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Torsion of Non-Uniform Cylindrical and Prismatic Rods Made of Ideally Plastic Material under Linearized Yield Criterion

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Abstract—The general relations of the torsion theory of inhomogeneous rods made of an ideal rigid-plastic material are considered. In the case of the linearized yield criterion, integrals are obtained that determine the stressed and deformed states of an ideal rigid-plastic inhomogeneous rod during torsion. The field of characteristics of the basic relations is constructed, the lines of stress rupture are found.

Keywords: Plasticity, heterogeneity, torsion, deformation, stress

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Torsion is one of the types of deformation of solids, characterized by mutual rotations of its cross sections under the influence of the moments acting in these sections. Torsion of rods is quite common in engineering practice, especially in mechanical engineering. The torsion theory of isotropic and anisotropic rods made of an ideal rigid-plastic material is described in [1–4]. The transition to the case of a rod made of an inhomogeneous material leads to certain difficulties: the problem in the general case cannot be integrated. The torsion of composite prismatic and cylindrical rods is considered in [5, 7]. In [6], general relations of the torsion theory of rods made of an ideal rigid-plastic material were investigated.

The relations between the torsion theory of inhomogeneous cylindrical and prismatic rods made of ideally plastic material [6] can be written in the form:

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0,\\ \tau_{xz} = \tau_{xz}(x,y), \quad \tau_{yz} = \tau_{yz}(x,y); \tag{1}$$

—equilibrium equation

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0; \tag{2}$$

—field criterion

$$f\left(\tau_{xz}, \tau_{yz}, x, y\right) = 0; \tag{3}$$

—the ratio of the associated plastic flow rule

$$\frac{\varepsilon_{xz}}{A} = \frac{\varepsilon_{yz}}{B}, \quad \varepsilon_x = \varepsilon_y = \varepsilon_z = \varepsilon_{xy} = 0, \tag{4}$$

where σ_{ij} are stress components, ε_{ij} are strain rate components,

$$A = \frac{\partial f}{\partial \tau_{xz}}(\tau_{xz}, \tau_{yz}, x, y), \quad B = \frac{\partial f}{\partial \tau_{yz}}(\tau_{xz}, \tau_{yz}, x, y). \tag{5}$$

Yield criterion (3) in the plane τ_{xz}, τ_{yz} for fixed x, y represents a closed convex curve (Fig. 1), the origin is in the region bounded by this curve.

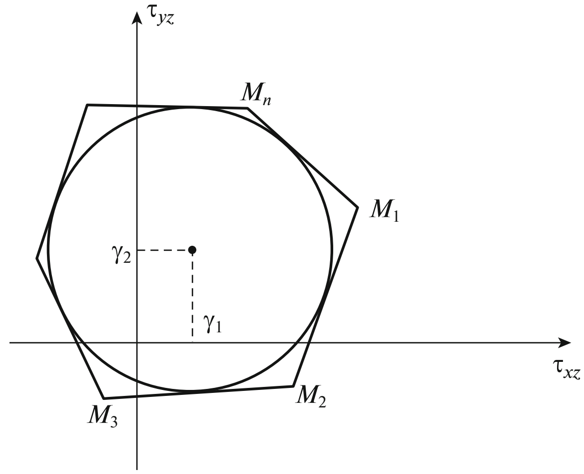

Fig. 1.

Let us assume that the yield curve (3) is replaced by a closed polygonal line $M_1M_2M_3\dots M_nM_1$ (Fig. 1)

$$A_i\tau_{xz} + B_i\tau_{yz} = K_i, \tag{6}$$

where $A_i = \dfrac{\partial f}{\partial\tau_{xz}}(\tau_{xz}^{i0}, \tau_{yz}^{i0}, x_0, y_0)$, $B_i = \dfrac{\partial f}{\partial\tau_{yz}}(\tau_{xz}^{i0}, \tau_{yz}^{i0}, x_0, y_0)$, $f(\tau_{xz}^{i0}, \tau_{yz}^{i0}, x_0, y_0) = 0,$

$$K_i = A_i\tau_{xz}^{i} + B_i\tau_{yz}^{i}, \quad A_i = \mu\frac{\partial f}{\partial\tau_{xz}}(\tau_{xz}^{i}, \tau_{yz}^{i}, x, y), \quad B_i = \mu\frac{\partial f}{\partial\tau_{yz}}(\tau_{xz}^{i}, \tau_{yz}^{i}, x, y),$$

$$f(\tau_{xz}^{i}, \tau_{yz}^{i}, x, y) = 0, \quad \mu = \text{const}, \quad i = 1, 2, \dots, n.$$

Criterion (6) represents on a certain segment the linearized yield criterion (3). Differentiating equation (6) with respect to the variable y, we obtain

$$A_i\frac{\partial\tau_{xz}}{\partial y} + B_i\frac{\partial\tau_{yz}}{\partial y} = \frac{\partial K_i}{\partial y}. \tag{7}$$

Taking into account (2), from equation (7), we have

$$A_i\frac{\partial\tau_{xz}}{\partial y} - B_i\frac{\partial\tau_{xz}}{\partial x} = \frac{\partial K_i}{\partial y}. \tag{8}$$

The system of equations for determining the characteristics of relation (8) and relations along the characteristics has the form

$$\frac{dx}{-B_i} = \frac{dy}{A_i} = \frac{d\tau_{xz}}{\dfrac{\partial K_i}{\partial y}} \tag{9}$$

It follows from (9) that the straight lines

$$A_ix + B_iy = C_{i1} \quad (C_{i1} = \text{const}) \tag{10}$$

are characteristics of relation (8). The following relations hold along characteristics (10)

$$A_i\tau_{xz} = \int\frac{\partial K_i}{\partial y}(\alpha, y)\,dy, \quad -B_i\tau_{xz} = \int\frac{\partial K_i}{\partial y}(x, \beta)\,dx, \tag{11}$$

where $\alpha = \dfrac{1}{A_i}(C_{i1} - B_iy)$, $\beta = \dfrac{1}{B_i}(C_{i1} - A_ix)$.

Similarly, differentiating equation (6) with respect to the variable x, taking into account (2), we obtain that along the characteristics (10) the following relations are valid for the stress components

$$-A_i\tau_{yz} = \int \frac{\partial K_i}{\partial x}(\alpha, y)\,dy, \quad B_i\tau_{yz} = \int \frac{\partial K_i}{\partial x}(x,\beta)\,dx. \tag{12}$$

Considering criterion (6) as a plastic potential, instead of (4) we obtain the relation

$$\frac{\varepsilon_{xz}}{A_i} = \frac{\varepsilon_{yz}}{B_i}. \tag{13}$$

Integrating relation (6) and part of relations (4) and taking into account that at the initial moment of torsion, the strain components e_{ij} are equal to 0, we obtain

$$\frac{e_{xz}}{A_i} = \frac{e_{yz}}{B_i}, \quad e_x = e_y = e_z = e_{xy} = 0. \tag{14}$$

From (14) it follows that

$$B_i e_{xz} - A_i e_{yz} = 0. \tag{15}$$

Let us assume that the displacement components u, v, w have the form

$$u = \theta yz, \quad v = -\theta xz, \quad w = w(x, y), \tag{16}$$

where θ is the twist, w is the warping.

Expressing strain components in terms of displacement components

$$e_{xz} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \quad e_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \tag{17}$$

from (15) and (16), we obtain

$$-B_i\frac{\partial w}{\partial x} + A_i\frac{\partial w}{\partial y} = \theta(A_i x + B_i y). \tag{18}$$

It follows from (18) that straight lines (10) are characteristics. Along characteristics (10), the following relations hold:

$$B_i w + \theta C_{i1}x = C_{i2} \quad \text{or} \quad A_i w - \theta C_{i1}y = C_{i3}, \tag{19}$$

where $C_{i2}, C_{i3} = \text{const}$ along the characteristic.

Differentiating relation (18) with respect to the variable x, we obtain the equation

$$-B_i\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) + A_i\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) = \theta A_i. \tag{20}$$

From equation (20) it follows that along characteristics (10) the following relations are valid:

$$B_i\frac{\partial w}{\partial x} + \theta A_i x = C_{i4} \quad \text{or} \quad \frac{\partial w}{\partial x} - \theta y = C_{i5}, \tag{21}$$

where $C_{i4}, C_{i5} = \text{const}$ along the characteristic.

Similarly, differentiating relation (18) with respect to the variable y, we obtain the equation

$$-B_i\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) + A_i\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) = \theta B_i. \tag{22}$$

From Eq. (22) it follows that along characteristics (10) the following relations are valid:

$$\frac{\partial w}{\partial x} + \theta x = C_{i6} \quad \text{or} \quad A_i\frac{\partial w}{\partial y} - \theta B_i y = C_{i7}, \tag{23}$$

where $C_{i6}, C_{i7} = \text{const}$ along the characteristic.

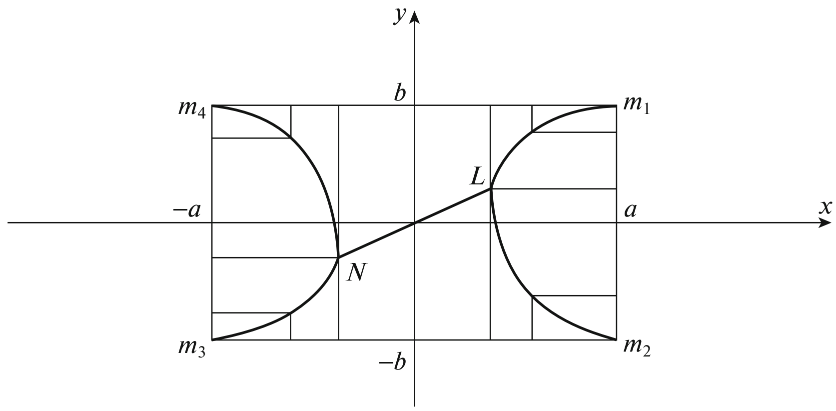

Fig. 2.

Using the second relation (21) and the first relation (23), we obtain that along the characteristics (10) the relations

$$e_{xz} - \theta y = \frac{1}{2}C_{i5}, \quad e_{yz} + \theta x = \frac{1}{2}C_{i6} \tag{24}$$

are valid.

It should be noted that relations (24), (10), (15) imply

$$A_i C_{i6} - B_i C_{i5} = 2\theta C_{i1}. \tag{25}$$

Let us consider the torsion of a rectangular cross-section rod $m_1 m_2 m_3 m_4$ with sides $2a$ and $2b$ (Fig. 2). On the contour of the section, the shear stress vector $\boldsymbol{\tau} = (\tau_{xz}, \tau_{yz})$ is parallel to the contour.

In the case of an isotropic ideally plastic material, the characteristics are directed perpendicular to the contour. In the case under consideration, the direction of characteristics (10) is fixed; therefore, for a given contour of the bar section, it is always possible to choose a linearized yield criterion (6) so that the characteristics remain perpendicular to the contour. For this, A_i, B_i in criterion (6) must be chosen so that the vector $\mathbf{n_i} = (A_i, B_i)$ is parallel to the segment $m_i m_{i+1}$ of the contour (Fig. 2).

Here we have four families of characteristics

$$A_1 x + B_1 y = C_{11}, \tag{26}$$

$$A_2 x + B_2 y = C_{21}, \tag{27}$$

$$A_3 x + B_3 y = C_{31}, \tag{28}$$

$$A_4 x + B_4 y = C_{41}. \tag{29}$$

In order for the characteristics (26) to be orthogonal to the segment $m_1 m_2$ of the contour of the cross-section of the rod, we should assume $A_1 = 0$, $B_1 = -1$. Yield criterion (6) takes the form

$$\tau_{yz} = -K_1. \tag{30}$$

Characteristics (26) will be written in the form

$$y = -C_{11}. \tag{31}$$

From (25), it follows that

$$C_{15} = 2\theta C_{11}. \tag{32}$$

Then from (11), (30), and (2) along characteristics (31) we have

$$\tau_{yz} = -K_1, \quad \tau_{xz} = \int \frac{\partial K_1}{\partial y}(x, -C_{11})\, dx = k_1. \tag{33}$$

In order for characteristics (27) to be orthogonal to the segment m_2m_3 of the contour of the cross-section of the rod, we should assume $A_2 = -1$, $B_2 = 0$. The yield criterion (6) takes the form

$$\tau_{xz} = -K_2 \tag{34}$$

Characteristics (27) will be written in the form

$$x = -C_{21} \tag{35}$$

From (25), it follows that

$$C_{26} = -2\theta C_{21} \tag{36}$$

Then, from (12), (34), and (2) along characteristics (35), we have

$$\tau_{xz} = -K_2, \quad \tau_{yz} = \int \frac{\partial K_2}{\partial x}(-C_{21}, y)\, dy = k_2. \tag{37}$$

In order for characteristics (28) to be orthogonal to the segment m_3m_4 of the contour of the cross-section of the rod, we should assume $A_3 = 0$, $B_3 = 1$. The yield criterion (6) takes the form

$$\tau_{yz} = K_3. \tag{38}$$

Characteristics (28) will be written in the form

$$y = C_{31}. \tag{39}$$

From (25), it follows that

$$C_{35} = -2\theta C_{31}. \tag{40}$$

Then, from (11), (38), and (2) along characteristics (39), we have

$$\tau_{yz} = K_3, \quad \tau_{xz} = -\int \frac{\partial K_3}{\partial y}(x, C_{31})\, dx = -k_3. \tag{41}$$

In order for characteristics (29) to be orthogonal to the segment m_4m_1 of the contour of the cross-section of the rod, we should assume $A_4 = 1$, $B_4 = 0$. The yield criterion (6) takes the form

$$\tau_{xz} = K_4. \tag{42}$$

Characteristics (29) will be written in the form

$$x = C_{41}. \tag{43}$$

From (25), it follows that

$$C_{46} = 2\theta C_{41}. \tag{44}$$

Then, from (12), (42), and (2) along characteristics (41), we have

$$\tau_{xz} = K_4, \quad \tau_{yz} = -\int \frac{\partial K_4}{\partial x}(C_{41}, y)\, dy = -k_4. \tag{45}$$

Special attention should be paid to the stress discontinuity lines (lines m_1L, m_2L,m_3N, m_4N, LN in Fig. 2), which arise when two or more characteristics pass through a given point of the cross-section.

Stress rupture lines are a trace of disappearing hard regions. They always satisfy the relations

$$e_{xz} = e_{yz} = 0. \tag{46}$$

Curve m_1L is the line of stress discontinuity emerging from the vertex m_1 of the contour of the cross-section of the rod and formed due to the intersection of the family of characteristics (31) and (43).

From (33) and (45), we have the equation for the stress discontinuity line m_1L

$$\frac{dx}{K_4 - k_1} = \frac{dy}{K_1 - k_4}. \tag{47}$$

Curve m_2L is the line of stress discontinuity emerging from the vertex m_2 of the contour of the cross-section of the rod and formed due to the intersection of the family of characteristics (31) and (35).

From (33) and (37), we have the equation for the stress discontinuity line m_2L

$$\frac{dx}{-K_2 - k_1} = \frac{dy}{K_1 + k_2}. \tag{48}$$

Curve m_3N is the line of stress discontinuity emerging from the vertex m_3 of the contour of the cross-section of the rod and formed due to the intersection of the family of characteristics (35) and (39).

From (37) and (41), we have the equation for the stress discontinuity line m_3N

$$\frac{dx}{K_2 - k_3} = \frac{dy}{K_3 - k_2}. \tag{49}$$

Curve m_4N is the line of stress discontinuity emerging from the vertex m_4 of the contour of the cross-section of the rod and formed due to the intersection of the family of characteristics (39) and (43).

From (41) and (45), we have the equation for the stress discontinuity line m_4N

$$\frac{dx}{-K_4 - k_3} = \frac{dy}{K_3 + k_4}. \tag{50}$$

The NL curve is the stress discontinuity line formed due to the intersection of the family of characteristics (35) and (43).

From (37) and (45) we get the equation of the stress discontinuity line NL

$$\frac{dx}{K_4 + K_2} = \frac{dy}{-k_4 - k_2}. \tag{51}$$

Let us consider the case when the yield criterion (3) has the form

$$(\tau_{xz} - \gamma_1)^2 + (\tau_{yz} - \gamma_2)^2 = k_0, \tag{52}$$

where $\gamma_1 = a_1x + b_1y, \gamma_2 = a_2x + b_2y, k_0 = \text{const}$.

According to (6), (33), (37), (41), (45), the stress components are determined as follows:

—in the region m_1Lm_2

$$\tau_{xz} = -b_2(x - a), \quad \tau_{yz} = -k_0 + a_2x + b_2y; \tag{53}$$

—in the region m_2LNm_3

$$\tau_{xz} = -k_0 + a_1x + b_1y, \quad \tau_{yz} = -a_1(y + b); \tag{54}$$

—in the region m_3Nm_4

$$\tau_{xz} = -b_2(x + a), \quad \tau_{yz} = k_0 + a_2x + b_2y; \tag{55}$$

—in the region m_4NLm_1

$$\tau_{xz} = k_0 + a_1x + b_1y, \quad \tau_{yz} = -a_1(y - b). \tag{56}$$

From (53), (54), (55), (56) we obtain the equations for the lines of discontinuity of stresses

$$m_1L\colon a_2x^2 + 2(a_1 + b_2)xy + b_1y^2 - 2(k_0 + a_1b)x + 2(k_0 - b_2a)y = r, \tag{57}$$

$$m_2L\colon a_2x^2 + 2(a_1 + b_2)xy + b_1y^2 - 2(k_0 - a_1b)x - 2(k_0 + b_2a)y = r, \tag{58}$$

$$m_3N\colon a_2x^2 + 2(a_1 + b_2)xy + b_1y^2 + 2(k_0 + a_1b)x - 2(k_0 - b_2a)y = r, \tag{59}$$

$$m_4N\colon a_2x^2 + 2(a_1 + b_2)xy + b_1y^2 + 2(k_0 - a_1b)x + 2(k_0 + b_2a)y = r, \tag{60}$$

$$LN\colon -a_1bx + k_0y = 0, \tag{61}$$

where $r = a_2a^2 - 2k_0(a - b) + b_1b^2$.

Thus, in the article:

—integrals were obtained that determine the stress and strain states of an inhomogeneous ideal rigid-plastic rod during torsion for the linearized yield criterion; characteristics of the basic relations and relations for the components of stresses and strains are found;

—the limiting state of an inhomogeneous ideal rigid-plastic bar with a rectangular cross-section were investigated: the field of characteristics of the basic relations is determined, relations are found along the characteristics and the stress rupture line.

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