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Simplified Method for Solving the Problem of Transversal Deflection of Micropolar Elastic Plates

S. V. Vardanyan

Moscow State University, Moscow, 119991 Russia

e-mail: vardanyan.sedrak@gmail.com

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Abstract—A simplified technique for solving the problem of transversal deflection of micropolar plates is developed within the framework of the micropolar theory of elasticity. The method is recommended for engineering calculations of micropolar structures, allowing a stress–strain state to be simply calculated via embedding a single function that brings a system of equations to the more convenient form. The task is successfully solved for a stress–strain state by the example of a long rectangular plate with different boundary conditions. The deflection plots are given, both in the context of micropolarity and in the framework of classical theory.

Keywords: micropolar plate, transversal deflection, simplified solution

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The analytical tools for solving the problems of plate deflection are given by the accurate spatial approaches developed by Nowacki [1, 2], as well as by the methods based on various hypotheses, e.g., proposed by Kirchhoff [3, 4], Mindlin [5], Reissner [6], and Ambartsumyan [7]. The theory of the micropolar shell was put forward by Zubov and Eremeev [8]. Finite strains of incompressible isotropic micropolar materials were investigated in work [9], and the equilibrium state of nonlinearly elastic micropolar bodies was within the scope of study [10].

The behavior of bending plates was analyzed via various finite-element approaches by Gevorkyan [11], whose method is reduced to quadratic programming problems, enabling one to formulate the more sophisticated boundary problems where the conditions of continuity of displacements and stresses under different boundary conditions are satisfied. This technique was extended to deflection problems within the framework of the theory of micropolar plates [12–14].

Another numerical method for solving the bending-plate problem based on the finite element approach was shown in [15]. Propositions for optimizing the plate computation algorithms have been made (e.g., in [16]).

The results, obtained in numerous experimental works, led to the breakthrough in the theory of bending of micropolar plates. Furthermore, state-of-the-art tools allow multifunctional problems to be solved via high-accuracy experiments [17].

Unlike the classical isotropic theory of elasticity, the micropolar theory of elasticity has six independent constants of the material. Their physical meaning, as well as the values, was specified empirically [18–22]. It is also worth mentioning works [23, 24], aimed at experimental study of the mechanical characteristics. Besides the experimental techniques, there are other ways to establish the parameters of the process, which still remain unknown for some materials.

1. Task formulation. Consider an elastic micropolar plate with a constant thickness h. A coordinate system has to be introduced so that the middle plane of the plate matches the coordinate plane xOy, and the z axis is perpendicular to the median plane. The discrete structure is assumed to be a solid medium. Let the power effects on the elementary area of the differential element be implemented by both the main force and the main moment vectors [27]. The main vector of external forces is expressed through the stress tensor components σ_{ij}, and the main moment of external forces is presented by the moment stress tensor components $\mu_{ij}\,(i, j = 1,2,3)$, where both tensors are asymmetric. Each point of the elementary particle has the bulk forces and the bulk moments represented by vectors $f = f\,(f_1, f_2, f_3)$ and $g = g\,(g_1, g_2, g_3)$, respectively.

Taking the moment stresses into account, the equilibrium equation and the law of small elastic deformations can be written, as follows:

$$\sigma_{ji,j} + f_i = 0, \quad \in_{ijk} \sigma_{jk} + \mu_{ji,j} + g_i = 0,$$
$$\sigma_{ji} = (\mu+\alpha)\gamma_{ji} + (\mu-\alpha)\gamma_{ij} + \lambda\gamma_{ii}\delta_{ij}, \quad \mu_{ji} = (\gamma+\varepsilon)\chi_{ji} + (\gamma-\varepsilon)\chi_{ij} + \beta\chi_{ii}\delta_{ij}, \tag{1.1}$$

where $\in_{ijk}$ are the Levi-Civita tensor components, χ_{ij} are the bending—torsion tensor components, δ_{ij} is the Kronecker symbol, λ and μ are the Lamé parameters, and $\alpha, \gamma, \beta, \varepsilon$ are the elastic micropolarity constants.

Let the plane $z = \pm h/2$ be loaded by surface forces in the absence of external moment effects, then the boundary conditions are as follows:

$$z = \pm h/2, \quad \sigma_{zz} = \pm Z^{\pm}, \quad \sigma_{31} = \pm X^{\pm}, \quad \sigma_{32} = \pm Y^{\pm}, \quad \mu_{33} = 0, \quad \mu_{31} = 0, \quad \mu_{32} = 0, \tag{1.2}$$

where $X^{\pm}(x,y)$, $Y^{\pm}(x,y)$ and $Z^{\pm}(x,y)$ are the tangential and normal components of surface force vectors applied to the planes $z = \pm h/2$, respectively.

The hypotheses that form the basis of the theory of plates are defined by Ambartsumyan [25–27], as follows:

1) Displacement w and rotation ω_3, being normal to the median plane of the plate, are independent of coordinate z.

2) Shear stresses σ_{xz} and σ_{yz} within a plate thickness vary in accordance with a specified law.

3) Force (σ_{zx}, σ_{zy}, σ_{zz}) and moment (μ_{zx}, μ_{zy}, μ_{zz}) stresses are negligibly small.

Assume the designations below:

$$A = \gamma + \varepsilon + D/h,$$
$$A_1 = (\mu+\alpha)(\gamma+\varepsilon) + 12D\frac{\alpha}{5h}, \quad A_2 = (\mu+\alpha)\left(2\gamma + \beta + \frac{\alpha}{5}h^2\right),$$
$$A_3 = 12D\nu\frac{\alpha}{5h} + \frac{\alpha(\mu-\alpha)}{5}h^2 - (\mu+\alpha)(\gamma+\beta-\varepsilon), \tag{1.3}$$
$$B = \frac{E}{1-\nu^2}, \quad B_{12} = \frac{E\nu}{1-\nu^2}, \quad D = \frac{E}{12(1-\nu^2)}h^3,$$
$$X_1 = \frac{1}{2}(X^+ - X^-), \quad X_2 = X^+ + X^- \quad X \leftrightarrow Y, \quad Z_2 = Z^+ + Z^-,$$

where E is the elastic modulus, ν is the Poisson coefficient, and D is the stiffness of the bending plate at zero moment stresses.

According to the above hypotheses, the transverse bending equations ($X^{\pm} = 0, Y^{\pm} = 0$) of a micropolar plate can be presented relatively the sought functions $w(x, y)$ and $\psi_i(x,y)$ $(i = 1,2)$ as

$$(\psi_{1x} + \psi_{2y})\alpha h^3/12 = -Z_2 \tag{1.4}$$

$$A\Delta w_x - (A_1\Delta\psi_{1xx} - A_3(\psi_{1yy} - \psi_{1xy}))h^2/(48\mu) + \psi_1\alpha h^2/12 = 0$$
$$(1 \leftrightarrow 2, x \leftrightarrow y). \tag{1.5}$$

It assumes that $Z_2 = q$ is the uniformly distributed load and, according to designations (1.3), $A_2 = A_1 - A_3$.

We have the following relationships [25] for the internal forces and moments:

$$N_{xz} = \psi_1\alpha h^3/12,$$
$$M_{xx} = -D(w_{xx} + \nu w_{yy}) + (\psi_{1x} + \nu\psi_{2y})D\alpha h^2/(20\mu),$$
$$H_{xy} = -w_{xy}2\mu h^3/12 + (\psi_{2x} + \bar{\eta}\psi_{1y})\alpha h^5(\mu+\alpha)/(240\mu), \tag{1.6}$$
$$\bar{\eta} = (\mu-\alpha)/(\mu+\alpha),$$

for the moments due to the moment stresses:

$$P_{xx} = 2\gamma h w_{xy} - ((2\gamma + \beta)\psi_{2x} - \beta\psi_{1y}) h^3 (\mu + \alpha)/(48\mu),$$
$$R_{xy} = -(\gamma + \varepsilon) h(w_{xx} - \eta w_{yy}) + (\psi_{1x} - \eta\psi_{2y}) h^2 (\mu + \alpha)(\gamma + \varepsilon)/(48\mu), \quad (1.7)$$
$$\eta = (\gamma - \varepsilon)/(\gamma + \varepsilon),$$

for two rotation tensor components ω_1 and ω_2

$$\omega_1 = w_y - (h^2/4 - z^2)(\mu + \alpha)\psi_2/(8\mu). \quad (1.8)$$

All formulas (1.6)–(1.8) have to be completed by the relationships that obey the rule $(1 \leftrightarrow 2, x \leftrightarrow y)$.

2. Solution. The equilibrium equations (1.4) and (1.5), characterizing the bending of a micropolar plate, can be solved using the earlier proposed technique [12] introducing the function F:

$$\psi_1 = -12A\Delta F_x/(\alpha h^2) \ (1 \leftrightarrow 2, x \leftrightarrow y), \quad w = F - A_1\Delta F/(4\mu\alpha). \quad (2.1)$$

Then, one has the identical satisfaction of Eqs. (1.5), and Eq. (1.4) results in:

$$D_1\Delta\Delta F = Z_2 \quad \text{and} \quad D_1 = D + (\varepsilon + \gamma) h. \quad (2.2)$$

Taking relationships (2.1) into account and based on Eqs. (1.6) and (1.7), the internal forces and moments are defined as follows (in the relationships (2.3) below $x \leftrightarrow y$):

$$N_{xz} = -D_1\Delta F_x,$$
$$M_{xx} = -D(F_{xx} + \nu F_{yy}) + D(\gamma + \varepsilon)(5\mu - 7\alpha)(\Delta F_{xx} + \nu\Delta F_{yy})/(20\mu\alpha),$$
$$H_{xy} = D(1 - \nu) F_{xy} + h^3 (\gamma + \varepsilon)(5\mu - 7\alpha)\Delta F_{xy}/(120\alpha), \quad (2.3)$$
$$P_{xx} = 2\gamma h F_{xy} + D(5\mu - 7\alpha)\Delta F_{xy}/(5h),$$
$$R_{xy} = -(\gamma + \varepsilon) h(F_{xx} - \eta F_{yy}) - D(\gamma + \varepsilon)(5\mu - 7\alpha)(\Delta F_{xx} - \eta\Delta F_{yy})/(20\mu\alpha).$$

Using relationships (1.8) and designations (1.3), the rotation tensor components ω_1 and ω_2 at $z = 0$ can be found as:

$$\omega_1 = F_y + ((\mu + \alpha)(\gamma + \varepsilon)/(8\mu\alpha) - 3D(3\alpha - 5\mu)/(40h\mu\alpha))\Delta F_y \ (1 \leftrightarrow 2, x \leftrightarrow y). \quad (2.4)$$

Hence, the problem of transversal bending of a plate is reduced to the integration of the differential equation (2.2) with the corresponding boundary conditions. Knowing the solution of this equation, the bending moments $(M_{xx} + R_{xy}, M_{yy} + R_{yx})$ and torques $(H_{xy} + P_{xx}, H_{yx} + P_{yy})$, as well as the transversal forces (N_{zx}, N_{yz}) can be calculated from relationships (2.3).

Introducing the designations below:

$$M_x = M_{xx} + R_{xy}, \quad M_{xy} = H_{xy} + P_{xx}, \quad (x \leftrightarrow y),$$

and accounting for relationships (2.3), one finds:

$$M_x = -D_1F_{xx} - D_2F_{yy} + D_3\Delta F_{yy}, \quad M_{xy} = D_4F_{xy} + (D_5 + D_6)\Delta F_{xy} \ (x \leftrightarrow y), \quad (2.5)$$

$$\omega_1 = F_y + D_7\Delta F_y \quad (x \leftrightarrow y) \quad (2.6)$$

with the following designations of coefficients with elastic constants of the plate material:

$$D_2 = D\nu + (\varepsilon - \gamma) h, \quad D_3 = D(\gamma + \varepsilon)(5\mu - 7\alpha)(\nu + \eta)/(20\mu\alpha),$$
$$D_4 = D(1 - \nu) - 2h\gamma, \quad D_5 = (5\mu - 7\alpha) h^3 (\gamma + \varepsilon)/(120\alpha), \quad (2.7)$$
$$D_6 = (5\mu - 7\alpha) D/5h, \quad D_7 = (\mu + \alpha)(\gamma + \varepsilon)/(8\mu\alpha) - 3D(3\alpha - 5\mu)/(40h\mu\alpha).$$

Substituting the transversal forces and moments in the expressions of shear force components

$$N_x = N_{xz} - M_{xyy} \quad (x \leftrightarrow y) \quad (2.8)$$

one obtains the transversal forces and torques:

$$N_x = -D_1\Delta F_x - (D_4 + D_5 - D_6)\Delta F_{xyy} \quad (x \leftrightarrow y). \quad (2.9)$$

Consider a long rectangular plate that is exposed to the distributed load ($Z_2 = q$). The plate is assumed to be uniformly fastened at the long sides, whereas the short sizes are fixed arbitrarily (Fig. 1). Placing the

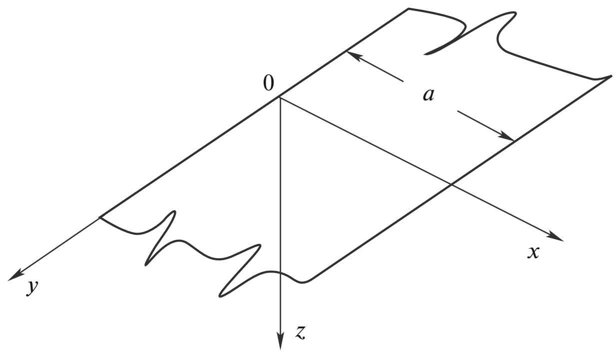

Fig. 1.

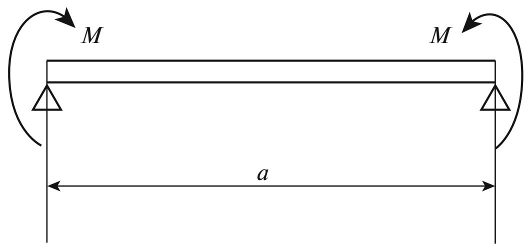

Fig. 2.

origin on the long side far from the short sides of the plate and directing the y axis along the long side, it implies that sought values for the bending plate (w, ψ_1, ψ_2) depend only on the x-coordinate. Then, Eqs. (2.2) yield:

$$F = \frac{q}{24D_1}x^4 + \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3x + C_4, \tag{2.10}$$

where $C_1, \ldots, C_4$ are the integral constants.

Consider some types of boundary conditions, introducing the designations below:

$$\tilde{A}_1 = A_1/(4\mu\alpha), \quad \tilde{q} = q/(24D_1).$$

The plate edges are firmly clamped. When the plate edges are firmly clamped, deflections and the corresponding rotation angles around the contour lines of edges are equal to zero, giving the following boundary conditions:

$$x = 0, \quad a: w = 0, \quad \omega_2 = 0, \quad \text{or} \quad F - \tilde{A}_1F_{xx} = F_x + D_7F_{xxx} = 0.$$

Finding the integrations constants results in:

$$w = [x^3 - 2ax^2 + (a^2 - 12D_7 - 12\tilde{A}_1)x + 12a(D_7 + \tilde{A}_1)]\tilde{q}x. \tag{2.11}$$

The plate edges are pivotally supported. In this case, deflections and bending moments along these edges are zero, allowing the boundary conditions to be written as

$$x = 0, \quad a: w = M_x = 0, \quad \text{or} \quad F - \tilde{A}_1F_{xx} = F_{xx} = 0.$$

Hence, finding the integration constants yields:

$$w = [x^2(x - 2a) + a^3 + 12\tilde{A}_1(a - x)]\tilde{q}x. \tag{2.12}$$

Pure bend of plates. In this case, the long sides of a pivotally supported plate are loaded by uniformly distributed moments with intensity M (Fig. 2). Thus, the boundary conditions can be written as

$$x = 0, \quad a: w = 0, \quad M_x = M.$$

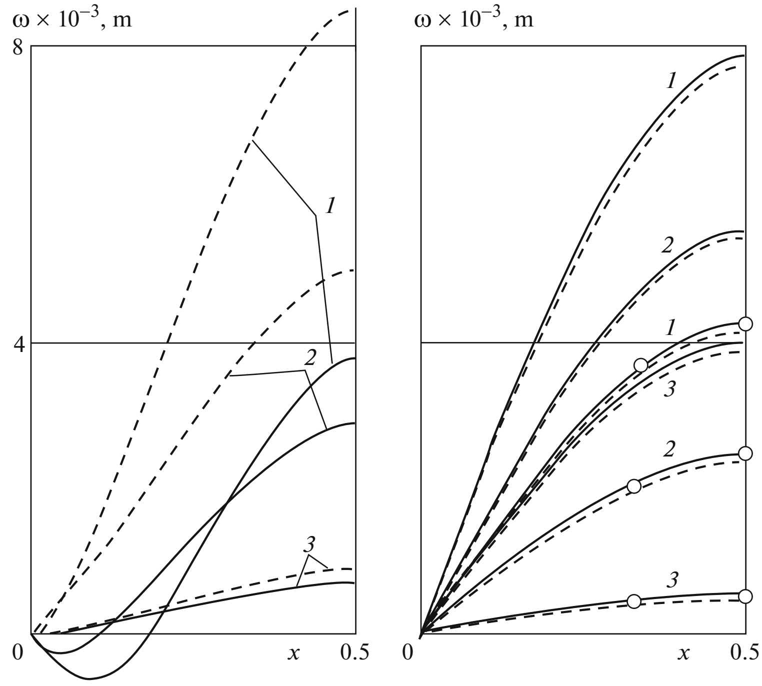

Fig. 3.

Hence, we obtain:

$$w = (x - 2a)\tilde{q}x^3 + Mx^2/2 + (\tilde{q}a^3 - Ma/2)x + 12\tilde{A}_1\tilde{q}x(a - x). \quad (2.13)$$

The classical theory is associated with expressions (2.11)–(2.13) at $\tilde{A}_1 = 0$, $D_7 = 0$ and $\tilde{q} = q/(24D)$.

3. Discussion of results. We consider graphs of the displacement functions $w(x)$ in the micropolar and classical theories of plate bending. We consider various materials whose micropolar parameters are described by Lakes [20, 22]. The micropolar constants can be found as follows:

$$\alpha = \frac{N^2\gamma}{2l_b^2(1 - N^2)}, \quad \beta = 2\mu(l_t^2 - 2l_b^2), \quad \gamma = 4l_b^2\mu, \quad \varepsilon = \frac{(\beta + \gamma)(1 - \Psi)}{\Psi}, \quad (3.1)$$

where l_b is the characteristic torsional width, l_t is the characteristic length for bending, N is the moment number, and Ψ is the polar coefficient. All these parameters are evaluated experimentally, and their values are listed in Table 1 with the numbers of the corresponding curves in Fig. 3.

Figure 3 illustrates the displacement w as a function of the x-coordinate in accordance with the micropolar (solid curves) and classical (dashed curves) theories of the bending of plates with the parameters below:

$$h = 0.1\,\text{m}, \quad a = 1\,\text{m}, \quad q = 0.1\,\text{MPa/m}, \quad M = -0.25 \times 10^{-6}\,\text{MPa/m}.$$

Table 1

Number of curve	Material	E, MPa	μ, MPa	ν	α, MPa	$\beta \times 10^5$, MN	$\gamma \times 10^5$, MN	$\varepsilon \times 10^5$, MN
1	Polyurethane	300	104	0.40	–8.66	3.00	4.00	–2.00
2	Poly (methacrylamide) foam	637	285	0.12	–23.75	–30.00	60.00	–10.00
3	Artificial foam	2758	1033	0.34	–229.55	0.45	0.42	–0.29

The left-hand part of Fig. 3 refers to a case of clamped edges and the right-hand part is associated with a pivotally supported plate, where curves, attributed to pure bending, are shown with a light-point marker. Due to the symmetry of curves relative to the axis $x = 0.5$, Fig. 3 is limited by only a range of $0 \leq x \leq 0.5$.

4. Conclusions. The above-proposed approach for studying the bending of plates greatly simplifies the solution. In these problems, the results are found to be consistent with those reported in the literature [27]. The practical divergence of deflections in the classical and micropolar theories of the bending of plates, given in Fig. 3, are likely due to the influence of moment stresses on the displacement function. The solution algorithm, developed in the present work, is suitable for the design of programming tools.

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