I. Babenko, "Investigation of hydrodynamic stability by means of computers," Fluid Dyn. Trans. **10**, 9–75 (1980).
25. K. I. Babenko and A. L. Afendikov, "On the occurrence of turbulence in viscous incompressible fluid flows," in *Mechanics and Scientific and Technical Progress* (Nauka, Moscow, 1987), Vol. 2 [in Russian].
26. K. I. Babenko and A. I. Rakhmanov, "Proof-based calculations in the surface wave problem," in *Designing Algorithms and Solving Problems of Mathematical Physics* (Keldysh Institute of Applied Mathematics, Moscow, 1989), pp. 123–127 [in Russian].
27. K. I. Babenko and V. Yu. Petrovich, "On the Rayleigh-Taylor instability," Preprint No. 68 (Keldysh Institute of Applied Mathematics, Moscow, 1978).
28. K. I. Babenko and V. Yu. Petrovich, "On the Rayleigh-Taylor instability," Sov. Phys. Dokl. **24**, 161–163 (1979).
29. K. I. Babenko and V. Yu. Petrovich, "Numerical study of the Rayleigh-Taylor instability problem," Sov. Phys. Dokl. **25**, 909–912 (1980).
30. K. I. Babenko and V. Yu. Petrovich, "On the Rayleigh-Taylor instability," in *To the 75th Anniversary of Academician A. N. Tikhonov. Topical Problems of Mathematical Physics and Computational Mathematics* (Nauka, Moscow, 1984) [in Russian].
31. K. I. Babenko, "Computational experiment and demonstrative calculations in the surface wave problem," Preprint No. 116 (Keldysh Institute of Applied Mathematics, Moscow, 1986).
32. K. I. Babenko, "Some remarks on the theory of surface waves of finite amplitude," Sov. Math. Dokl. **35**, 599–603 (1987).
33. K. I. Babenko, "A local existence theorem in the theory of surface waves of finite amplitude," Sov. Math. Dokl. **35**, 647–650 (1987).
34. K. I. Babenko, V. Yu. Petrovich, and A. I. Rakhmanov, "A computational experiment in the theory of surface waves of finite amplitude," Sov. Math. Dokl. **38**, 327–331 (1988).
35. K. I. Babenko, V. Yu. Petrovich, and A. I. Rakhmanov, "On a demonstrative experiment in the theory of surface waves of finite amplitude," Sov. Math. Dokl. **38**, 626–630 (1988).
36. P. I. Plotnikov and J. F. Toland, "Convexity of Stokes waves of extreme form," Arch. Ration. Mech. Anal. **171**, 349–416 (2004).
37. A. I. Dyachenko, E. A. Kuznetsov, M. D. Spector, and V. E. Zakharov, "Analytic description of the free surface dynamics of an ideal fluid (canonical formalism and conformal mapping)," Phys. Lett. A **221** (1–2), 73–79 (1996).
38. E. Shargorodsky and J. F. Toland, "Riemann-Hilbert theory for problems with vanishing coefficients that arise in nonlinear hydrodynamics," J. Funct. Anal. **197**, 283–300 (2003).
39. A. I. Dyachenko, S. A. Dyachenko, P. M. Lushnikov, and V. E. Zakharov, "Traveling capillary waves on the boundary of a disc of poles in two-dimensional hydrodynamics with free surface: new constants of motion," J. Fluid Mech. **874**, 891–925 (2019).
40. P. M. Lushnikov, S. A. Dyachenko, and D. A. Silantyev, "New conformal mapping for adaptive resolving of the complex singularities of Stokes wave," Proc. R. Soc. A. **473**, 20170198 (2017). https://doi.org/10098/rspa.2017.0198

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