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On the Main Hypotheses of the General Mathematical Theory of Plasticity and the Limits of Their Applicability

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Abstract—The article discusses and analyzes the issues of applicability and the limits of applicability of some of the main hypotheses of the general mathematical theory of plasticity. In the theory of elastoplastic deformation processes, this is the postulate of the isotropy of initially isotropic bodies, in which the invariance of orthogonal transformations of the process images is established when a relationship between stresses and deformations is established. In the theory of flow, this is a hypothesis about the decomposition of total deformations into elastic and plastic deformations and the influence on its relationship between stresses and deformations under complex loading.

Keywords: elasticity, plasticity, stress and strain tensors, invariants, complex loading, stress and strain vectors, isotropy postulate, deformation decomposition hypothesis

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1. STRESS AND STRAIN TENSORS AND THEIR INVARIANTS

The stress-strain state (SSS) of continuous media and bodies referred to the coordinate axes x_i $(i = 1,2,3)$ for each point of the physical space with the coordinate orthonormal reference frame $\hat{e}_i$, is characterized by the specification of symmetric stress (σ_{ij}) and strain (ε_{ij}) tensors, where σ_{ij} and ε_{ij} $(i, j = 1,2,3)$ are their components. Geometrically, stress and strain tensors can be represented by trivectors

$$\bar{S}_i = \sigma_{ji}\hat{e}_j, \quad \bar{E}_i = \varepsilon_{ji}\hat{e}_j \quad (i,\ j = 1,\ 2,\ 3)$$

at each point of physical space. The stress vector at a given point on an arbitrary area with a unit normal $\hat{n} = n_i\hat{e}_i$ s represented by the Cauchy formula

$$\bar{S}_n = \bar{S}_i n_i = X_i\hat{e}_i, \quad X_i = \sigma_{ij}n_j. \tag{1.1}$$

The stress vector $\bar{S}_n$ is called the eigen or principal normal stress vector if its direction coincides with the direction of the normal $\hat{n} = n_i\hat{e}_i$, i.e.

$$\bar{S}_n = \sigma_k\hat{n} = \sigma_k\delta_{ij}n_j\hat{e}_i. \tag{1.2}$$

The modulus of this stress vector is simply called the eigenvalue or principal voltage.

Comparing (1.1) and (1.2), we obtain a system of equations for n_j

$$(\sigma_{ij} - \delta_{ij}\sigma_k)n_j = 0, \quad n_jn_j = 1. \tag{1.3}$$

Equating determinant (1.3) to zero, we obtain the characteristic equation for determining the eigenstresses σ_k $(k = 1, 2, 3)$

$$|\sigma_{ij} - \delta_{ij}\sigma_k| = 0,$$

whence the cubic equation follows

$$\sigma_k^3 - I_1\sigma_k^2 + I_2\sigma_k - I_3 = 0, \tag{1.4}$$

where

$$I_1 = \sigma_{ii} = \sigma_1 + \sigma_2 + \sigma_3 = 3\sigma_0,$$
$$I_2 = \sigma_{ii}\sigma_{jj} - (\sigma_{ij}\sigma_{ij}) = 9\sigma_0^2 = S^2,$$
$$I_3 = |\sigma_{ij}| = \sigma_1\sigma_2\sigma_3, \quad S^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2,$$

which are invariants of the stress tensor relative to the rotation of the coordinate axes x_i ($i = 1, 2, 3$). Components of deviators

$$S_{ij} = \delta_{ij} - \delta_{ij}\sigma_0, \quad Э_{ij} = \varepsilon_{ij} - \delta_{ij}\varepsilon_0.$$

Stress deviator tensor invariants

$$J_1 = S_{ii} = S_{11} + S_{22} + S_{33} = S_1 + S_2 + S_3 = 0,$$
$$2J_2 = S_{ij}S_{ij} = S_1^2 + S_2^2 + S_3^2 = \sigma^2,$$
$$J_3 = |S_{ij}| = \frac{\sigma^3 \cos 3\varphi}{3\sqrt{6}}.$$

Stress tensor module

$$S = \sqrt{3\sigma_0^2 + \sigma^2},$$

where $\sigma_0 = \sigma_{ii}/3$ is the modulus of the ball tensor.

$$\sigma = \sqrt{3}\tau_{\text{oct}} = \frac{1}{\sqrt{3}}\sqrt{(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2)}$$

is the modulus of the stress deviator, $\tau_{\text{oct}} = \sigma/\sqrt{3}$ is the octahedral shear stress. General solution of cubic equation (1.4)

$$\sigma_1 = \sigma_0 + \sqrt{\frac{2}{3}}\sigma\cos\varphi,$$
$$\sigma_2 = \sigma_0 + \sqrt{\frac{2}{3}}\sigma\cos\left(\frac{2\pi}{3} - \varphi\right),$$
$$\sigma_3 = \sigma_0 + \sqrt{\frac{2}{3}}\sigma\cos\left(\frac{2\pi}{3} + \varphi\right).$$

Principal shear stresses

$$T_{12} = \frac{\sigma_1 - \sigma_2}{2} = \frac{\sigma}{\sqrt{2}}\sin\left(\frac{2\pi}{3} + \varphi\right),$$
$$T_{23} = \frac{\sigma_2 - \sigma_3}{2} = \frac{\sigma}{\sqrt{2}}\sin\varphi, \tag{1.5}$$
$$T_{13} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma}{\sqrt{2}}\sin\left(\frac{2\pi}{3} - \varphi\right),$$

where φ is the angle of the stress state of the deformation on the octahedral area. Similar formulas hold for the strain tensor.

Along with the main shear stresses in the theory of plasticity T_{ij}, the octahedral shear stresses

$$\tau_{\text{oct}} = \frac{\sigma}{\sqrt{3}} = \frac{2}{3}\sqrt{T_{12}^2 + T_{23}^2 + T_{13}^2}$$

introduced by A. Nadai [8, 9] are of great importance. These stresses are equal on all faces of the octahedral particle, which is important. A. Nadai assumed that a material passes from an elastic state to a plastic one when τ_{oct} reaches a certain limiting value $\tau_{\text{oct}} = \tau^T$ or $\sigma = \sigma^T$,

$$\tau^T = \sqrt{2}\tau_*, \quad \sigma^T = \sqrt{\frac{2}{3}}\sigma_T$$

Table 1

Sector	$T_{\max}$	ω	$\varphi°$
I, IV	T_{13}, T_{31}	$\frac{2\pi}{3} - \varphi$	$0° \le \varphi \le 60°$ $180° \le \varphi \le 240°$
II, V	T_{23}, T_{32}	φ	$120° \le \varphi \le 180°$ $300° \le \varphi \le 360°$
II, VI	T_{12}, T_{21}	$\frac{2\pi}{3} + \varphi$	$60° \le \varphi \le 120°$ $240° \le \varphi \le 300°$

where τ_* is the pure spatial shear yield point, σ_T is the tensile yield point. Thus, A. Nadai gave a completely understandable interpretation of the plasticity criterion of Mises. On the deviatorial plane in the space of principal stresses, the Mises criterion is depicted by a circle of radius $\sigma = \sigma^T = \sqrt{2/3}\sigma_T = \sqrt{3}\tau^T$.

The deviatorial plane can be divided into six sectors and, according to formulas (1.5), compile a table for $T_{\max}$ [12].

It follows from Table 1 that the pattern of changes in $T_{\max}$ in each sector coincides and can be represented by the formula

$$T_{\max} = \frac{\sigma}{\sqrt{2}} \sin\omega = \sqrt{\frac{3}{2}}\tau_{\text{oct}} \sin\omega \quad (60° \le \omega \le 90°).$$

At the extreme points we get

$$\frac{\tau_{\text{oct}}}{\tau_{\max}} = \frac{2\sqrt{2}}{3} = 0.945, \quad \frac{\tau_{\text{oct}}}{\tau_{\max}} = \sqrt{\frac{2}{3}} = 0.816.$$

Thus, of the two well-known plasticity criteria Treska and Mises-Nadai, the Mises-Nadai criterion is correct (true), since $\tau_{\text{oct}} < \tau_{\max}$.

A natural generalization of the Mises–Nadai criterion for the processes of elastoplastic deformation is a unified universal curve of material hardening under simple loading by Roche and Eichinger and a unified universal curve of material hardening for trajectories of small and medium curvature of Il'yushin [7].

2. ON THE INFLUENCE OF ORTHOGONAL TRANSFORMATIONS OF THE COORDINATE FRAME AND THE STRESS VECTOR ON THE INVARIANTS OF STRESS AND STRAIN TENSORS IN PHYSICAL SPACE

This problem is studied in the works [1–3, 10–12]. When orthogonal transformation of the coordinate basis $\{\hat{e}_i\}$ to a new position $\hat{e}'_i = l_{ij}e_j$ $(i, j = 1, 2, 3)$, where $A = (l_{ij})$ is the matrix of this transformation. In the new position, the stationary stress vector $\overline{S}_n = X_i\hat{e}_i$ changes its coordinates so that

$$\overline{S}_n = X_j\hat{e}_j = X'_i\hat{e}'_i.$$

However, $\hat{e}_j = l_{ij}\hat{e}'_i$ and therefore

$$X'_i l_{ij} X_j \quad (i, j = 1, 2, 3).$$

The stress vector retains its length. Hence,

$$X'_i X'_i = (l_{ij}X_j)(l_{ik})X_k = X_j\delta_{ij}X_k,$$

whence the relation follows

$$l_{ij}l_{ik} = \delta_{jk} \quad (i, j, k = 1, 2, 3), \tag{2.1}$$

from which it follows that the coordinate basis remains orthonormal.

When deformation and loading processes are implemented at a point of the body, all three invariants of tensors (trivectors) will change. Vector $\overline{S}_n$ and trivectors will change their orientation. New projections of the vector $\bar{S}_n = X_i\hat{e}_i$ are determined by the formula $X_i' = \beta_{ij}X_j$, where β_{ij} are the components of the orthogonal transformation matrix. The vector length is unchanged and therefore

$$X_i'X_i' = (\beta_{ij}X_j)(\beta_{ik})X_k = X_j\delta_{jk}(X_k),$$

whence we obtain the relation

$$\beta_{ij}\beta_{ik} = \delta_{jk} \quad (i, j, k = 1, 2, 3). \tag{2.2}$$

This relation (2.2) coincides in form with (2.1) and therefore $l_{ij} = \beta_{ij}$. This means that the orthogonal transformations of the coordinate axes and the vector $\bar{S}_n$ coincide. However, in the first case, all three invariants of tensors are preserved, and in the second, only one (the length of the vector $\bar{S}_n$). The other two invariants of the type of SSS remain undefined. This creates some problem in determining the defining laws of the relationship between stresses and deformations.

The tensor form of the constitutive relations in continuum mechanics between stresses and strains is one of the most general, since it does not depend on the coordinate system. V. Prager and A.A. Il'yushin [1, 2, 7] for complex loading processes proposed, respectively, the relations

$$\frac{dS_{ij}}{ds} = A\frac{dЭ_{ij}}{ds} + BS_{ij} + CЭ_{ij},$$

$$S_{ij} = \sum_{n=0}^{4} A_n \frac{d^nЭ_{ij}}{ds^n},$$

where the coefficients A, B, C, A_n depend on the invariants.

As noted above, in the processes of deformation and loading under orthogonal transformations of the trivectors $\bar{S}_n$, the invariants can change, which can lead to non-invariance of the constitutive relations themselves [12].

3. VECTOR REPRESENTATION OF STRESS TENSORS, DEFORMATIONS AND LOADING PROCESSES

For a geometrically visual display of deformation and loading processes at a point in physical space, Il'yushin A.A. proposed to represent the stress (σ_{ij}) and deformation (ε_{ij}) tensors as vectors in the coordinate six-dimensional space of linear algebra [1–3, 11, 12]

$$\bar{S} = X_i\hat{e}_i, \quad \bar{E} = Y_i\hat{e}_i \quad (i = 1, 2, \ldots, 6),$$

where $\hat{e}_i$ are the components of the orthonormal coordinate frame,

$$X_1 = \sigma_{11}, \quad X_2 = \sigma_{22}, \quad X_3 = \sigma_{33}, \quad X_4 = \sqrt{2}\sigma_{12}, \quad X_5 = \sqrt{2}\sigma_{23}, \quad X_6 = \sqrt{2}\sigma_{13},$$
$$Y_1 = \varepsilon_{11}, \quad Y_2 = \varepsilon_{22}, \quad Y_3 = \varepsilon_{33}, \quad Y_4 = \sqrt{2}\varepsilon_{12}, \quad Y_5 = \sqrt{2}\varepsilon_{23}, \quad Y_6 = \sqrt{2}\varepsilon_{13}$$

are the coordinates of the vectors.

$$S = \sqrt{\sigma_{ij}\sigma_{ij}}, \quad E = \sqrt{\varepsilon_{ij}\varepsilon_{ij}}$$

are their modules, which are equal to tensor modules.

In the combined six-dimensional space E_6, the concepts of the image of the deformation process and the image of the loading process were also introduced in [1–3, 11]. The stress tensor (σ_{ij}) can be expanded into a direct sum of normal and shear stresses. They can be associated with two three-dimensional subspaces of normal and tangential stresses. If the initial tensor (σ_{ij}) is referred to the main natural axes, then the subspace of normal stresses becomes a subspace of natural stresses, and the subspace of tangential stresses is empty. Therefore, the space E_6 can have at most three proper directions. Thus, the stress vector $\bar{S}$ will be identical to the tensor (σ_{ij}), if their moduli are equal, but the vector $\bar{S}$ can have no more than three eigen directions and principal eigenstresses. Similarly for the vector $\bar{E}$ and strain tensor (ε_{ij}). Herefore, three invariants of tensors (σ_{ij}) and (ε_{ij}) in physical space remain invariants of vectors $\bar{S}$ and $\bar{E}$ in space E_6.

The image of the deformation process in E_6 is the deformation trajectory described by the end of the deformation vector $\overline{E}$ and the vectors $\overline{S}$ constructed at each of its points, as well as the scalar properties assigned to these points, such as temperature T and pressure p. In the theory of plasticity, the volumetric deformation $\theta = 3\varepsilon_0$ is considered elastic and obeys Hooke's law

$$\sigma_0 = 3K\varepsilon_0,$$

where K is the elastic Bridgman modulus. For this case, A.A. Il'yushin [1–3] introduced the following transformations of tensor components into coordinates of stress and strain vectors

$$\overline{S} = S_0\hat{i}_0 + S_k\hat{i}_k, \quad E = \mathcal{Э}_0\hat{i}_0 + \mathcal{Э}_k\hat{i}_k \quad (k = 1, 3, \dots, 5),$$

where new coordinates and coordinate basis

$$S_0 = \sqrt{3}\sigma_0, \quad S_1 = \sqrt{\frac{3}{2}}S_{11}, \quad S_2 = \frac{S_{22} - S_{33}}{\sqrt{2}}, \quad S_3 = \sqrt{2}\sigma_{12},$$
$$S_4 = \sqrt{2}\sigma_{23}, \quad S_5 = \sqrt{2}\sigma_{13},$$
$$\mathcal{Э}_0 = \sqrt{3}\varepsilon_0, \quad \mathcal{Э}_1 = \sqrt{\frac{3}{2}}\mathcal{Э}_{11}, \quad \mathcal{Э}_2 = \frac{\mathcal{Э}_{22} - \mathcal{Э}_{33}}{\sqrt{2}}, \quad \mathcal{Э}_3 = \sqrt{2}\varepsilon_{12},$$
$$\mathcal{Э}_4 = \sqrt{2}\varepsilon_{23}, \quad \mathcal{Э}_5 = \sqrt{2}\varepsilon_{13},$$
$$\hat{i}_0 = \frac{1}{\sqrt{3}}(\hat{e}_1 + \hat{e}_2 + \hat{e}_3), \quad \hat{i}_1 = \sqrt{\frac{2}{3}}\left[\hat{e}_1 - \frac{1}{2}(\hat{e}_2 + \hat{e}_3)\right], \quad \hat{i}_2 = \frac{\hat{e}_2 - \hat{e}_3}{\sqrt{2}},$$
$$\hat{i}_3 = \hat{e}_4, \quad \hat{i}_4 = \hat{e}_5, \quad \hat{i}_5 = \hat{e}_6.$$

In this case, the image of the process is constructed in the five-dimensional E_5 deviatoric subspace. The image of the deformation process is understood as a trajectory described in E_5 by the end of the deformation vector of deformation $\overline{\mathcal{Э}}$, the stress vectors $\overline{\sigma}$, $d\overline{\sigma}/ds$ constructed at each of its points and the parameters of temperature T and pressure p assigned to these points.

At each point of the deformation trajectory, it is also possible to construct a coordinate reference $\{d^n\overline{\mathcal{Э}}/ds^n\}$ and expand the stress vector in this frame

$$\overline{\sigma} = \sum_{n=0}^{4} A_n \frac{d^n\overline{\mathcal{Э}}}{ds^n}, \tag{3.1}$$

where the coefficients A_n are functionals of the process depending on the invariants of the tensors.

Instead of an oblique frame, it is possible to construct at each point of the trajectory a movable orthonormal Frenet–Ilyushin frame $\hat{p}_k$ $(k = 1, 2, \dots, 5)$ whose unit vectors satisfy the recurrent formulas

$$\frac{d\hat{p}_k}{ds} = -æ_{k-1}\hat{p}_{k-1} + æ_k\hat{p}_{k+1}. \tag{3.2}$$

In this frame, the vectors $\overline{\sigma}$, $d\overline{\sigma}/ds$ can be expanded in the form

$$\overline{\sigma} = P_k\hat{p}_k, \quad \frac{d\overline{\sigma}}{ds} = P_k^*\hat{p}_k, \quad \frac{d\hat{\sigma}}{ds} = P_k^0\hat{p}_k, \tag{3.3}$$

where the defining relations (3.1) and (3.3) in [1–3] are called the postulate of isotropy: the defining relations for the connection between stresses and deformations are invariant with respect to orthogonal transformations of the process image in coordinate spaces E_5 and E_6.

In [6], Professor D.D. Ivlev noted that the third invariant of stress and strain deviators during orthogonal transformation of the process image can change, which leads to violation of the isotropy postulate. This change in the invariants of deviators was previously noted in [1–3, 10]. The remark of D.D. Ivlev gave rise to a discussion in 1960–61 on a new direction in the development of the theory of plasticity proposed by Il'yushin [1–3]. In [4] Il'yushin noted that (the change in the third invariants of tensors is) as shown by numerous experiments of domestic and foreign scientists, the influence of the third invariants on the fulfillment of the isotropy postulate is weak and can be neglected at small elastoplastic deformations.

In [3] Il'yushin noted that violations of the isotropy postulate are possible in the nonlinear theory of elasticity.

Like any hypothesis, the isotropy postulate has its limits of applicability. However, no systematic experimental research has been carried out to establish this boundary.

4. THE THEORY OF PROCESSES OF ELASTOPLASTIC DEFORMATION AND EXTENDED THEORY OF FLOW

In the relations of the isotropy postulate (3.3), we change the orthonormal basis $\{\hat{p}_k\}$. Replace in it the unit vector $\hat{p}_2$ by the unit vector

$$\hat{\sigma} = p_k \cos\beta_k \quad (k = 1, 2, \ldots, 5), \tag{4.1}$$

where β_k are angular coordinates of $\hat{\sigma}$. hen the relations for $d\bar{\sigma}/ds$ can be represented as

$$\frac{d\bar{\sigma}}{ds} = M_m\hat{p}_m + M\hat{\sigma} \quad (m = 1, 3, 4, 5). \tag{4.2}$$

After multiplying (4.2) by $\hat{\sigma}$, we find M and transform (4.2) to the form

$$\frac{d\bar{\sigma}}{ds} = M_m\hat{p}_m + \left(\frac{d\sigma}{ds} - M_m\cos\beta_k\right)\hat{\sigma}, \tag{4.3}$$

where $M_m(s) = \sigma M_m^*(s, æ_m)$ are the functionals of the deformation process.

We represent the stress vector taking into account (4.1) in the form

$$\bar{\sigma} = \sigma\hat{\sigma} = \sigma(\hat{p}_k\cos\beta_k).$$

Differentiating the resulting expression, we find

$$\frac{d\bar{\sigma}}{ds} = \frac{d\sigma}{ds}\hat{\sigma} + \sigma\left(\frac{d\hat{\sigma}}{ds}\right) = \frac{d\sigma}{ds}\hat{\sigma} + \sigma\left[\frac{dp_k}{ds}\cos\beta_k + \hat{p}_k\frac{d}{ds}(\cos\beta_k)\right].$$

Using formulas (3.2), we obtain

$$\frac{d\bar{\sigma}}{ds} = \frac{d\sigma}{ds}\hat{\sigma}[-æ_{k-1}\hat{p}_{k-1} + æ_k\hat{p}_{k+1}]\cos\beta_k + \hat{p}_k\frac{d}{ds}(\cos\beta_k). \tag{4.4}$$

Eliminating from the obtained expression (4.4)

$$\hat{p}_2 = \frac{\hat{\sigma} - \hat{p}_m\cos\beta_m}{\cos\beta_2} \quad (m = 1, 3, 4, 5),$$

we arrive at an expression of the form (4.3).

Let us further restrict ourselves to a particular case of the theory of processes, that is, the coplanarity hypothesis. In this case, three vectors $\bar{\sigma}$, $d\bar{\sigma}$, $d\bar{Э}$ always lie in the same contiguous plane of the Frenet–Ilyushin frame and $k = m = 1$, $\beta_2 = 90° - \beta_1$. In this case, from equations (4.3)–(4.4) we obtain

$$\frac{d\bar{\sigma}}{ds} = M_1\hat{p}_1 + \left(\frac{d\sigma}{ds} - M_1\cos\beta_1\right)\hat{\sigma}, \tag{4.5}$$

$$\frac{d\beta_1}{ds} + æ_1 = -\frac{M_1}{\sigma}\sin\beta_1. \tag{4.6}$$

Equation (4.5) can also be written in the form

$$\frac{d\bar{Э}}{ds} = \frac{1}{M_1}\frac{d\bar{\sigma}}{ds} + \left(\cos\beta_1 - \frac{1}{M_1}\frac{d\sigma}{ds}\right)\hat{\sigma}. \tag{4.7}$$

The system of equations (4.5) and (4.6) of the theory of processes contains two functionals. For trajectories of mean curvature, these functionals have the form

$$M_1 = 2G_p + (2G - 2G_p^0)f^q(\beta_1),$$
$$\sigma = \Phi(s)$$

where

$$f = \frac{1 - \cos\beta_1}{2}$$

is the functional of complex loading, q is the experimentally determined parameter, $\Phi(s)$ is the universal Odqvist–Ilyushin hardening function, G_p is the plastic shear modulus, G_p^0 is its value at the bend point of the trajectory.

In the theory of flow, a fundamental hypothesis was introduced about the possibility of decomposing total deformations into elastic and plastic parts

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p, \quad \mathit{Э}_{ij} = \mathit{Э}_{ij}^e + \mathit{Э}_{ij}^p.$$

From our point of view, with complex loading and unloading, this is impossible. This hypothesis also contradicts the concepts of complete and incomplete plasticity by Haar and Karman [7]. In our conversations with Professor Ivlev D. D. about the hypothesis of the decomposition of total deformations into elastic $\bar{\mathit{Э}}^e$ and plastic $\bar{\mathit{Э}}^p$ parts, we agreed that, like any hypothesis, it has limits of its applicability. If we put in the equations of the theory of processes (4.6) and (4.7) $M_1 = G$, then we obtain the extended basic equations of the theory of flow

$$\begin{gathered} \frac{d\bar{\mathit{Э}}}{ds} = \frac{1}{2G}\frac{d\bar{\sigma}}{ds} + \left(\cos\beta_1 - \frac{1}{2G}\frac{d\sigma}{ds}\right)\hat{\sigma}, \\ \frac{d\beta_1}{ds} + æ_1 = -\frac{2G}{\sigma}\sin\beta_1, \end{gathered} \tag{4.8}$$

where $\sigma = \Phi(s)$ is a universal hardening function. From (4.8) it follows

$$\frac{d\bar{\mathit{Э}}^e}{ds} = \frac{1}{2G}\frac{d\bar{\sigma}}{ds}, \quad \frac{d\bar{\mathit{Э}}^p}{ds} = \left(\cos\beta_1 - \frac{1}{2G}\frac{d\sigma}{ds}\right).$$

Equations (4.8) satisfy the postulate of isotropy for trajectories of average curvature, contain the parameter $æ_1$ of complex loading and the approach angle β_1, which characterizes the vector properties of the material. In general, equations (4.8) are equations of an extended version of the theory of flow. The classical version of the theory of free plastic flow is obtained for $\beta_1 = 0$, $\hat{\sigma} = \hat{p}_1$ ($\cos\beta_1 = \hat{\sigma}\hat{p}_1$), $æ_1 \approx 0$ for a deformation path of small curvature or close to simple loading. For trajectories of large curvature deformation, the flow theory becomes unacceptable.

REFERENCES

1. A. A. Il'yushin, *Selected Works*, Vol. 2: *Plasticity (1946–1966)* (Fizmatlit, Moscow, 2004) [in Russian].
2. A. A. Il'yushin, *Plasticity. Foundations of the General Mathematical Theory* (Izd-vo AN SSSR, Moscow, 1963) [in Russian].
3. A. A. Il'yushin, *Continuum Mechanics* (MGU, Moscow, 1990) [in Russian].
4. A. A. Il'yushin, "Once again on the isotropy postulate," Izv. Akad. Nauk SSSR, Mekh. Mashinostr. No. 1, 201–204 (1962).
5. A. Yu. Ishlinskii and D. D. Ivlev, *Mathematical Theory of Plasticity* (Fizmatlit, Moscow, 2001)[in Russian].
6. D. D. Ivlev, "On the postulate of isotropy in the theory of plasticity," Izv. Akad. Nauk SSSR, Mekh. Mashiriostr. No. 2, 125–127 (1960).
7. *Theory of Plasticity*, Eb. by Yu. N. Rabotnov, (Izdat. Inostr. Lit., Moscow, 1948) [in Russian].
8. A. Nadai, *Theory of Flow and Fracture of Solids* (McGraw-Hill, New York, 1950).
9. R. Hill, *The Mathematical Theory of Plasticity* (The Clarendon Press, Oxford, 1950).
10. I. S. Sokol'nikov, *Tensor Analysis* (Nauka, Moscow, 1971) [in Russian].
11. V. G. Zubchaninov, *Mechanics of Processes of Plastic Media* (Fizmatlit, Moscow, 2010) [in Russian].
12. V.G. Zubchaninov, "The general mathematical theory of plasticity and the Il'yushin postulates of macroscopic definability and isotropy," Moscow Univ. Mech. Bull. **73**, 101–116 (2018).

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