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Quaternion Solution of the Problem of Optimal Rotation of the Orbit Plane of a Variable-Mass Spacecraft Using Thrust Orthogonal to the Orbit Plane

Ya. G. Sapunkov[1] and Yu. N. Chelnokov[1*]

[1] Institute of Precision Mechanics and Control Problems of the Russian Academy of Sciences, ul. Rabochaya 24, Saratov, 410028 Russia

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Abstract—The problem of the optimal rotation of the orbital plane of a spacecraft (SC) of variable mass in an inertial coordinate system is solved in a nonlinear formulation using the quaternionic differential equation of orientation of the orbital coordinate system and the Pontryagin maximum principle. The problems of speed, minimization of the thrust impulse, the spacecraft characteristic speed, and also the problems of minimizing the combined quality functionals: time and total momentum of the thrust value spent on the control process, time and the spacecraft characteristic speed are considered. Rotation of the orbital plane of the spacecraft to any angles of magnitude is controlled using the reactive thrust limited in absolute value, orthogonal to the plane of the osculating spacecraft orbit. The change in the mass of the spacecraft due to the flow of the working fluid to the control process is taken into account. A special case of the problem under study is the problem of optimal correction of the angular elements of the spacecraft orbit. The results of calculations of the optimal control of the spacecraft orbital plane by means of a small limited reactive thrust with a large number of passive and active sections of the trajectory are presented.

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Keywords: *spacecraft, variable mass, rotation of the orbit plane, maximum principle, speed, minimization of the thrust impulse, characteristic speed, combined quality functional, quaternion.*

1. INTRODUCTION

In this article, we consider in a nonlinear formulation the problem of optimal reorientation of the orbit plane of a spacecraft in an inertial coordinate system using reactive thrust orthogonal to the plane of the osculating spacecraft orbit, in continuous formulations using limited (small) reactive thrust and using the quaternion differential orientation equation of the orbital coordinate system. The plane of the osculating orbit passes through the radius vector and the velocity vector of the center of mass of the spacecraft for the current point in time. With this control, the orbit of the spacecraft rotates in an inertial space as an unchanging (non-deformable) figure (ellipse or circle), rotating in an inertial coordinate system with an instantaneous angular velocity directed along the radius vector of the center of mass of the spacecraft. A special case of this problem is the well-known and of great practical importance in space flight mechanics problem of correction of the angular elements of the orbit of the spacecraft, when changes in the angular elements of the orbit in the control process are small. Using control orthogonal to the plane of the osculating SC orbit, allows adjusting the elements of the SC orbit, keeping the shape and dimensions of the SC orbit unchanged. This valuable property of the studied process of reorientation of the spacecraft's orbit is useful both in solving the problem of correcting the angular elements of the spacecraft's orbit and in other problems of space flight mechanics, for example, when controlling the configuration of a satellite group.

The problem of optimal reorientation of the orbital plane of the spacecraft is solved in the general case when changes in the angular elements of the orbit in the control process can take any finite values. We study the problems of speed, minimization of the thrust impulse of a jet engine, the characteristic

* e-mail: chelnokovyun@gmail.com

speed of the spacecraft, and also problems when the functional that determines the quality of the control process is a linear convolution with weight factors of two criteria: 1) the time and the total impulse of the thrust spent on the control process, 2) spacecraft time and characteristic speed.

In our earlier published and cited below papers, we studied the problem of optimal reorientation of the orbit of the spacecraft by reactive acceleration of the center of mass of the spacecraft, orthogonal to the plane of the osculating orbit, in various settings using different quaternion models of orbital motion (the quaternionic differential equation of orientation of the orbital coordinate system or the quaternionic differential equation of orientation of the orbit of the spacecraft) and various quality criteria. The solution to the problem using either a small thrust engine (in a continuous setting), or using a large thrust engine (in a pulse setting) was considered. As a control, we used the acceleration vector of the center of mass of the spacecraft from the thrust of the jet engine, orthogonal to the plane of the osculating orbit of the spacecraft.

In this article, a vector of reactive thrust limited by absolute value orthogonal to the plane of the osculating spacecraft orbit is used as a control. The change in the mass of the spacecraft due to the flow of the working fluid to the control process is taken into account. The mathematical model used to solve the problem describing the reorientation of the orbital plane of the spacecraft includes, as already noted, the quaternion differential equation of orientation of the orbital coordinate system, the differential equation for the true anomaly characterizing the position of the center of mass of the spacecraft in orbit and the rotation of the orbital coordinate system relative to the system coordinates associated with the orbit of the spacecraft, as well as the differential equation describing the change in the mass of the spacecraft in the process of reorienting its plane orbits.

2. DIFFERENTIAL EQUATIONS OF ORIENTATION OF THE ORBIT OF THE SPACECRAFT AND THE ORBITAL COORDINATE SYSTEM AND THE PROBLEM OF OPTIMAL REORIENTATION OF THE ORBIT AND THE PLANE OF THE ORBIT OF THE SPACECRAFT

We assume that the vector $\mathbf{a}$ of the reactive acceleration of the center of mass of the spacecraft from the thrust of the spacecraft's reaction engine, and, consequently, the thrust vector $\mathbf{u}*$ of the reaction engine during the entire controlled motion of the spacecraft are directed orthogonally to the plane of the osculating orbit of the spacecraft, i.e. orthogonal to the radius vector $\mathbf{r}$ and vector $\mathbf{v}$ of the velocity of the center of mass of the spacecraft (collinear to the vector $\mathbf{c} = \mathbf{r} \times \mathbf{v}$ of the velocity of the center of mass of the spacecraft). Then the differential equations of motion of the center of mass of the spacecraft in the Newtonian gravitational field, describing the change in the size and shape of the instantaneous orbit of the spacecraft, are integrated, giving the equation of the conical section. Therefore, the controlled motion of the spacecraft's center of mass in this case is described by differential equations describing the change in the instantaneous orientation of the spacecraft's orbit or the used (for example, orbital) rotating coordinate system in which the initial equations of motion of the spacecraft's center of mass are written, and by the differential equation for the true anomaly characterizing the position of the spacecraft's center of mass in orbit.

The orbit of the spacecraft in the process of such control of the motion of the center of mass of the spacecraft does not change its shape and size, but rotates in space under the action of control $\mathbf{a}$ or $\mathbf{u}^*$ as an unchangeable (non-deformable) figure.

The motion of the center of mass of the spacecraft will be considered in the inertial coordinate system X − the geocentric equatorial coordinate system $OX_1X_2X_3(X)$ with the origin at the center O of the Earth's gravity. The axis OX_3 of this coordinate system is directed along the axis of the diurnal rotation of the Earth, the axes OX_1 and OX_2 lie in the plane of the equator of the Earth, the axis OX_1 is directed to the vernal equinox for the Earth, the axis OX_2 complements the system to the right three vectors.

We also introduce the coordinate system ξ, associated with the plane and pericenter of the spacecraft orbit. The origin of this coordinate system is located in the center O (or in the pericenter of the orbit), the ξ_1 axis is directed along the radius vector of the orbit pericenter, the ξ_3 axis is perpendicular to the orbit plane and has the direction of the constant modulus vector $\mathbf{c}$ of the velocity of the spacecraft center of mass relative to the center O, and the axis ξ_2 forms the right triple with axes ξ_1 and ξ_3. The orientation of the coordinate system ξ in the inertial coordinate system X characterizes the orientation of the spacecraft orbit in the inertial space and is traditionally defined by three angular osculating elements of the orbit [1, 2]: the longitude of the ascending node Ω_u, the inclination of the orbit I and the angular distance of the pericenter from the node ω_π.

Differential equations describing the instantaneous orientation of the orbit of the spacecraft in the inertial coordinate system in the corner elements of the orbit in the considered case of orthogonality of the reactive thrust vector of the plane of the osculating orbit of the spacecraft have the form [1, 2]

$$\begin{aligned} &\frac{d\Omega_u}{dt} = \frac{r}{c} a \sin(\omega_\pi + \varphi) \operatorname{cosec} I, \quad \frac{dI}{dt} = \frac{r}{c} a \cos(\omega_\pi + \varphi), \\ &\frac{d\omega_\pi}{dt} = -\frac{r}{c} a \sin(\omega_\pi + \varphi) \cot I, \quad \frac{d\varphi}{dt} = \frac{c}{r^2}, \quad r = \frac{p}{1 + e\cos\varphi}, \quad c = \text{const}, \end{aligned} \tag{2.1}$$

where φ is the true anomaly (the angular variable measured in the plane of the orbit from its pericenter and characterizing the position of the center of mass of the spacecraft in orbit), $r = |\mathbf{r}|$ is the modulus of the radius vector of the center of mass of the spacecraft, p and e are the parameter and eccentricity of the orbit, $c = |\mathbf{c}| = |\mathbf{r} \times \mathbf{v}|$ is the area constant (modulus of the velocity vector of the velocity $\mathbf{v}$ of the center of mass of the spacecraft), a is the projection of the acceleration vector $\mathbf{a} = \mathbf{u}^*/m^*$ of the center of mass of the spacecraft on the direction of its velocity vector of the spacecraft (the algebraic value of jet acceleration perpendicular to the plane of the osculating orbit of the spacecraft), m^* is the mass of the spacecraft.

The problem of reorienting the spacecraft orbit in angular variables is formulated as follows: it is required to construct a control a that transfers the orbit, the orientation change of which is described by equations (2.1) from a given initial position

$$\Omega_u = \Omega_u(t_0) = \Omega_u^0, \quad I = I(t_0) = I^0, \quad \omega_\pi = \omega_\pi(t_0) = \omega_\pi^0, \quad I^0 \neq 0, \, \pi,$$

to the desired final position

$$\Omega_u = \Omega_u(t_1) = \Omega_u^*, \quad I = I(t_1) = I^*, \quad \omega_\pi = \omega_\pi(t_1) = \omega_\pi^*, \quad I^* \neq 0, \, \pi.$$

In this case, the selected functional of the quality of the reorientation of the spacecraft orbit should be minimized.

A special case of this problem was considered in the works of Yu. M. Kopnin [3–7]. In [4], the rotation of the plane of the near-Earth circular orbit was studied using traction normal to the instantaneous plane of the orbit, using the averaged equations in the angular elements of the orbit. In [6], the rotation of the plane of the satellite's circular orbit by a transverse thrust (thrust directed perpendicular to the instantaneous plane of the orbit, also called "binormal thrust" in [6]) was studied. To describe the motion, we used equations (2.1) written in dimensionless variables. The problem was considered under the assumption that the satellite's initial orbit lies in the equatorial plane and that the required inclination of the orbit is small (therefore, $\sin I \approx I$, $\cos I \approx 1$ was assumed in what follows).

In [7], the problem of turning the plane of the SC osculating orbit of the spacecraft using the "binormal force" creating the "binormal acceleration" using the equations for angular osculating elements (2.1) was considered. The consideration is limited to the case of a circular orbit, which, according to the authors of this work, was investigated in [3, 5]. An analysis is made of the rotation of the plane of a circular orbit that is optimal in the sense of minimizing the characteristic speed by an angle of inclination of the orbit ΔI for an unlimited time. The rotation of the orbit by the angle ΔI was considered in [7] in an approximate formulation using only the second equation of system (2.1) for inclining the orbit.

In paper [8], the secular variation of the angular elements of the orbit Ω_u, I, ω_π under the influence of reactive acceleration, orthogonal to the plane of the osculating spacecraft orbit, is considered. This task is called in the article the task of correcting the elements of the orbit Ω_u, I, ω_π "binormal reactive acceleration". The initial equations of motion of the spacecraft used in [8] have the form of equations in angular elements (2.1). To solve the problem, the authors of the article go over in these equations to a new independent variable (true anomaly φ) by the formula $dt = (r^2/c)d\varphi$ and supplement them with a differential equation for the characteristic velocity v_{ch}. The boundary conditions of the correction maneuver are written as

$$\begin{aligned} &t = t_0 = t(\varphi_0) = 0, \quad \Omega_u = \Omega_{u0}, \quad I = I_0, \quad \omega_\pi = \omega_{\pi 0}, \quad v_{ch} = 0, \\ &t = t_1 = t(\varphi_1), \quad \Omega_u = \Omega_{u1}, \quad I = I_1, \quad \omega_\pi = \omega_{\pi 1}, \quad v_{ch} = v_{ch1} \to \min. \end{aligned}$$

The problem is solved using the principle of maximum and averaging of equations. From the averaged equations, a series of analytical relationships is obtained to determine the costs of the characteristic velocity in particular cases of correction of one or two elements of the orbit (orbital inclination, longitude of the ascending node), provided that the changes in the orbital inclination and longitude of the ascending node are small.

The solution of the problem of optimal reorientation of the spacecraft orbit by means of reactive acceleration, orthogonal to the plane of the osculating orbit, using equations (2.1) in the angular elements of the orbit in a strict nonlinear formulation is quite difficult due to the non-linearity of these equations, the presence of singular points $I = 0, \pi$, and also due to cumbersome equations for conjugate factors. Therefore, to solve this problem, instead of the angular elements of the orbit, it is advisable to use the Euler (Rodrigue–Hamilton) parameters.

The differential equations of orientation of the spacecraft orbit in the Euler parameters are of the form [9–13]

$$2\frac{d\Lambda_0}{dt} = -\Omega_1\Lambda_1 - \Omega_2\Lambda_2,\ 2\frac{d\Lambda_1}{dt} = \Omega_1\Lambda_0 - \Omega_2\Lambda_3,\ 2\frac{d\Lambda_2}{dt} = \Omega_2\Lambda_0 + \Omega_1\Lambda_3,\ 2\frac{d\Lambda_3}{dt} = \Omega_2\Lambda_1 - \Omega_1\Lambda_2, \quad (2.2)$$

$$\frac{d\varphi}{dt} = \frac{c}{r^2}, \quad r = \frac{p}{1 + e\cos\varphi}, \quad c = \text{const}, \quad \Omega_1 = \frac{r}{c}a\cos\varphi, \quad \Omega_2 = \frac{r}{c}a\sin\varphi, \quad (2.3)$$

where Λ_j $(j = 0, 1, 2, 3)$ are the Euler parameters characterizing the orientation of the spacecraft orbit (coordinate system ξ) in the inertial coordinate system X; $\Omega_1, \Omega_2, \Omega_3 = 0$ are projections of vector $\boldsymbol{\Omega}$ of the instantaneous absolute angular velocity of the orbit onto the coordinate axes $O\xi_i$ associated with it.

The Euler parameters Λ_j are related to the angular elements of the orbit by relations

$$\begin{aligned} \Lambda_0 &= \cos\left(\frac{I}{2}\right)\cos\left(\frac{\Omega_u + \omega_\pi}{2}\right), \quad \Lambda_1 = \sin\left(\frac{I}{2}\right)\cos\left(\frac{\Omega_u - \omega_\pi}{2}\right), \\ \Lambda_2 &= \sin\left(\frac{I}{2}\right)\sin\left(\frac{\Omega_u - \omega_\pi}{2}\right), \quad \Lambda_3 = \cos\left(\frac{I}{2}\right)\sin\left(\frac{\Omega_u + \omega_\pi}{2}\right). \end{aligned} \quad (2.4)$$

Equations (2.2) in the quaternion record take the form [9–13]

$$\frac{2d\boldsymbol{\Lambda}}{dt} = \boldsymbol{\Lambda} \circ \boldsymbol{\Omega}, \quad \boldsymbol{\Omega} = \Omega_1\mathbf{i} + \Omega_2\mathbf{j} = \frac{r}{c}a(\cos\varphi\mathbf{i} + \sin\varphi\mathbf{j}), \quad (2.5)$$

where $\boldsymbol{\Lambda} = \Lambda_0 + \Lambda_1\mathbf{i} + \Lambda_2\mathbf{j} + \Lambda_3\mathbf{k}$ is the quaternion of the orientation of the spacecraft orbit (the quaternion osculating (slowly changing) element of the spacecraft orbit); $\boldsymbol{\Omega}$ is the mapping of vector $\boldsymbol{\Omega}$ onto basis ξ (vector $\boldsymbol{\Omega}$ of the instantaneous absolute angular velocity of the orbit is directed along the radius vector $\mathbf{r}$ of the center of mass of the spacecraft and is determined by the formula: $\boldsymbol{\Omega} = (a/c)\mathbf{r}$); $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are Hamilton's imaginary vector units, 0 is a symbol of quaternion multiplication.

Equations (2.2), (2.3) or (2.5), (2.3) are a system of five nonlinear stationary differential equations of the first order with respect to the Euler parameters Λ_j and the true anomaly φ. These equations, in contrast to the four nonlinear differential equations (2.1) of the orbit orientation in the angular elements of the orbit Ω_u, I, ω_π, do not have singular points $I = 0, \pi$, moreover, when passing from time t to a new independent variable φ in them, in accordance with the differential relation $d\varphi = (c/r^2)dt$ a system of four linear non-stationary differential equations with respect to the Euler parameters Λ_j (while the differential equations in the angular elements of the orbit remain essentially non-linear) is obtained (for $a = a(\varphi)$).

These circumstances make the use of equations (2.2), (2.3) or (2.5), (2.3) for solving the problems of reorienting the orbit, the plane of the orbit, and correcting the angular elements of the orbit more convenient and efficient compared to using the equations in angular osculating elements (2.1). Such a solution to the problem of reorienting the spacecraft orbit in a continuous formulation (using limited (low) thrust) was considered in [14–17]. In them, the combined functional equal to the weighted sum of the reorientation time and the integral of the square of the control module, as well as the combined functional equal to the weighted sum of the reorientation time and the control pulse (characteristic velocity) during the reorientation of the spacecraft orbit, are considered as the quality functional of the reorientation process of the spacecraft. In [18, 19], using equations (2.5), (2.3), a theory was developed for solving the problem of optimal reorientation of the spacecraft orbit in a pulse formulation (using pulsed (large) reactive thrust). Algorithms for solving boundary value problems of the optimal two-pulse and multi-pulse reorientation of the spacecraft orbit (for an unfixed number of pulses of reactive propulsion) and examples of numerical solutions of the boundary value problems of optimal reorientation of the spacecraft orbit using limited (small) or pulsed (large) thrust are presented, in which to describe the orientation of the spacecraft orbit a quaternion osculating element of the orbit orientation is used.

Note that in [14–19], the maximum principle was used to solve the reorientation problems of the spacecraft orbit, and the reactive acceleration a of the center of mass of the spacecraft was used as control.

Along with the differential equations (2.2), (2.3) or (2.5), (2.3) of the orientation of the spacecraft orbit in the Euler parameters Λ_j to solve the problems of reorienting the orbit, the plane of the spacecraft's orbit and correcting the spacecraft's angular elements, differential equations of orientation of the orbital coordinate system η can be used in the Euler parameters λ_j, having the form [11, 12, 20, 21]

$$2\frac{d\lambda_0}{dt}=-\omega_1\lambda_1-\omega_3\lambda_3,\ 2\frac{d\lambda_1}{dt}=\omega_1\lambda_0+\omega_3\lambda_2,\ 2\frac{d\lambda_2}{dt}=-\omega_3\lambda_1+\omega_1\lambda_3,\ 2\frac{d\lambda_3}{dt}=\omega_3\lambda_0-\omega_1\lambda_2, \tag{2.6}$$

$$\frac{d\varphi}{dt}=\frac{c}{r^2},\quad r=\frac{p}{1+e\cos\varphi},\quad c=\mathrm{const},\quad \omega_1=\frac{r}{c}a,\quad \omega_2=0,\quad \omega_3=\frac{c}{r^2}. \tag{2.7}$$

where λ_j $(j=0,\ 1,\ 2,\ 3)$ are the Euler parameters characterizing the orientation of the orbital coordinate system η in the inertial coordinate system X (the axis η_1 of this coordinate system is directed along the radius vector $\mathbf{r}$ of the spacecraft's center of mass, and the axis η_3 is perpendicular to the orbit plane (parallel to axis ξ_3)); ω_1, $\omega_2=0$, ω_3 are projections of vector $\boldsymbol{\omega}$ of the instantaneous absolute angular velocity of the orbital coordinate system onto its coordinate axes.

Note that control a enters in equations (2.6) multiplicatively into one of the two components of the absolute angular velocity of rotation of the orbital coordinate system, directed along the radius vector of the spacecraft's center of mass, and in equations (2.2) this control enters into two components of the absolute vector angular velocity of rotation of the orbit of the spacecraft.

Parameters λ_j are related to the angular variables Ω_u, I, ω_π, φ by relations similar to relations (2.4):

$$\begin{aligned}\lambda_0&=\cos\left(\frac{I}{2}\right)\cos\left(\frac{\Omega_u+\omega_\pi+\varphi}{2}\right),\quad &\lambda_1&=\sin\left(\frac{I}{2}\right)\cos\left(\frac{\Omega_u-\omega_\pi-\varphi}{2}\right),\\ \lambda_2&=\sin\left(\frac{I}{2}\right)\sin\left(\frac{\Omega_u-\omega_\pi-\varphi}{2}\right),\quad &\lambda_3&=\cos\left(\frac{I}{2}\right)\sin\left(\frac{\Omega_u+\omega_\pi+\varphi}{2}\right).\end{aligned} \tag{2.8}$$

Equations (2.6) in the quaternion record take the form [11, 12, 20, 21]

$$\frac{2d\boldsymbol{\lambda}}{dt}=\boldsymbol{\lambda}\circ\boldsymbol{\omega}_\eta,\boldsymbol{\omega}_\eta=\omega_1\mathbf{i}+\omega_3\mathbf{k}=\frac{r}{c}a\mathbf{i}+\frac{c}{r^2}\mathbf{k}, \tag{2.9}$$

where $\boldsymbol{\lambda}=\lambda_0+\lambda_1\mathbf{i}+\lambda_2\mathbf{j}+\lambda_3\mathbf{k}$ is the quaternion of inertial orientation of the orbital coordinate system associated with the quaternion $\boldsymbol{\Lambda}$ of the orientation of the spacecraft orbit by the relation

$$\boldsymbol{\lambda}=\boldsymbol{\Lambda}\circ\left[\cos\left(\frac{\varphi}{2}\right)+\mathbf{k}\sin\left(\frac{\varphi}{2}\right)\right], \tag{2.10}$$

$\boldsymbol{\omega}_\eta$ is the mapping of vector $\boldsymbol{\omega}$ onto basis $\boldsymbol{\eta}$.

We also note that the quaternion differential equation of orientation of the orbital coordinate system, similar to (2.9), was also used to describe the orbital motion in [21, 22].

Equations (2.6), (2.7) or (2.7), (2.9) were used in [12, 24–26] to solve using the Pontryagin maximum principle the problem of optimal reorientation of the spacecraft's orbit by means of reactive thrust orthogonal to the orbit plane in a continuous formulation (using limited (low) thrust and using reactive acceleration as a control a). As an optimality criterion, the combined functional equal to the weighted sum of the reorientation time and the integral quadratic (with respect to control) quality functional, or the functional equal to the weighted sum of the reorientation time and the control pulse (characteristic speed) during the reorientation of the spacecraft orbit was used. Control was supposed to be limited in modulus.

In [27, 28], equations (2.7), (2.9) were used to solve the problem of optimal reorientation of the spacecraft orbit by means of pulsed (large) reactive thrust orthogonal to the plane of the osculating orbit. The acceleration a of the center of mass of the spacecraft from the thrust of the reactive engine was used as control. The combined functional is minimized, equal to the weighted sum of the reorientation time and the jet acceleration impulse of the spacecraft's center of mass (characteristic velocity) during the reorientation of the spacecraft's orbit. The solution to the problem is constructed using limit transitions in equations and relations obtained as a result of solving the problem of optimal reorientation of the spacecraft orbit in a continuous formulation (using limited (small) reactive thrust and the Pontryagin maximum principle). The algorithms for the impulsive solution of the problem, constructed in [28], allow one to determine the optimal moments of turning on the reactive engine, the optimal values of the pulses of the reactive acceleration of the spacecraft and their optimal number. Examples of a numerical solution to the problem of optimal pulsed reorientation of the spacecraft orbit are presented, demonstrating the capabilities of the proposed method. It was also shown in [28] that the problem of optimal pulsed reorientation of the spacecraft orbit in the case when the optimal control consists of two reactive acceleration pulses applied to the spacecraft at initial and final moments of time of motion is solved analytically.

Using both equations (2.2), (2.3) (or (2.5), (2.3)), which describe the orientation of the spacecraft orbit and the position of the spacecraft in orbit, and equations (2.6), (2.7) (or (2.9), (2.7)), which describe the orientation of the orbital coordinate system and the position of the spacecraft in orbit, has its own advantages for solving problems of optimal reorientation of the spacecraft's orbit. Thus, equations (2.6), (2.7) are, for $r = \text{const}$ (in the case of a circular orbit) and $a = \text{const}$ linear differential equations with constant coefficients, while equations (2.2), (2.3) in this case are linear differential equations with variable coefficients. Therefore, equations (2.6), (2.7) (or (2.9), (2.7)) are more convenient and efficient in comparison with equations (2.2), (2.3) (or (2.5), (2.3)) from an analytical point of view. However, equations (2.2), (2.3) include equations (2.2) in scalar osculating elements Λ_j, and equations (2.5), (2.3) include equation (2.5) for the quaternion osculating element of orbit $\mathbf{\Lambda}$. Variables Λ_j, as well as quaternion $\mathbf{\Lambda}$, composed of them, as already noted, directly characterize the orientation of the spacecraft orbit in inertial space. If the control a (reactive acceleration) is equal to zero, these variables become constant values; therefore, the variables Λ_j are scalar osculating (slowly changing) elements of the orbit of the spacecraft, and the variable $\mathbf{\Lambda}$ is the quaternion osculating (slowly changing) element of the orbit of the spacecraft. The scalar variables λ_j and the equivalent quaternion variable $\boldsymbol{\lambda}$, that appear in equations (2.6), (2.7) and (2.9), (2.7) do not possess such properties (they are rapidly changing time functions). Therefore, the study of the problems of optimal reorientation of the orbit, the plane of the orbit, and the correction of the angular elements of the orbit of the spacecraft seems relevant both using the quaternion differential equation of orientation of the orbital coordinate system (2.9), and using the quaternion differential equation of orientation of the orbit of the spacecraft (2.5).

In this article, the problem of optimal reorientation of the orbital plane of a spacecraft is studied using equations (2.6), (2.7) (or (2.9), (2.7)), which are more convenient from an analytical point of view. In this case, it is not the algebraic value a of the reactive acceleration of the center of mass of the spacecraft, as was done in all the papers cited above, is taken as control, but the algebraic value $u*$ of the reactive thrust vector $\mathbf{u}^* = m^*(t)\mathbf{a}$ of the spacecraft engine, orthogonal to the plane of the osculating orbit of the spacecraft. The mass of the spacecraft m^* is assumed to be a piecewise differentiable function of time t: $m^* = m^*(t)$. The mathematical model describing the reorientation of the orbital plane of the spacecraft additionally includes a differential equation describing the change in the mass of the spacecraft in the process of controlled motion.

3. THE STATEMENT OF THE PROBLEM OF OPTIMAL ROTATION OF THE ORBITAL PLANE OF A SPACECRAFT OF VARIABLE MASS BY MEANS OF A LIMITED (SMALL) REACTIVE THRUST

Differential equations of the reorientation problem of the orbit of the spacecraft of variable mass (and the problem of reorientation of the plane of the orbit of the spacecraft) in the inertial coordinate system using reactive thrust orthogonal to the plane of the osculating orbit of the spacecraft, in the case of using the quaternion differential equation of orientation of the orbital coordinate system to solve the problem, have the following form:

$$2\frac{d\lambda}{dt} = \boldsymbol{\lambda} \circ \left(\frac{r}{cm^*}u^*\mathbf{i} + \frac{A}{r^2}\mathbf{k}\right), \quad \frac{d\varphi}{dt} = \frac{A}{r^2}, \quad r = \frac{p}{1 + e\cos\varphi}, \tag{3.1}$$

$$\frac{dm^*}{dt} = -\beta^*|u^*|, \tag{3.2}$$

where β^* is a constant coefficient of proportionality equal to the reciprocal of the velocity of the expiration of the working fluid from the reactive engine nozzle [29].

The orientation of the orbital plane of the spacecraft in the inertial coordinate system is characterized by the direction cosines n_k of the unit vector $\mathbf{n}$ normal to the plane of the orbit in this coordinate system, which we write through the Euler parameters λ_j and the angular elements of the spacecraft's orbit in the following form:

$$n_1 = 2(\lambda_1\lambda_3 + \lambda_0\lambda_2) = \sin I \sin\Omega_u, \quad n_2 = 2(\lambda_2\lambda_3 - \lambda_0\lambda_1) = -sin I\cos\Omega_u,$$
$$n_3 = \lambda_0^2 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2 = \cos I. \tag{3.3}$$

We pass to dimensionless variables ρ, τ, m, u, u_m, β by formulas

$$r = R\rho, \quad t = \frac{R^2}{c}\tau, \quad m^* = m_0^* m, \quad u^* = \frac{c^2 m_0^*}{R^3}u, \quad u_m^* = \frac{c^2 m_0^*}{R^3}u_m, \quad R = p, \quad \beta^* = \frac{R}{c}\beta, \tag{3.4}$$

where m_0^* is the initial mass of the spacecraft, $u_m^* = u_{\max}^*$ is the maximum thrust value.

The spacecraft motion equations (3.1), (3.2) in dimensionless variables take the form

$$2\frac{d\lambda}{d\tau} = \boldsymbol{\lambda}\circ\Big[\frac{u}{(1+e\cos\varphi)m}\mathbf{i} + (1+e\cos\varphi)^2\mathbf{k}\Big], \tag{3.5}$$

$$\frac{d\varphi}{d\tau} = (1+e\cos\varphi)^2, \quad \frac{dm}{d\tau} = -\beta|u|. \tag{3.6}$$

Problem statement: it is required to determine the control u limited by modulus::

$$-u_m \le u \le u_m < \infty, \quad |u| = |\mathbf{u}|, \tag{3.7}$$

orthogonal to the plane of the osculating orbit of the spacecraft, translating the spacecraft, the motion of the center of mass of which is described by equations (3.5), (3.6), from a given initial state

$$\tau = \tau_0 = 0, \quad \varphi(0) = \varphi_0, \quad \boldsymbol{\lambda}(0) = \boldsymbol{\Lambda}(0)\circ\Big[\cos\Big(\frac{\varphi_0}{2}\Big) + \mathbf{k}\sin\Big(\frac{\varphi_0}{2}\Big), m = 1, \tag{3.8}$$

to the final state belonging to the manifold

$$\tau = \tau 1, \quad 2[\lambda_1(\tau_1)\lambda_3(\tau_1) + \lambda_0(\tau_1)\lambda_2(\tau_1)] - \sin I(\tau_1)\sin\Omega_u(\tau_1) = 0,$$
$$2[\lambda_2(\tau_1)\lambda_3(\tau_1) - \lambda_0(\tau_1)\lambda_1(\tau_1)] + \sin I(\tau_1)\cos\Omega_u(\tau_1) = 0, \tag{3.9}$$

and minimizing the combined quality functional (3.10) or (3.11):

$$J = \int_0^{\tau_1}(\alpha_1 + \alpha_2|u|)d\tau, \quad \alpha_1 = \text{const} \ge 0, \quad \alpha_2 = \text{const} \ge 0, \tag{3.10}$$

$$J = \int_0^{\tau_1}\Big(\alpha_1 + \alpha_2\frac{|u|}{m}\Big)d\tau, \quad \alpha_1 = \text{const} \ge 0, \quad \alpha_2 = \text{const} \ge 0. \tag{3.11}$$

Functionals (3.10) and (3.11) are linear convolutions with constant weighting factors α_1 and α_2 of two criteria: (3.10) – time and momentum of the thrust of reactive engine spent on the control process, (3.11) – time and characteristic speed of the spacecraft.

The dimensionless time τ_1 of the control process is not predefined. In the particular case, when $\alpha_1 \ne 0$, $\alpha_2 = 0$, the optimality criterion is the duration of the control process and the optimal control problem becomes a performance problem. In other special cases, when $\alpha_1 = 0$, $\alpha_2 = 1$, the optimality criterion is the momentum of the thrust of a reactive engine (functional (3.10)) or the characteristic speed of the spacecraft (functional (3.11)).

Quaternion $\boldsymbol{\Lambda}(0)$ and scalar quantity φ_0 appearing in the initial conditions characterize the initial orientation of the spacecraft orbit and initial position of the spacecraft in orbit. The quantities c, p, e, φ_0, $\boldsymbol{\Lambda}(0)$ are given (the initial $\boldsymbol{\Lambda}(0)$ value of the quaternion $\boldsymbol{\Lambda}$ of the orientation of the spacecraft orbit can be found through the given initial values of the angular elements of the orbit Ω_u, I, ω_π according to formulas (2.4)). The finite values $\Omega_u(\tau_1)$, $I(\tau_1)$ of the angular elements of the orbit Ω_u, I, are also specified, which characterize the required (final) orientation of the orbital plane of the spacecraft and are included in equations (3.9) of the manifold to which the final state of the controlled system belongs. The optimal control law $u = u(\tau)$, under the action of which the orbital plane of the spacecraft will occupy the required position, and the quantities τ_1, φ_1 characterizing the time of the control process and the final position of the spacecraft in orbit are subject to determination.

The task is the optimal control problem with the moving right end of the trajectory. Conditions (3.9) at the right end of the trajectory are written in accordance with relations (3.3) for the directing cosines n_k of unit vector $\mathbf{n}$ normal to the plane of the spacecraft's orbit. Note that the direction cosines n_k are related by the condition $n_1^2+ n_2^2 + n_3^2 = 1$. Therefore, to solve the problem, only the first two of the three conditions (3.3) at the right end of the trajectory are taken. Instead, the first and third of conditions (3.3) or the second and third of these conditions can be taken.

4. LAWS (STRUCTURE) OF OPTIMAL CONTROL AND TRANSVERSALITY CONDITIONS

We will solve the problem with the help of the maximum principle. To this end, we introduce additional variables $\boldsymbol{\mu}$, χ and l conjugate with respect to the phase variables $\boldsymbol{\lambda}$, φ and m. The Hamilton–Pontryagin function in the case of minimizing the functional (3.10) has the form

$$\begin{aligned} H = \psi_0(\alpha_1 + \alpha_2|u|) + \frac{1}{2}[\mu_0(-\omega_{1d}\lambda_1 - \omega_{3d}\lambda_3) + \mu_1(\omega_{1d}\lambda_0 + \omega_{3d}\lambda_2 t) \\ + \mu_2(-\omega_{3d}\lambda_1 + \omega_{1d}\lambda_3) + \mu_3(\omega_{3d}\lambda_0 - \omega_{1d}\lambda_2)] + \chi(1 + e\cos\varphi)^2 + l(-\beta|u|t), \end{aligned} \tag{4.1}$$

and in the case of minimizing the functional (3.11), the form

$$\begin{aligned} H = \psi_0(\alpha_1 + \alpha_2\frac{|u|}{m}) + \frac{1}{2}[\mu_0(-\omega_{1d}\lambda_1 - \omega_{3d}\lambda_3) + \mu_1(\omega_{1d}\lambda_0 + \omega_{3d}\lambda_2) \\ + \mu_2(-\omega_{3d}\lambda_1 + \omega_{1d}\lambda_3) + \mu_3(\omega_{3d}\lambda_0 - \omega_{1d}\lambda_2)] + \chi(1 + e\cos\varphi)^2 + l(-\beta|u|). \end{aligned} \tag{4.2}$$

Here μ_j are the components of the conjugate quaternion variable $\boldsymbol{\mu}$,

$$\omega_{1d} = \frac{u}{(1 + e\cos\varphi)m}, \quad \omega_{3d} = (1 + e\cos\varphi)^2. \tag{4.3}$$

According to the maximum principle, the constant $\psi_0 \leq 0$. In what follows, we consider the case where the constant $\psi_0 < 0$. In this case, due to the homogeneity of the Hamilton–Pontryagin function with respect to the conjugate variables, the constant ψ_0 can be chosen arbitrarily, therefore, in the future we will set $\psi_0 = -1$.

The system of equations for conjugate variables in the case of minimizing the functional (3.10) has the form of equations (4.4)–(4.6):

$$2\frac{d\mu}{d\tau} = \mu \circ \Big[\frac{u}{(1 + e\cos\varphi)m}\mathbf{i} + (1 + e\cos\varphi)^2\,\mathbf{k}\Big], \tag{4.4}$$

$$\begin{aligned} \frac{d\chi}{d\tau} = e\sin\varphi[(1 + e\cos\varphi)(\lambda_0\mu_3 - \lambda_1\mu_2 + \lambda_2\mu_1 - \lambda_3\mu_0 + 2\chi) \\ - \frac{1}{2}\frac{u}{(1 + e\cos\varphi)^2 m}(\lambda_0\mu_1 - \lambda_1\mu_0 - \lambda_2\mu_3 + \lambda_3\mu_2)], \end{aligned} \tag{4.5}$$

$$\frac{dl}{d\tau} = \frac{1}{2}\frac{u}{(1 + e\cos\varphi)m^2}(\lambda_0\mu_1 - \lambda_1\mu_0 - \lambda_2\mu_3 + \lambda_3\mu_2). \tag{4.6}$$

In the case of minimizing functional (3.11), the system of equations for conjugate variables has the form of equations (4.4), (4.5) and equations (4.7):

$$\frac{dl}{d\tau} = \frac{1}{m^2}\Big[\alpha_2|u| + \frac{1}{2}\frac{u}{(1 + e\cos\varphi)}(\lambda_0\mu_1 - \lambda_1\mu_0 - \lambda_2\mu_3 + \lambda_3\mu_3)\Big]. \tag{4.7}$$

We introduce a new quaternion variable

$$\boldsymbol{\nu} = \nu_0 + \nu_1\mathbf{i} + \nu_2\mathbf{j} + \nu_3\mathbf{k} = \overline{\boldsymbol{\lambda}} \circ \boldsymbol{\mu}, \tag{4.8}$$

which is the multiplicative composition of the phase $\boldsymbol{\lambda}$ and the conjugate $\boldsymbol{\mu}$ quaternion variables. In the formula (4.8) and below, the upper line is the symbol of the quaternion pairing.

Components ν_j of quaternion $\boldsymbol{\mu}$ in accordance with (4.8) are determined by the formulas

$$\begin{aligned} &\nu_0 = \lambda_0\mu_0 + \lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3, \quad \nu_1 = \lambda_0\mu_1 - \lambda_1\mu_0 - \lambda_2\mu_3 + \lambda_3\mu_2, \\ &\nu_2 = \lambda_0\mu_2 + \lambda_1\mu_3 - \lambda_2\mu_0 - \lambda_3\mu_1, \quad \nu_3 = \lambda_0\mu_3 - \lambda_1\mu_2 + \lambda_2\mu_1 - \lambda_3\mu_0. \end{aligned} \tag{4.9}$$

Using the new variables ν_j, expressions (4.1), (4.2), (4.3) for the Hamilton–Pontryagin function and the conjugate equations (4.5), (4.6) and (4.7) take on a simpler form:

$$H = -[\alpha_1 + (\alpha_2 + \beta l)|u|] + \frac{1}{2}\Big[\frac{u}{(1+e\cos\varphi)m}\nu_1 + (1+e\cos\varphi)^2(\nu_3 + 2\chi)\Big], \tag{4.10}$$

$$H = -(\alpha_1 + \frac{\alpha_2}{m} + \beta l)|u| + \frac{1}{2}\Big[\frac{u}{(1+e\cos\varphi)m}\nu_1 + (1+e\cos\varphi)^2(\nu_3 + 2\chi)\Big], \tag{4.11}$$

$$\frac{d\chi}{d\tau} = e\sin\varphi\Big[(1+e\cos\varphi)(\nu_3 + 2\chi) - \frac{1}{2}\frac{u}{(1+e\cos\varphi)^2 m}\nu_1\Big], \tag{4.12}$$

$$\frac{dl}{d\tau} = \frac{1}{2}\frac{u}{(1+e\cos\varphi)m^2}\nu_1, \tag{4.13}$$

$$\frac{dl}{d\tau} = \frac{1}{m^2}\Big[\alpha_2|u| + \frac{1}{2}\frac{u}{(1+e\cos\varphi)}\nu_1\Big]. \tag{4.14}$$

Note that the quaternion conjugate equation (4.4) coincides in its form with the quaternion phase equation (3.5), since the quaternion equation (3.5) has the property of self-adjointness. We also note that the property of self-adjointness of the quaternionic kinematic equation was first established by V. N. Branz and I. P. Shmyglevsky in solving problems of optimal spatial turns of a rigid body [30].

It can be seen from (4.5) or (4.12) that in the case of a circular orbit, when the eccentricity of the orbit is $e = 0$, conjugate variable

$$\chi = \chi_0 = \chi(0) = \text{const}. \tag{4.15}$$

The optimal control law (i.e., control law satisfying the necessary optimality conditions) is found from the condition of the maximum of function H with respect to variable u taking into account the imposed constraint (3.7) and, in the case of minimizing the functional (3.10), takes the form (4.16):

$$\begin{aligned} &u = u_m \operatorname{sign}\nu_1, \quad \text{if} \quad \frac{|\nu_1|}{2(1+e\cos\varphi)m} \geq \alpha_2 + \beta l, \\ &u = 0, \quad \text{if} \quad \frac{|\nu_1|}{2(1+e\cos\varphi)m} < \alpha_2 + \beta l, \end{aligned} \tag{4.16}$$

and in the case of minimizing the functional (3.11) the form (4.17):

$$\begin{aligned} &u = u_m \operatorname{sign}\nu_1, \quad \text{if} \quad \frac{1}{m}\Big[\frac{|\nu_1|}{2(1+e\cos\varphi)} - \alpha_2\Big] \geq \beta l, \\ &u = 0, \frac{1}{m}\Big[\frac{|\nu_1|}{2(1+e\cos\varphi)} - \alpha_2\Big] < \beta l. \end{aligned} \tag{4.17}$$

Note that special control mode is not considered in the article.

The transversality conditions corresponding to the manifold of the final state (3.9), after excluding from them two indefinite Lagrange multipliers, take the following form:

for $\tau = \tau_1$:

$$\lambda_0\mu_3 - \lambda_1\mu_2 + \lambda_2\mu_1 - \lambda_3\mu_0 = \nu =_{3=} = 0, \quad \chi = 0, \quad l = 0, \tag{4.18}$$

$$\lambda_0\mu_0 - \lambda_1\mu_1 - \lambda_2\mu_2 + \lambda_3\mu_3 = \nu =_0 -2(\lambda_1\mu_1 + \lambda_2\mu_2) = 0. \tag{4.19}$$

5. ANALYSIS OF THE PROBLEM OF OPTIMAL REORIENTATION OF THE SPACECRAFT ORBIT

In the case of minimizing functional (3.10), the problem is reduced to a boundary value problem with a moving right end of the trajectory described by a system of nonlinear differential equations (3.5), (3.6), (4.4), (4.12), (4.13), (4.16) of the twelfth order and eight boundary conditions (3.8), (3.9), which must be supplemented by four transversality conditions (4.18), (4.19) and the equality

$$H = -[\alpha_1 + (\alpha_2 + \beta l)|u|] + \frac{1}{2}\Big[\frac{u}{(1 + e\cos\varphi)m}\nu_1 + (1 + e\cos\varphi)^2(\nu_3 + 2\chi)\Big] = 0 \qquad (5.1)$$

which holds for optimal control u and an optimal trajectory at its right end.

In the case of minimizing functional (3.11), the problem is reduced to a boundary-value problem with a moving right end of the trajectory described by a system of nonlinear differential equations (3.5), (3.6), (4.4), (4.12), (4.14), (4.17) of the twelfth order and eight boundary conditions (3.8), (3.9), which must be supplemented by four transversality conditions (4.18), (4.19) and the equality

$$H = -(\alpha_1 + \frac{\alpha_2}{m} + \beta l)|u| + \frac{1}{2}\Big[\frac{u}{(1 + e\cos\varphi)m}\nu_1 + (1 + e\cos\varphi)^2(\nu_3 + 2\chi)\Big] = 0 \qquad (5.2)$$

which holds for optimal control u and an optimal trajectory at a finite moment τ_1 of dimensionless time.

Differential equations of the problem have first integrals

$$\| \boldsymbol{\lambda} \|^2 = \lambda =_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \| \boldsymbol{\lambda}(0) \|^2 = 1, \qquad (5.3)$$

$$\| \boldsymbol{\mu} \|^2 = \mu_0^2 + \mu_1^2 + \mu_2^2 + \mu_3^2 = \| \boldsymbol{\mu}(0) \|^2 = \mathrm{const}, \qquad (5.4)$$

$$\boldsymbol{\mu} \circ \overline{\boldsymbol{\lambda}} = \boldsymbol{\nu}^* = \nu_0^* + \nu_1^*\mathbf{i} + \nu_2^*\mathbf{j} + \nu_3^*\mathbf{k} = \mathrm{const}, \qquad (5.5)$$

$$\lambda_0\mu_0 + \lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 = \nu_0 = \nu_0(0) = \mathrm{const}. \qquad (5.6)$$

Integrals (5.3)–(5.6) hold for any control u. Integrals (5.3), (5.4) reflect the constancy of the norms of the quaternion variables $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. The integral (5.5) holds due to the self-adjointness of the quaternionic phase equation (3.5).

Due to the autonomy of the system of equations for phase and conjugate variables, equalities (5.1) and (5.2) will be fulfilled at any time of optimal controlled motion, i.e. will be the first integrals of these equations for optimal control u. In the case of a circular orbit, the equations of the problem also have the first integral (4.15): $\chi = \chi_0 = \mathrm{const}$.

In the active segment of the trajectory, the optimal control u is a constant value; therefore, differential equation for mass m (the second equation of subsystem (3.6)) is integrated and gives the relation

$$m(\tau) = m_0 - \beta|u|\tau, \qquad (5.7)$$

which describes the change in the mass of the spacecraft in this section and shows that the dimensionless mass m of the spacecraft is a linear function of dimensionless time τ (decreases linearly over time τ).

Dimensional q^* and dimensionless q characteristic velocities of the spacecraft are related by relation $q^* = (c/R)q$. The change in the dimensionless characteristic velocity is described by the differential equation

$$\frac{dq}{d\tau} = \frac{|u|}{m(\tau)}. \qquad (5.8)$$

A comparison of this equation with the combined functional (3.11) shows that the characteristic velocity q is a special case of functional (3.11), when the coefficients of this functional are $\alpha_1 = 0,\ \alpha_2 = 1$.

Integration of equation (5.8) taking into account (5.7) in the active section of the trajectory gives the following law of variation of the characteristic velocity in the process of controlled spacecraft motion in this section of the trajectory:

$$q(\tau) = q_0 + \frac{1}{\beta}\ln\frac{m_0}{|m_0 - \beta|u|\tau|},$$

where m_0 and q_0 are values of m and q at the moment of time corresponding to the beginning of the considered active section of the trajectory.

Both functionals (3.10) and (3.11) can be expressed in terms of a finite dimensionless motion time τ_1 and a finite value of the dimensionless mass of the spacecraft m_1. We have:

$$J = \int_0^{\tau_1}(\alpha_1 + \alpha_2|u|)d\tau = \alpha_1\tau_1 + \frac{\alpha_2}{\beta}(1 - m_1),$$

$$J = \int\limits_0^{\tau_1} \Big(\alpha_1 + \frac{\alpha_2}{m}|u|\Big) d\tau = \alpha_1 \tau_1 - \frac{\alpha_2}{\beta} \ln m_1.$$

Thus, the Lagrange optimal control problem posed above is reduced to the Mayer problem. It follows from the last formulas given that, for $\alpha_1 = 0$ the tasks of minimizing the thrust momentum and characteristic speed are reduced to the problem of minimizing fuel. This is also seen from the Tables 4, 5 and 7 of the numerical calculations below.

From the Hamilton–Pontryagin functions (4.10), (4.11) and the optimal control laws (4.16), (4.17) it can be seen that they are actually functions not of the conjugate variables μ_j ($j = 0, 1, 2, 3$) (components of the quaternionic conjugate variable $\boldsymbol{\mu}$), but functions of the scalar variables ν_1 and ν_3 (components of quaternion $\boldsymbol{\nu}$), which in turn are multiplicative compositions of phase λ_j and conjugate μ_j variables of the form (4.9). Moreover, the quantity $\nu_0 = \text{const}$ (it is equal to component ν_0^* of quaternionic constant $\boldsymbol{\nu}^*$ of the controlled motion defined by (5.5)) and it appears only in the transversality condition (4.19). Therefore, instead of the above twelve differential equations of the boundary value optimal control problem with respect to phase and conjugate variables λ_j ($j = 0, 1, 2, 3$), φ, m, μ_j, χ, l, we can use a system of eleven differential equations for the variables λ_j, φ, m, ν_k ($k = 1, 2, 3$), χ, l to solve the problem of optimal reorientation of the spacecraft's orbit, omitting the differential equation for the variable ν_0, passing into relation $\nu_0 = \text{const}$, and the transversality condition (4.19) containing ν_0.

Differentiating the quaternion relation (4.8) with respect to the variable τ and taking into account equations (3.5), (4.4), we obtain the following differential equations for the scalar ν_0 and vector $\boldsymbol{\nu}_v = \nu_1 \boldsymbol{i} + \nu_2 \boldsymbol{j} + \nu_3 \boldsymbol{k}$ parts of the quaternion variable $\boldsymbol{\nu}$:

$$\frac{d\nu_0}{d\tau} = 0, \tag{5.9}$$

$$\frac{d\nu_v}{d\tau} = \nu_v \times \Big[\frac{u}{(1 + e\cos\varphi)m}\mathbf{i} + (1 + e\cos\varphi)^2 \mathbf{k}\Big]. \tag{5.10}$$

From equation (5.9) we have: $\nu_0 = \nu_0(0) = \text{const}$. From the vector equation (5.10) we obtain the following system of three scalar differential equations for the components ν_k of the vector variable $\boldsymbol{\nu}_v$:

$$\begin{aligned} &\frac{d\nu_1}{d\tau} = (1 + e\cos\varphi)^2 \nu_2, \quad \frac{d\nu_2}{d\tau} = \frac{u}{(1 + e\cos\varphi)m}\nu_3 - (1 + e\cos\varphi)^2 \nu_1, \\ &\frac{d\nu_3}{d\tau} = -\frac{u}{(1 + e\cos\varphi)m}\nu_2. \end{aligned} \tag{5.11}$$

Equations (5.11) ((5.10)) have (for any control) the first integral

$$\nu_1^2 + \nu_2^2 + \nu_3^2 = \boldsymbol{\nu}_v^2 = \boldsymbol{\nu}_v^{*2} = \text{const}, \quad \boldsymbol{\nu}_v^* = \text{vect}\boldsymbol{\nu}^* = \text{vect}(\boldsymbol{\mu} \circ \overline{\boldsymbol{\lambda}}).$$

Equations (5.11) are differential equations of the switching function $\nu_1 = \nu_1(\tau)$, since the control, as can be seen from relations (4.16) and (4.17), switches from one branch to another depending on the value of the coordinate ν_1. Variables ν_k, in equations (5.11) are related to the quaternionic first integral (5.5) of the equations of the boundary value problem by the rotation transformation: $\boldsymbol{\nu}_v = \overline{\boldsymbol{\lambda}} \circ \boldsymbol{\nu}_v^* \circ \boldsymbol{\lambda}$.

Thus, the problem of optimal reorientation of the orbital plane of a spacecraft in a continuous formulation (using limited (small) reactive thrust) reduces to solving a boundary-value problem with a moving right end of the trajectory described in the case of minimizing functional (3.10) by a system of eleven nonlinear stationary differential equations (3.5) , (3.6), (5.11), (4.12), (4.13), (4.16), and in the case of minimizing the functional (3.11) — by a system of eleven nonlinear stationary differential equations (3.5), (3.6), (5.11), (4.12), (4.14), (4.17) with respect to the variables λ_j, φ, m, ν_k, χ, l and eight boundary conditions (3.8), (3.9), which must be supplemented by three transversality conditions (4.18) and equality (5.1) in the first case or equality (5.2) in the second case.

The optimal control, as can be seen from (4.16) and (4.17), consists of two possible modes: the active section of motion with the constant maximum possible modulus of reactive thrust $u = \pm u_m$, the direction (sign) of which is determined by the sign of the variable ν_1, and a passive section of movement with zero thrust (recall that the special control mode is not considered in the article).

As can be seen from the transversality conditions (4.18), variables ν_3 and χ at the right end of the trajectory are equal to zero; therefore, condition (5.1) (in the case of minimizing the functional (3.10)) and condition (5.2) (in the case of minimizing the functional (3.11)) which hold for the optimal control u and the optimal trajectory at this end of the trajectory, take the form (5.11) and (5.13), respectively:

$$-\left[\alpha_1 + (\alpha_2 + \beta l)|u|\right] + \frac{u}{2(1 + e\cos\varphi)m}\nu_1 = 0, \tag{5.12}$$

$$-\left(\alpha_1 + \frac{\alpha_2}{m} + \beta l\right)|u| + \frac{u}{2(1 + e\cos\varphi)m}\nu_1 = 0. \tag{5.13}$$

Conjugate variable χ and the second transversality condition (4.18) are excluded from consideration, and the boundary-value problem will be described by a system of nonlinear stationary differential equations (3.5), (3.6), (5.11), (4.13), (4.16) or (3.5), (3.6), (5.11), (4.14), (4.17) of the tenth order with respect to the variables λ_j, φ, m, ν_k, l and eight boundary conditions (3.8), (3.9), which must be supplemented by two transversality conditions $\nu_3 = 0$, $l = 0$ and equality (5.11) or (5.13) at the right end of the trajectory.

Note that it is possible to reduce the dimension of the initial boundary-value problem described by differential equations for eleventh order variables λ_j, φ, m, ν_k, χ, l by two units due to the transition to a new independent variable correspond to the true anomaly φ (in this transition, differential equations for the true anomaly φ and the variable χ conjugate to it are excluded from consideration, and the differential equations of the boundary value problem become unsteady).

We also note that the differential equations (5.11), which describe the control switching function $\nu_1 = \nu_1(\varphi)$ and the second differential equation of the subsystem (3.6) for mass m when passing to a new independent variable φ (true anomaly), are allocated for optimal control $u = \pm u_m$ into a closed subsystem of four nonlinear non-stationary differential equations of the first order with respect to the variables ν_k, m, having the form

$$\frac{d\nu_1}{d\varphi} = \nu_2, \quad \frac{d\nu_2}{d\varphi} = \frac{u}{(1 + e\cos\varphi)^3 m}\nu_3 - \nu_1, \quad \frac{d\nu_3}{d\varphi} = -\frac{u}{(1 + e\, cos\varphi)^3 m}\nu_2, \quad \frac{dm}{d\varphi} = -\frac{\beta|u|}{(1 + e\cos\varphi)^2}.$$

6. RESULTS OF NUMERICAL SOLUTION OF THE PROBLEM OF OPTIMAL REORIENTATION OF THE ORBITAL PLANE OF THE SPACECRAFT AND THEIR ANALYSIS

The solutions of the problem given below for the same initial data, but for different functionals of the quality of the control process, are presented in dimensionless variables, angular quantities are expressed in degrees. The eccentricity of the orbit is $e = 0.1$, he coefficient characterizing the fuel consumption for creating a unit of thrust is $\beta = 4.0$, and the initial mass of the spacecraft is $m_0 = 1.0$. The initial orientation of the spacecraft orbit is determined by the angular elements of the orbit $I_n = 65.8°$, $\Omega_{un} = 217.0°$, $\omega_{\pi n} = 0.0°$; position of the spacecraft in orbit at the initial moment $\tau = 0$ is determined by the true anomaly $\varphi_n = 30.0°$. The orientation of the plane onto which the spacecraft orbit needs to be transferred is determined by the angular elements $I_k = 64.8°$, $\Omega_{uk} = 215.25°$.

6.1. Solution to the Performance Problem for Maximum Thrust

6.1.1. Solution to the performance problem for maximum thrust $u_m = 0.004$ Optimal control consists of 4 stages with constant controls u_m or $-u_m$. During the control process, the spacecraft makes one full turn. Table 1 describes the control stages: the times τ_i of the end of the stages, the true anomaly φ_i, the angular elements of the orbit, the mass m_i of the spacecraft at the end of the i^{th} stage, the value u_i of control at this stage are given. From Table 1 it can be seen that the duration of the optimal process is 10.998431 units of dimensionless time and during this time 0.175975 fuel was consumed (from the initial mass of the apparatus).

6.1.2. Solution to the performance problem for maximum thrust $u_m = 0.002$ Optimal control consists of 8 stages with constant controls u_m or $-u_m$. During the control process, the spacecraft makes three full turns. Table 2 describes the control stages. It can be seen that the duration of the optimal process is 22.948044 units of dimensionless time and during this time 0.183584 fuel was consumed (from the initial mass of the apparatus). At each internal stage (i.e., with the exception of the first and last stages) with control $u = 0.002$ the true anomaly changes by 180.0004°, and at the stage with control $u = -0.002$ by 180.0002°. The duration of the internal stages with control $u = 0.002$ gradually decreased by 0.000004, and the duration of the stages with control $u = -0.002$ increased by 0.000002 units of dimensionless time.

Table 1

Stage number	τ_i	φ_i	I_i	Ω_{ui}	$\omega_{\pi i}$	m_i	u_i
1	2.096597	149.2167	65.8433	216.5530	0.1837	0.966454	-0.004
2	5.494421	329.2171	65.4720	216.0780	0.3815	0.912089	0.004
3	8.475241	149.2176	65.3290	215.5773	0.5915	0.864396	-0.004
4	10.998431	274.0070	64.8000	215.2500	0.7309	0.824025	0.004

Table 2

Stage number	τ_i	φ_i	I_i	Ω_{ui}	$\omega_{\pi i}$	m_i	u_i
1	2.105966	149.6650	65.8217	216.7777	0.0912	0.983152	-0.002
2	5.501053	329.6654	65.6409	216.5487	0.1858	0.955992	0.002
3	8.484604	149.6656	65.5726	216.3139	0.2832	0.932123	-0.002
4	11.879687	329.6659	65.3827	216.0709	0.3846	0.904963	0.002
5	14.863240	149.6662	65.3113	215.8218	0.4888	0.881094	-0.002
6	18.258319	329.6664	65.1112	215.5630	0.5979	0.853933	0.002
7	21.241875	149.6667	65.0366	215.2979	0.7101	0.830065	-0.002
8	22.948044	230.5039	64.8000	215.2500	0.7306	0.816416	0.002

Table 3

u_m	τ_k	Number of stages	Number of full turns	Fuel consumption	φ_k°	$\omega_{\pi k}^\circ$
0.004	10.998431	4	1	0.1759752	74.0070	0.7309
0.002	22.948043	8	3	0.183584	230.5039	0.7306
0.001	45.697142	15	7	0.182789	95.1737	0.7314
0.0005	92.432460	30	14	0.184865	197.6370	0.7314

Table 4

Duration of process τ_k	52.523586	74.803842	115.606753	151.105858
Number of stages	33	47	74	94
Number of full turns φ_k	8	11	18	23
True anomaly ϕ_k	119.3230	275.1223	80.4386	261.3726
Angular distance to pericenter $\omega_{\pi k}$	0.7319	0.7315	0.7315	0.7314
Fuel consumption Δm_k	0.147081	0.130290	0.124351	0.122125
Functional value J	0.036773	0.032572	0.031088	0.030531

Table 3 shows the results of solving the performance problem for various values of maximum thrust. The subscript k for quantities τ, φ, ω_π means that their values are given for a finite point in time τ_k. It is seen that with a decrease in the maximum control value u_m the time τ_k of the control process, the number of control stages, and the full turns of the spacecraft trajectory increase. So, for $u_m = 0.0005$ the number of control stages and full turns of the spacecraft trajectory becomes 30 and 14, respectively.

Table 5

Duration of process τ_k	214.875157	297.675157	552.675945
Number of control stages	68	94	174
Number of full turns φ_k	33	46	86
True anomaly ϕ_k	260.4280	253.6861	246.0391
Angular distance to pericenter $\omega_{\pi k}$	0.7315	0.7314	0.7314
Fuel consumption Δm_k	0.119739	0.118224	0.117122
Functional value J	0.029935	0.029557	0.029280

Table 6

τ_k	Δm_k	φ_k	J
49.320972	0.153777	276.9730°	0.087765
51.746055	0.148850	75.9973°	0.088959

6.2. Solution to the Problem in the Case of Minimizing the Impulse of the Thrust of Reactive Engine of the Spacecraft

In this case, the coefficients of the optimization functionals $\alpha_1 = 0$, $\alpha_2 = 1$. It is established that the boundary-value optimization problem admits in this case, under the same boundary conditions, countless solutions. In the obtained solutions for the maximum thrust u_m= 0.001 with an increase in the duration of the control process, the momentum of the thrust value decreases and, as the duration increases, it tends to a certain limit. Table 4 shows the solutions to the optimal control problem in which four stages with controls $u = 0.001$, $u = 0.0$, $u = -0.001$, $u = 0.0$ are stacked in one full turn of the trajectory.

With an increase in the duration of the control process, the structure of optimal control changes. Only two stages with controls $u = 0.0$, $u = 0.001$ fit into one full turn. With a duration of $\tau_k = 214.875157$ the value of the functional is $J = 0.029935$, and with $\tau_k = 297.674094$, $J = 0.029557$. Results for these durations of the control process are presented in Table 5.

6.3. Solution to the Problem in the Case of Minimizing the Combined Functionality, which Includes the Time and Momentum of the Thrust of Reactive Engine of the Spacecraft

It is established that the boundary-value optimization problem allows in this case, with the same initial data of the problem, several solutions from which it is necessary to choose the optimal one (with a lower functional value). Table 6 (in it Δm_k is the dimensionless fuel consumption, φ_k is the final true anomaly) provides two solutions to this problem for the maximum thrust u_m= 0.001 in the case of minimizing the functional

$$J = \int_0^{\tau_1} (\alpha_1 + \alpha_2|u|)d\tau, \quad \tau_1 = \tau_k, \quad \alpha_1 = 0.001, \quad \alpha_2 = 1.0$$

The table shows that the optimal solution is presented in the first row. The solutions for optimal controls on each complete turn of the trajectory contain four stages: $u = 0.001$, $u = 0.0$, $u = -0.001$, $u = 0.0$.

6.4. Solution to the Problem in the Case of Minimizing the Characteristic Speed of the Spacecraft

It is established that the boundary value optimization problem admits in this case many solutions for the same initial data of the problem. Table 7 gives several such solutions for the case $u_m = 0.001$. Solutions for optimal controls presented in the upper two lines in each full turn contain four stages: $u = 0.001$, $u = 0.0$, $u = -0.001$, $u = 0.0$. In solutions presented in the third, fourth, and fifth rows, at the time τ^* indicated in the table, there is a transition from a four-stage mode in one full turn of the trajectory to a two-stage mode in one full turn: $u = 0.0$, $u = -0.001$.

Table 7

τ_k	φ_k°	Δm_k	Number of stages	Number of full turns	$\omega_{\pi k}$	J
52.418780	113.8536	0.147491	33	8	0.7318	0.039893
74.635590	265.4729	0131242	47	11	0.7314	0.035173
112.690580	253.5030	0.126391	$\tau^* = 80.901470$		0.7310	0.033781
150.808957	245.3223	0.124428	$\tau^* = 68.162150$		0.7310	0.033219
201.653104	235.7468	0.123696	$\tau^* = 68.151568$		0.7309	0.033012

Table 8

α_1	τ_k	**J**	φ_k°	Number of stages	Number of turns	$\omega_{\pi k}^\circ$	Fuel consumption
0.001	49.230108	0.091174	271.6881	31	7	0.7313	0.154458

6.5. Solution to the Problem in the Case of Minimizing the Combined Functionality

$$J = \int_0^{\tau_1} (\alpha_1 + \frac{\alpha_2}{m}|u|)d\tau, \quad \tau_1 = \tau_k, \quad u_m = 0.001, \quad \alpha_1 = 0.001, \quad \alpha_2 = 1.0.$$

The solution for optimal control, shown in Table 8, in each full turn contains four stages: $u = 0.001$, $u = 0.0$, $u = -0.001$, $u = 0.0$.

7. CONCLUSION

Using the quaternion differential orientation equation of the orbital coordinate system and the Pontryagin maximum principle, the problem of optimal rotation of the orbital plane of a spacecraft of variable mass in an inertial coordinate system is solved. The rotation of the orbital plane of the spacecraft to any angles of magnitude is controlled using reactive thrust limited in absolute value, orthogonal to the plane of the osculating spacecraft orbit. Under the influence of such a thrust, the shape and dimensions of the spacecraft's orbit remain unchanged throughout the control process, and the orbit itself rotates in an inertial space as an unchanging (non-deformable) figure (circle, ellipse). The change in the mass of the apparatus due to the flow of the working fluid to the control process is taken into account. A special case of the problem under study is the optimal correction of the angular elements of the spacecraft orbit, which is important in the mechanics of space flight.

The problems of performance, minimization of the thrust impulse, the characteristic speed of the spacecraft are considered, and also problems when the functional determining the quality of the control process is a linear convolution with weighting factors of two criteria: 1) time and total impulse of the thrust value spent on the control process, 2) time and characteristic speed of the spacecraft. The structure of optimal control, depending on the selected functional of the quality of the control process, is of a different nature and includes passive sections of the trajectory where there is no thrust and the orbit plane remains unchanged, and the active sections of the trajectory in which the jet thrust assumes the maximum value and the orbital plane of the spacecraft rotates.

Numerical solutions to the problem of optimal control of the spacecraft orbital plane by means of a small limited reactive thrust with a large number of passive and active sections of the trajectory (control stages) are obtained. The calculations show that in cases where the thrust of the thrust control or the characteristic speed of the spacecraft is the criterion for the quality of the control process, the Pontryagin maximum principle satisfies many solutions in which, with an increase in the time interval of the control process, the value of the quality functional of the control process decreases and tends to a certain limit. In this case, with an increase in the time interval of the control process, the number of control stages increases, the durations of the active stages decrease, the durations of the passive stages increase, the structure of the optimal control changes. In cases where the combined quality functional is minimized, including the time and momentum of the thrust spent on the control process, or the spacecraft time and characteristic speed, the optimal control boundary value problems allow several solutions, from which it is necessary to choose a solution with a minimum functional value.

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