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Van der Pol Oscillator. Technical Applications

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Abstract—Van der Pol equations describing self-oscillations in a quasilinear one-dimensional oscillator are generalized to the case when the generating isotropic oscillator has an arbitrary number of degrees of freedom. Two-dimensional (flat) and three-dimensional (spatial) cases are specifically considered. In contrast to the classical problem, in which a given amplitude of oscillations was stabilized, in the general case it is possible to stabilize not only the oscillation energy, but also the area of a flat elliptical trajectory, its orientation in space, the frequency of the oscillatory process, etc.

The technical applications of the respective models are indicated.

Keywords: wave solid-state gyroscope, strapdown inertial navigation system

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1. ONE-DIMENSIONAL VAN DER POL OSCILLATOR

Consider the Van der Pol equation in the form given, for example, in [1]

$$\ddot{q} + q = \mu(1 - q^2)\dot{q} \quad (\mu > 0), \tag{1.1}$$

here μ is a small parameter, which allows us to solve this equation, as is often done, by the averaging method.

For the purposes of our further discussion, it is convenient for us to interpret equation (1.1) as the equation of a linear spring oscillator in one-dimensional space (Fig. 1) with feedback superimposed on it by a formalized nonlinear right-hand side of equation (1.1). In [1], a solution to this equation was also given; in particular, it was shown that the equation has two stationary solutions: unstable $q = 0$, and asymptotically stable

$$q = 2\cos(t - t_0) \tag{1.2}$$

where t_0 is an arbitrary constant.

The only reason for introducing feedback in the one-dimensional case is the desire to provide a periodic process, despite the inevitable presence of dissipative forces in real systems.

Note that the type of feedback

$$\mu(1 - q^2)\dot{q} \tag{1.3}$$

chosen by Van der Pol and used in various subsequent works, according to nonlinear methods, where this equation is used as an example, is not the best for technical applications. Instead, you should write

$$\mu(1 - q^2 - \dot{q}^2)\dot{q}. \tag{1.4}$$

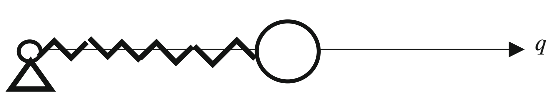

Fig. 1.

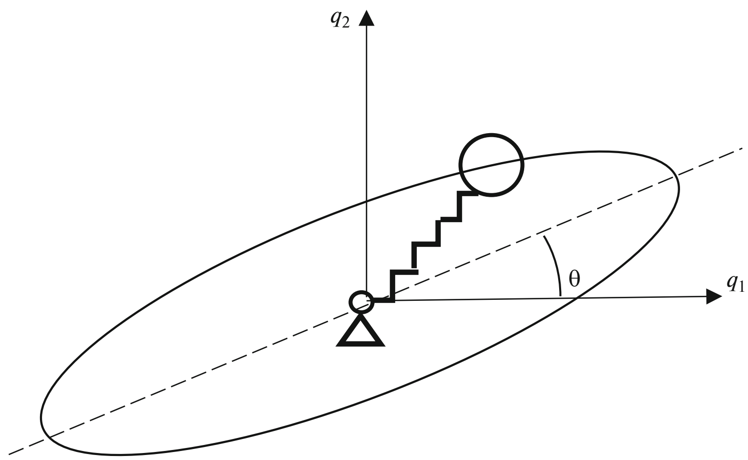

Fig. 2.

Indeed, for $\mu = 0$ equation (1.1) has the first integral $q^2 + \dot{q}^2 = \text{const}$ and, therefore, for $\mu \neq 0$ it changes slowly, in contrast to the variable q^2, which is rapidly changing in all cases. This not only simplifies the solution, but also improves the quality of management.

We will show this. Instead of equation (1.1), we consider the system

$$\begin{aligned} \dot{q} &= p, \\ \dot{p} &= -q + \mu(1 - q^2 - p^2)p. \end{aligned} \tag{1.5}$$

For $\mu = 0$, the solution to system (1.5) is

$$q = r\cos\varphi, \quad p = -r\sin\varphi, \quad \varphi = t - t_0. \tag{1.6}$$

Using this solution to move to the new variables $(q, p) \to (r, \varphi)$, we find

$$\begin{aligned} \dot{r} &= \mu(1 - r^2)r\sin^2\varphi, \\ \dot{\varphi} &= 1. \end{aligned} \tag{1.7}$$

We got a system with one slow variable r and one fast variable φ. Averaging over the fast variable gives $\dot{r} = \mu(1 - r^2)r/2$, whence two stationary solutions $r = 0$ (unstable) and $r = 1$ (asymptotically stable) follow.

Numerous technical applications of the one-dimensional oscillator model (1.1) are known, in particular, the tube generator model can be described in this way (however, the variable q in this case is not a spatial variable and plays the role of the anode strain in the electron tube).

2. TWO-DIMENSIONAL VAN DER POL OSCILLATOR

In such an oscillator (Fig. 2), instead of feedback of type (1.3), we will also use feedback (1.4), which in the two-dimensional case takes the form

$$\begin{aligned} \ddot{q}_1 + q_1 &= \mu(1 - q_1^2 - q_2^2 - \dot{q}_1^2 - \dot{q}_2^2)\dot{q}_1, \\ \ddot{q}_2 + q_2 &= \mu(1 - q_2^2 - q_2^2 - \dot{q}_1^2 - \dot{q}_2^2)\dot{q}_2. \end{aligned} \tag{2.1}$$

Or, in a shorter form

$$\ddot{q} + q = \mu(1 - q^2 - \dot{q}^2)\dot{q}, \quad q = (q_1, q_2)^{\mathrm{T}}.$$

In the plane (q_1, q_2), a free oscillator ($\mu = 0$) escribes an elliptical trajectory. As in the one-dimensional case, the feedback should fix the value of the doubled total energy.

The control problems in system (2.1) are much more meaningful than in system (1.1). In addition to controlling the amplitude of plane vibrations, it is possible to control the area of the described ellipse, the ratio of its semiaxes and the inclination of the major axis to the q_1 axis. In order to understand how to do this, we consider system (2.1) in a more general form

$$\ddot{q} + q = Aq + B\dot{q}. \tag{2.2}$$

The feedbacks linear in coordinates and velocities are defined by the to be determined square matrices A and B. In the most general form, these matrices look like this

$$A = C + N + H, \quad B = D + \Gamma + G, \tag{2.3}$$

$$C = c\begin{Vmatrix}1 & 0\\ 0 & 1\end{Vmatrix}, \quad N = n\begin{Vmatrix}0 & 1\\ -1 & 0\end{Vmatrix}, \quad H = h\begin{Vmatrix}\cos 2\alpha & \sin 2\alpha\\ \sin 2\alpha & -\cos 2\alpha\end{Vmatrix},$$

$$D = d\begin{Vmatrix}1 & 0\\ 0 & 1\end{Vmatrix}, \quad \Gamma = \gamma\begin{Vmatrix}0 & 1\\ -1 & 0\end{Vmatrix}, \quad G = g\begin{Vmatrix}\cos 2\beta & \sin 2\beta\\ \sin 2\beta & -\cos 2\beta\end{Vmatrix},$$

where C is a symmetric matrix of potential forces of a spherical type (scalar matrix), N is a skew-symmetric matrix of circular forces, H is a symmetric matrix of potential forces of a hyperbolic type, D is a symmetric matrix of dissipative forces of a spherical type, Γ is a skew-symmetric matrix of gyroscopic forces, G is a symmetric matrix of gyroscopic forces dissipative forces of hyperbolic type. The matrices H and G have a trace equal to zero (deviators), the forces determined by these matrices are not used for controlling the oscillator trajectory.

We proceed from equations (2.2) to equations in phase variables, using as a change of variables the general solution of the homogeneous part of the system in (2.2)

$$(q,\dot{q}) \to x : \begin{pmatrix} q_1\\ q_2\\ \dot{q}_1\\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} x_1\cos t + x_3\sin t\\ x_2\cos t + x_4\sin t\\ -x_1\sin t + x_3\cos t\\ -x_2\sin t + x_4\cos t \end{pmatrix}. \tag{2.4}$$

Assuming the right-hand sides of the system to be small, and applying time averaging, we find

$$\dot{x} = X(x) = \frac{1}{2\pi}\int_0^{2\pi} (-Q_1\sin t, -Q_2\sin t, Q_1\cos t, Q_2\cos t)^{\mathrm{T}} dt. \tag{2.5}$$

The right-hand sides of this system for the forces listed above in (2.2) are as follows

$$\begin{aligned} C &: X(x) = \frac{c}{2}(-x_3, -x_4, x_1, x_2),\\ N &: X(x) = \frac{n}{2}(-x_4, x_3, x_2, -x_1),\\ D &: X(x) = \frac{d}{2}(x_1, x_2, x_3, x_4),\\ \Gamma &: X(x) = \frac{\gamma}{2}(x_2, -x_1, x_4, -x_3). \end{aligned} \tag{2.6}$$

In [2, 4], for the system (2.2), a basis for the infinitesimal evolutions of its phase state in the vicinity of the zero quadrature ($K = q_1\dot{q}_2 - \dot{q}_1 q_2 = 0$), corresponding to rectilinear oscillations in the (q_1, q_2) plane is given. The components of this basis make it possible to find out what transformations the elliptic trajectory of a free two-dimensional oscillator undergoes under the influence of the listed forces

(a) Rotation group: $e_1 = (x_2, -x_1, x_4, -x_3)^{\mathrm{T}}$.

(b) Sprain group: $e_2 = (x_1, x_2, x_3, x_4)^{\mathrm{T}}$.

(c) Shift group: $e_3 = (x_4, -x_3, -x_2, x_1)^{\mathrm{T}}$. (2.7)

(d) Translation group: $e_4 = (x_3, x_4, -x_1, -x_2)^{\mathrm{T}}$.

It is easy to verify that the basis is orthogonal, therefore, projecting forces (2.6) onto it, we find the following table 1 of local evolutions

From Table 1 it follows that the potential spherical forces (matrix of these forces C) lead only to a change in the oscillation frequency. Circular forces (matrix N) cause a quadrature variation (destruction

Table 1.

	C	N	D	Γ
Precession (e_1)	0	0	0	$\gamma/2$
Amplitude (e_2)	0	0	$d/2$	0
Quadrature (e_3)	0	$-n/2$	0	0
Frequency (e_4)	$-c/2$	0	0	0

of the rectilinear waveform). The amplitude is changed by the velocity forces of a spherical type (matrix D). Gyroscopic forces (matrix Γ) lead to precession.

The specified table allows us to form control of these evolutions. As follows from the table, to control the frequency of the generated oscillations, positional forces of a spherical type should be applied

$$\begin{Vmatrix} Q_1 \\ Q_2 \end{Vmatrix} = c \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \begin{Vmatrix} q_1 \\ q_2 \end{Vmatrix}. \tag{2.8}$$

If it is necessary to stabilize the frequency of steady-state oscillations (in the dimensionless form of system (2.1), this frequency is equal to one), put

$$c = \mu_1 \left(1 - \frac{T}{\Pi}\right), \quad T = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2), \quad \Pi = \frac{1}{2}(q_1^2 + q_2^2). \tag{2.9}$$

By T and Π the kinetic and potential energies of the oscillator are indicated separately.

To stabilize the quadrature K equal to zero, again following the table, circular forces should be applied

$$\begin{Vmatrix} Q_1 \\ Q_2 \end{Vmatrix} = \mu_2 \frac{K}{E} \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix} \quad \left(n = \mu_2 \frac{K}{E}, K = q_1\dot{q}_2 - \dot{q}_1 q_2\right). \tag{2.10}$$

The stabilization of the total vibrational energy (amplitude) is achieved by applying dissipative forces of a spherical type

$$\begin{Vmatrix} Q_1 \\ Q_2 \end{Vmatrix} = \mu_3(1 - E) \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \begin{Vmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Vmatrix} \quad (d = \mu_3(1 - E)). \tag{2.11}$$

Finally, the precession is controlled by the application of gyroscopic forces

$$\begin{Vmatrix} Q_1 \\ Q_2 \end{Vmatrix} = \gamma \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix} \begin{Vmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Vmatrix}. \tag{2.12}$$

The coefficient γ is selected depending on the control objectives. If it is necessary to provide a given precession of the oscillation form with an angular velocity ω, then

$$\gamma = \mu_4(\omega - \dot{\theta}), \tag{2.13}$$

where θ is the angle of inclination of the main axis of the ellipse of the oscillation form to the abscissa axis. This angle is calculated in [4]:

$$\tan 2\theta = \frac{x_1^2 - x_2^2 + x_3^2 - x_4^2}{2(x_1x_2 + x_3x_4)}. \tag{2.14}$$

The two-dimensional Van der Pol equations (2.2), taking into account the stabilization of the total energy and quadrature, as well as with the control of the precession and frequency, take the form

$$\begin{cases} \ddot{q}_1 + q_1 = d(1/2 - E)\dot{q}_1 - n(K/E)q_2 - \gamma\dot{q}_2 + cq_1 \\ \ddot{q}_2 + q_2 = d(1/2 - E)\dot{q}_2 + n(K/E)q_1 + \gamma\dot{q}_1 + cq_2 \end{cases} \tag{2.15}$$

In equations (2.15), we pass from the variables (q_1, q_2) to variables (S, K) by formulas

$$\begin{gathered} S = 2E - 1/2 = \dot{q}_1^2 + \dot{q}_2^2 + q_1^2 + q_2^2 - 1/2, \\ K = q_1\dot{q}_2 - \dot{q}_1 q_2. \end{gathered} \tag{2.16}$$

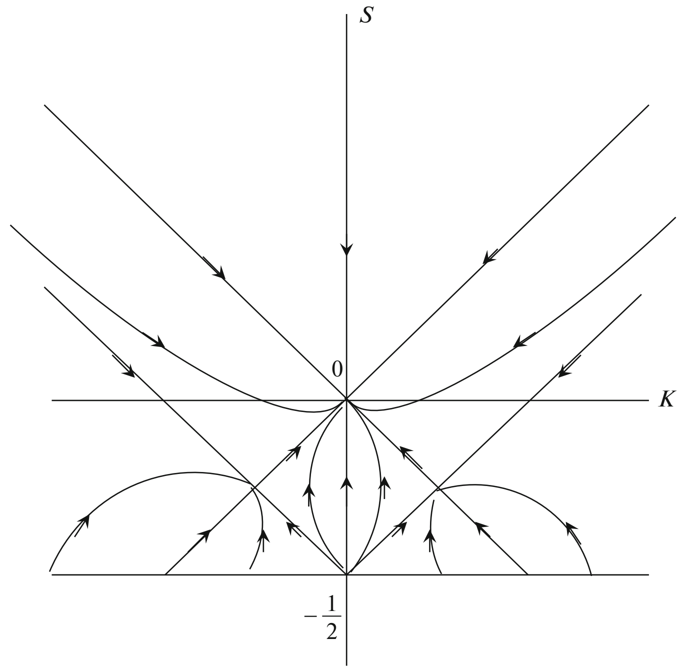

Fig. 3.

After averaging over time, we obtain equations two orders of magnitude smaller than the original

$$\dot{S} = -dS(2S+1) - 4n\frac{K^2}{2S+1},$$
$$\dot{K} = -2K(dS+n). \tag{2.17}$$

The phase portrait of the system (2.17) is shown in Fig. 3. There is an asymptotically stable singular point ($K = 0$, $S = 0$), which means the steady-state value of zero quadrature and 1/4 of the vibrational energy.

The presented model of a two-dimensional Van der Pol oscillator finds technical application as a model of a wave solid-state gyroscope [2, 5].

3. THREE-DIMENSIONAL VAN DER POL OSCILLATOR (Fig. 4)

The generalization of equation (1.1) to the three-dimensional case looks like this

$$\begin{cases} \ddot{q}_1 + q_1 = \mu(1-2E)\dot{q}_1, \\ \ddot{q}_2 + q_2 = \mu(1-2E)\dot{q}_2, \\ \ddot{q}_3 + q_3 = \mu(1-2E)\dot{q}_3, \end{cases}$$
$$2E = q_1^2 + q_2^2 + q_3^2 + \dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2,$$
$$K_1 = q_2\dot{q}_3 - \dot{q}_2 q_3, \quad K_2 = q_1\dot{q}_3 - \dot{q}_1 q_3, \quad K_3 = q_1\dot{q}_2 - \dot{q}_1 q_2.$$

A three-dimensional isotropic oscillator has the property that the elliptic trajectory described by it, as well as in the case of the Foucault pendulum, is stationary in inertial space (Fig. 4). The (I_1, I_2, I_3) axes formed by the major and minor semi-axes of the ellipse and the perpendicular to them do not rotate relative to the stars, no matter how the system (q_1, q_2, q_3) rotates, tightly connected with a moving

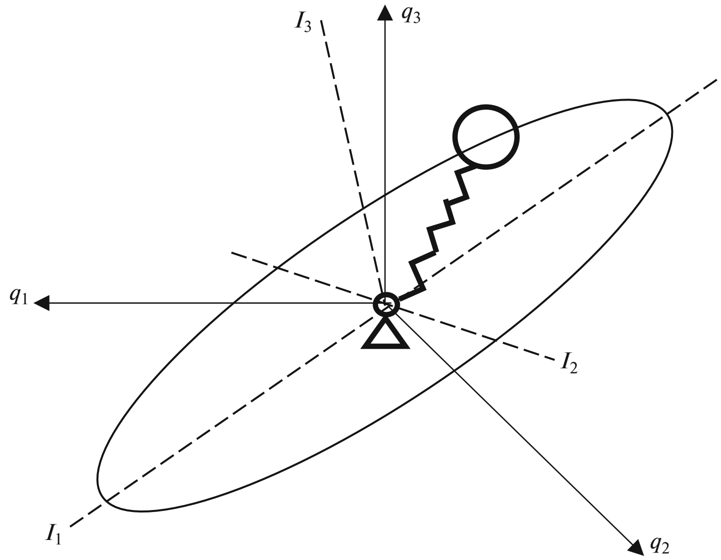

Fig. 4.

object. This makes it possible to use the three-dimensional Van der Pol oscillator as a strapdown inertial navigation system [3]. In such a system, there is no need to have three gyroscopes on board and integrate the Poisson equations to obtain information on the relative orientation of the (I_1, I_2, I_3) and (q_1, q_2, q_3) axes.

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