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This work is dedicated to the memory of V. V. Beletskii

Evolution of the Rotational Movement of a Dynamically Symmetric Satellite with Inner Damping in a Circular Orbit

N. I. Amel'kin[a,]* and V. V. Kholoshchak[a,]**

[a] *Moscow Institute of Physics and Technology, Dolgoprudny, 141701 Russia*

**e-mail: namelkin@mail.ru*

***e-mail: khoviktoriya@yandex.ru*

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Abstract—In this paper, we studied the effect of internal dissipation on the rotational motion of a satellite in a central gravitational field using the Lavrent'ev model. Evolution equations are derived, and the results of an evolution analysis of the rotational motion of a dynamically symmetric satellite moving in a Keplerian circular orbit depending on the parameter values and initial conditions are presented.

Keywords: satellite, central field, circular orbit, stationary rotations, stability, evolution of rotational motion

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In the past, the effect of internal dissipative forces on the rotational motion of a satellite was considered in many studies. One of three satellite models was mostly used to simulate internal dissipation: (1) a solid body with a cavity filled with a viscous fluid [1–3], (2) a solid body with a spherical damper (Lavrent'ev model) [4–6], and (3) a viscoelastic body [7, 8]. For a dynamically symmetric satellite in a circular orbit, the evolution of rotational motion was earlier studied in terms of model 1 for the case of a highly viscous fluid and large values of the reduced angular velocity of the satellite [3]. Below, the evolution of satellite rotations is studied using the Lavrent'ev model and in a much wider range of parameters and satellite angular velocities compared with the previous study [3].

1. STABILITY ANALYSIS OF STATIONARY ROTATIONS OF A SATELLITE CLOSE TO SPHERICALLY SYMMETRIC

The rotational motion of a satellite with a spherical damper in the central gravitational field in a circular orbit can be described by a system of equations [6]

$$(\mathbf{J} - I\,\mathbf{E})\dot{\mathbf{U}} + \mathbf{U}\times\mathbf{J}\mathbf{U} = 3\mathbf{r}\times\mathbf{J}\mathbf{r} + \mu I(\mathbf{V} - \mathbf{U}),$$
$$\dot{\mathbf{V}} + \mathbf{U}\times\mathbf{V} = -\mu(\mathbf{V} - \mathbf{U}), \qquad (1.1)$$
$$2\dot{\boldsymbol{\Lambda}} = \boldsymbol{\Lambda}\circ\mathbf{U}.$$

Here, $\mathbf{J}$ is the central inertia tensor of the entire satellite, I is the moment of inertia of the damper relative to its central axis, $\mathbf{E}$ is the unit matrix, $\mathbf{r} = \mathbf{R}/R$ is the unit vector codirected with the radius-vector of the satellite mass center, $\mathbf{U} = \boldsymbol{\omega}/\omega_0$, $\mathbf{V} = \boldsymbol{\Omega}/\omega_0$, where $\boldsymbol{\omega}$ is the absolute angular velocity of the shell, $\boldsymbol{\Omega}$ is the absolute angular velocity of the damper, ω_0 is the angular velocity of the orbital basis directed along the normal $\mathbf{n}$ to the orbit plane, μ is the dimensionless damping coefficient, and $\boldsymbol{\Lambda}$ is the quaternion of the unit norm that specifies the position of the shield-related basis of the main axes $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of satellite inertia relative to the König basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$. The dot denotes the derivative with respect to dimensionless time $\tau = \omega_0 t$. In Eqs. (1.1), all vectors are defined by their components in the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

For a dynamically symmetric satellite, the motion relative to the orbital basis formed by the vectors $\mathbf{r}$, $\boldsymbol{\tau} = \mathbf{n} \times \mathbf{r}$, and $\mathbf{n}$ is described by the following autonomous system of equations [6]:

$$\begin{gathered}\dot{\mathbf{e}} = \mathbf{u} \times \mathbf{e}, \\ (A - I)(\mathbf{n} \times \mathbf{u} + \dot{\mathbf{u}}) + (C - A)[((\mathbf{n} \times \mathbf{u} + \dot{\mathbf{u}}) \cdot \mathbf{e})\mathbf{e} + ((\mathbf{n} + \mathbf{u}) \cdot \mathbf{e})(\mathbf{n} + \mathbf{u}) \times \mathbf{e}] \\ = \mu I(\mathbf{v} - \mathbf{u}) + 3(C - A)(\mathbf{r} \cdot \mathbf{e})(\mathbf{r} \times \mathbf{e}), \\ \mathbf{n} \times \mathbf{v} + \dot{\mathbf{v}} = -\mu(\mathbf{v} - \mathbf{u}).\end{gathered} \tag{1.2}$$

Here, $\mathbf{u} = (\boldsymbol{\omega} - \boldsymbol{\omega}_0)/\omega_0$ and $\mathbf{v} = (\boldsymbol{\Omega} - \boldsymbol{\omega}_0)/\omega_0$ are the reduced angular velocity of the carrier body (shell) and the reduced angular velocity of the damper relative to the orbital basis, $\mathbf{e}$ is the satellite symmetry axis, C and A are the axial and equatorial satellite moments of inertia.

The limiting movements of a satellite were shown to be only equilibrium positions with respect to the orbital basis and stationary rotations around the symmetry axis aligned with the normal to the orbital plane (cylindrical regular precessions) [6]:

$$\mathbf{e}^* = \mathbf{n}, \quad \mathbf{v}^* = \mathbf{u}^* = u\,\mathbf{n}; \quad u \in (-\infty, +\infty). \tag{1.3}$$

The problem of the motion stability (1.3) is reduced to the study of the stability of the characteristic polynomial of the system obtained by linearizing Eqs. (1.2) in the vicinity of solutions (1.3). This polynomial has the following form [6]:

$$P(\lambda) = a_0\lambda^6 + a_1\lambda^5 + a_2\lambda^4 + a_3\lambda^3 + a_4\lambda^2 + a_5\lambda + a_6 \tag{1.4}$$

and its coefficients are determined by the expressions:

$$\begin{gathered}a_0 = 1, \quad a_1 = 2m, \quad a_2 = [2 + k^2 + m^2] + 3\varepsilon, \\ a_3 = 2m[1 + k^2] + \varepsilon[6m + \mu\gamma(2\beta(k+1) - 3)], \\ a_4 = [k^2 + (m^2 + 1)(1 + k^2)] + \varepsilon[3(m^2 + 1) - 3k + \mu\gamma m(2\beta k - 3)] + \varepsilon^2\beta^2(\mu\gamma)^2, \\ a_5 = 2mk^2 + \varepsilon\{-6mk + \mu\gamma[2\beta(k+1) + 3(2k+1)]\}, \\ a_6 = (m^2 + 1)k^2 - \varepsilon[3(m^2 + 1) - \mu\gamma m(2\beta + 3)]k + \varepsilon^2\mu\gamma[\beta^2\mu\gamma - 3\beta(m - \mu\gamma)],\end{gathered} \tag{1.5}$$

where the following notation is used:

$$\begin{gathered}\alpha = (C - I)/(A - I), \quad \gamma = I/(A - I), \quad \beta = u + 1, \\ m = \mu(1 + \gamma), \quad k = 1 - \beta(1 + \varepsilon), \quad \varepsilon = \alpha - 1; \quad \varepsilon \in [-1, 1].\end{gathered} \tag{1.6}$$

Here, $\alpha \in [0, 2]$ is the "flattening" factor of the auxiliary body formed by the shell and a point mass equal to the damper mass and located at its center, $\gamma \in [0, \infty)$ is the ratio of the damper inertia moment to the equatorial inertia moment of the auxiliary body, and β is the ratio of the absolute angular velocity of the satellite stationary rotation to the angular velocity of the orbital basis.

Note that the flattening factor of the entire satellite is determined by the following expression:

$$\alpha^* = C/A = (\alpha + \gamma)/(1 + \gamma). \tag{1.7}$$

If $\alpha > 1$, then $\alpha^* > 1$ (oblate satellite) while $\alpha^* < \alpha$. If $\alpha < 1$, then $\alpha^* < 1$ (prolate satellite) while $\alpha^* > \alpha$.

An analytical study of the stability conditions of the polynomial (1.4) was earlier performed for the values of $\mu \ll 1$ and $\mu \gg 1$ [6]. For the remaining values of the μ parameter, the roots of the polynomial (1.4) at γ-values comparable with unity were numerically analyzed.

In this section, we perform an analytical study of the stability conditions of stationary rotations (1.3) for an oblate satellite close to spherically symmetric in the entire range of μ and γ parameter values:

$$\varepsilon > 0, \quad \varepsilon \ll 1. \tag{1.8}$$

According to the Routh–Hurwitz criterion in the Liénard–Chipart form, the stability conditions are described by a system of inequalities:

$$a_k > 0; \quad k = 1, \ldots, 6, \quad \Delta_3 > 0, \quad \Delta_5 > 0, \tag{1.9}$$

where Δ_3 and Δ_5 are the Hurwitz matrix minors of the third and fifth order. As follows from formulas (1.5) and (1.6), the coefficients a_1, a_2, a_3, a_4 will be positive at sufficiently small ε-values for any values of $m > 0$

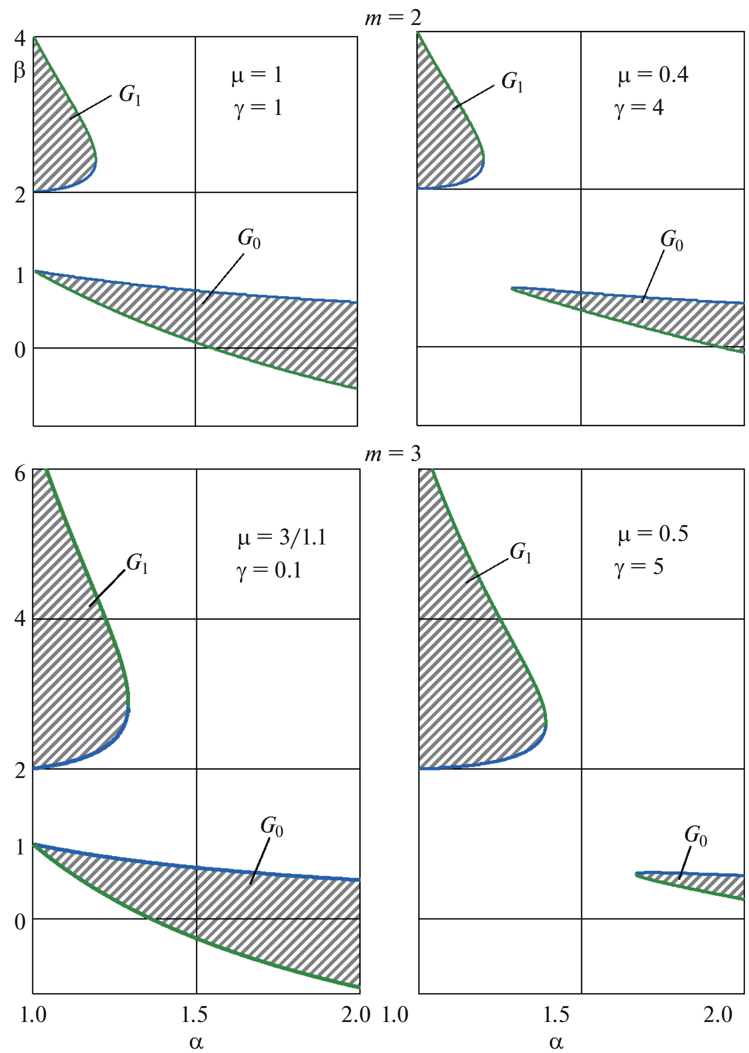

Fig. 1.

and β. Taking into account relations (1.6), the coefficient a_5 is written as a second-degree polynomial with respect to k as follows:

$$a_5 = \mu[2(1+\gamma+\varepsilon)k^2 - 6\varepsilon(1+\varepsilon)k + \varepsilon\gamma(5+3\varepsilon)]/(1+\varepsilon). \tag{1.10}$$

At $k = 0$ and $\varepsilon > 0$, we have $a_5 > 0$ while the polynomial, as it is easy to see, has no real roots at sufficiently small ε-values. Therefore, $a_5 > 0$ at $\varepsilon \ll 1$.

The coefficient a_6 is also expressed by a second-degree polynomial relative to k. Leaving only the principal terms in the coefficients of this polynomial, we obtain the following:

$$a_6 = (m^2+1)k^2 - \varepsilon(3 - 2m^2 + 5\mu m)k + \varepsilon^2(m-\mu)[(m-4\mu)]. \tag{1.11}$$

The discriminant of the polynomial is written as follows:

$$D = [3(1+\mu m) + 4\mu\gamma)][3(1+\mu m) - 4\mu\gamma].$$

If the condition

$$3(1+\mu m) - 4\mu\gamma > 0 \tag{1.12}$$

is satisfied, then the polynomial has two real roots:

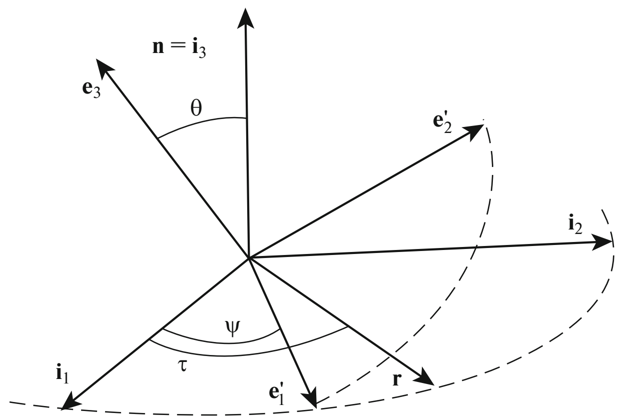

Fig. 2.

$$k_{1,2} = \frac{\varepsilon(3 - 2m^2 + 5\mu m \pm \sqrt{D})}{2(m^2+1)},$$

which correspond to two curves on the plane ε, β:

$$\beta_{1,2}(\varepsilon) = \frac{1 - k_{1,2}}{1+\varepsilon} \approx 1 - \varepsilon\left(1 + \frac{3 - 2m^2 + 5\mu m \pm \sqrt{D}}{2(m^2+1)}\right)$$

that intersect at a point (0, 1). In the range of $\beta_1 < \beta < \beta_2$, we have $a_6 < 0$, i.e., stationary rotations are unstable. At $\beta < \beta_1$ and $\beta > \beta_2$, the coefficient $a_6 > 0$. Curves $\beta_1(\varepsilon)$ and $\beta_2(\varepsilon)$ limit the instability region G_0 in the plane of parameters α and β, as shown in Fig. 1 on the left.

If inequality (1.12) has the opposite sign, which occurs at simultaneous fulfillment of the conditions

$$\mu < \frac{4}{3}, \quad \gamma > \frac{3(1+\mu^2)}{(4-3\mu)\mu}, \tag{1.13}$$

then $a_6 > 0$ at $\varepsilon \ll 1$. In this case, the instability region G_0 "detaches" from the axis $\varepsilon = 0$ and has the form shown in Fig. 1 on the right.

The minor Δ_3 is determined by the following expression:

$$\Delta_3 = 2\mu\gamma m[3 + 2(1-k)^2][(1+k)^2 + m^2]\varepsilon + O(\varepsilon^2) \tag{1.14}$$

and takes positive values at $\mu\gamma > 0$ and $\varepsilon \ll 1$.

The minor Δ_5 is written as follows:

$$\Delta_5 = 18\mu^3\gamma^2[3 + 2(1-k)^2][(1+k)^2 + m^2](m^2 + 2 + 2k)(1+k)\varepsilon^3 + O(\varepsilon^4). \tag{1.15}$$

At $\varepsilon \ll 1$, it takes positive values if $\beta < 2$ or $\beta > (m^2+4)/2$. In the range of

$$2 < \beta < (m^2+4)/2 = \beta^*, \tag{1.16}$$

stationary rotations are unstable. The points (1, 2) and (1, β*) belong to the curve limiting the instability region G_1 in the plane of the parameters α and β (Fig. 1).

Thus, for an oblate dynamically symmetric satellite close to spherically symmetric, the stability conditions for stationary rotations are determined by the value of a single parameter $m = \mu(\gamma+1)$. The only exception is a narrow range of rotations at an angular velocity close to the angular velocity of the orbital basis, for which the nature of stability is determined by the specific combinations of the two parameters μ and γ.

Figure 1 presents a system of diagrams of asymptotic stability (not shaded) and instability (shaded) domains obtained by a numerical study of the characteristic equation roots in the interval of $1 < \alpha < 2$ on the plane of parameters α and β at $m = 2$ and $m = 3$, where two different combinations of parameters μ and γ correspond to each of the indicated m-values. These diagrams, as well as the results of other calculations [6], completely confirm the above conclusions about the nature of the stability of stationary rotations of a satellite close to spherically symmetric. The linear dimensions of the region G_1 are proportional to m^2 while the dimension of the region G_1 along the α axis weakly depends on the specific combination of parameters μ and γ, as follows from the diagrams. It was also established that there is a value of γ^* at $\mu < 4/3$ (1.13) such that the instability region G_0 completely disappears in the interval of $1 < \alpha < 2$ at $\gamma > \gamma^*$.

2. EVOLUTION EQUATIONS

In an analytical study of the evolution of the rotational motion of a dynamically symmetric satellite, the motion equations are written in projections on the axes of the Resal base $\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3$ ($\mathbf{e}_3$ is the satellite symmetry axis) specified by angles ψ and θ (Fig. 2). Denoting a vector of the relative angular velocity of the damper as $\mathbf{W} = \mathbf{V} - \mathbf{U}$, we obtain the following equations:

$$\begin{gathered}(\mathbf{J} - I\,\mathbf{E})\dot{\mathbf{U}} + \mathbf{u}'\times(\mathbf{J} - I\,\mathbf{E})\mathbf{U} = 3\mathbf{r}\times\mathbf{J}\mathbf{r} + \mu I\,\mathbf{W},\\ \dot{\mathbf{U}} + \dot{\mathbf{W}} + \mathbf{u}'\times(\mathbf{U} + \mathbf{W}) = -\mu\,\mathbf{W},\end{gathered} \tag{2.1}$$

where $\mathbf{u}' = \boldsymbol{\omega}'/\omega_0$ is the reduced angular velocity of the Resal basis while all the vectors are specified by their components in the Resal basis. Accounting for the equality

$$\mathbf{u}' = \mathbf{e}_1'\dot{\theta} + \dot{\psi}(\mathbf{e}_2'\sin\theta + \mathbf{e}_3\cos\theta) = \mathbf{U} - \dot{\phi}\mathbf{e}_3, \tag{2.2}$$

where ϕ is the eigen-rotation angle, we obtain the following:

$$\dot{\theta} = U_1, \quad \dot{\psi}\sin\theta = U_2, \quad \dot{\phi} = U_3 - U_2\cot\theta, \tag{2.3}$$

$$\mathbf{u}' = U_1\mathbf{e}_1' + U_2\mathbf{e}_2' + U_2\cot\theta\,\mathbf{e}_3. \tag{2.4}$$

The gravitational moment acting on the satellite is determined by the expression:

$$\mathbf{m}_g = 3\mathbf{r}\times\mathbf{J}\mathbf{r} = \frac{3}{2}(C - A)\sin\theta[\mathbf{e}_2'\sin 2(\tau - \psi) + \mathbf{e}_1'\cos\theta(\cos 2(\tau - \psi) - 1)]. \tag{2.5}$$

Projecting equations (2.1) on the axes of the Resal basis, we obtain with allowance for relations (2.3), (2.4), and (1.6) the following closed system of eight equations:

$$\begin{gathered}\dot{U}_1 = -(1 + \varepsilon)U_3U_2 + U_2^2\cot\theta + \mu\gamma W_1 + F_1[\cos 2(\tau - \psi) - 1],\\ \dot{U}_2 = (1 + \varepsilon)U_3U_1 - U_2U_1\cot\theta + \mu\gamma W_2 + F_2\sin 2(\tau - \psi),\\ \dot{W}_1 = \varepsilon U_2U_3 + U_2W_2\cot\theta - U_2W_3 - \mu(1 + \gamma)W_1 - F_1[\cos 2(\tau - \psi) - 1],\\ \dot{W}_2 = -\varepsilon U_1U_3 + U_1W_3 - U_2W_1\cot\theta - \mu(1 + \gamma)W_2 - F_2\sin 2(\tau - \psi),\\ \dot{W}_3 = U_2W_1 - U_1W_2 - \frac{\mu(1 + \gamma + \varepsilon)}{1 + \varepsilon}W_3,\\ \dot{U} = \frac{\mu\gamma W_3}{1 + \varepsilon}, \quad \dot{\theta} = U_1, \quad \dot{\psi}\sin\theta = U_2,\end{gathered} \tag{2.6}$$

Functions F_1 and F_2 are determined by the following formulas:

$$F_1 = \frac{3\varepsilon\sin 2\theta}{4} \quad \text{and} \quad F_2 = \frac{3\varepsilon\sin\theta}{2} = \frac{F_1}{\cos\theta}. \tag{2.7}$$

We will further consider the oblate satellite ($\alpha > 1$) close to spherically symmetric, i.e., we will assume that $\varepsilon \ll 1$ (small parameter). An analysis of equations (2.6) and the results of numerical integration of equations (1.1) showed that, at various initial values of the angular velocity of the shell and damper, a relatively fast transient process (rapid evolution) is observed at $m = \mu(1 + \gamma) \geq \sqrt{\varepsilon}$, at the end of which the movement is set close to the rotation of the satellite as a single solid body around the symmetry axis codirected with the initial value of the satellite kinetic moment vector. Then, there is a slow evolution due to the action of the gravitational and dissipative moments. In this case, the variables U_k, W_k, and θ in the slow evolution mode, on average, change slowly and have harmonic components with the frequency close to the value of two while the average values of the variables U_1, U_2, W_1, W_2, W_3 and their harmonic components are limited by functions of ε.

In the evolution problem of satellite rotational motion, the main focus of interest is on the behavior of the satellite rotation axis and the magnitude of the angular velocity. Assuming that the oblate satellite movement in the slow evolution mode is close to its rotation around the symmetry axis, the evolution analysis is reduced to the study of the behavior of phase variables U_3, θ, and ψ.

The presence of a small parameter in Eqs. (2.6) provides a reason to apply the averaging method in order to obtain evolutionary equations. However, the "classical" scheme of the averaging method [9, 10] for system (2.6) cannot be directly used since reducing the system (2.6) to the standard form is problematic. Below, the variant of the averaging method is applied for the problem under consideration without reducing the system (2.6) to the standard form.

Let us introduce the following notation for phase variables:

$$\mathbf{x}\colon \mathbf{x}^T = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (U_1, U_2, W_1, W_2, U_3, W_3, \theta, \psi). \tag{2.8}$$

We rewrite system (2.6) as follows:

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}, \tau) = \mathbf{L}(\mathbf{x}) + \mathbf{G}(\mathbf{x}) + \tilde{\mathbf{X}}(\mathbf{x}, \tau), \tag{2.9}$$

where $\mathbf{L}(\mathbf{x})$ are the linear terms in the variables U_1, U_2, W_1, W_2, W_3, $\tilde{\mathbf{X}}(\mathbf{x}, \tau)$ is the explicitly time dependent term (in the considered problem, it is a harmonic function of time), and $\mathbf{G}(\mathbf{x})$ are the remaining terms in the right-hand side of system (2.6).

The solution will be sought in the following form:

$$\mathbf{x} = \mathbf{y} + \mathbf{S}(\mathbf{y}, \tau), \tag{2.10}$$

where the components of the function $\mathbf{S}(\mathbf{y}, \tau)$ are selected from the following conditions: if $\tilde{X}_j = 0$, then $S_j = 0$, and if $\tilde{X}_k \neq 0$, then S_k satisfies the equation

$$\frac{\partial S_k}{\partial \tau} = L_k(\mathbf{S}) + \tilde{X}_k(\mathbf{y}, \tau), \tag{2.11}$$

and when taken into account, we obtain the following equations after substituting expression (2.10) into system (2.6):

$$\begin{gathered} \dot{y}_k + \frac{\partial S_k}{\partial \mathbf{y}^T}\dot{\mathbf{y}} = L_k(\mathbf{y}) + G_k(\mathbf{y} + \mathbf{S}) + \tilde{X}_k(\mathbf{y} + \mathbf{S}, \tau) - \tilde{X}_k(\mathbf{y}, \tau); \quad k = 1,2,3,4, \\ \dot{y}_j = \dot{x}_j = X_j(\mathbf{y} + \mathbf{S}); \quad j = 5,6,7,8. \end{gathered} \tag{2.12}$$

Let us find the components of the function $\mathbf{S}(\mathbf{y}, \tau)$. We have

$$S_5 = S_6 = S_7 = S_8 = 0 \Rightarrow y_5 = U_3, \quad y_6 = W_3, \quad y_7 = \theta, \quad y_8 = \psi. \tag{2.13}$$

The remaining components are determined from the following system (hereinafter, $U = U_3$ and $m = \mu(1+\gamma)$):

$$\begin{aligned} \frac{\partial S_1}{\partial \tau} &= -(1+\varepsilon)US_2 + \mu\gamma S_3 + F_1\cos 2(\tau - \psi), \\ \frac{\partial S_2}{\partial \tau} &= (1+\varepsilon)US_1 + \mu\gamma S_4 + F_2\sin 2(\tau - \psi), \\ \frac{\partial S_3}{\partial \tau} &= \varepsilon US_2 - mS_3 - F_1\cos 2(\tau - \psi), \\ \frac{\partial S_4}{\partial \tau} &= -\varepsilon US_1 - mS_4 - F_2\sin 2(\tau - \psi). \end{aligned} \tag{2.14}$$

The solution of this system is expressed through time harmonic functions:

$$S_k = p_k\sin 2(\tau - \psi) + q_k\cos 2(\tau - \psi); \quad k = 1,2,3,4. \tag{2.15}$$

The coefficients are determined with accuracy up to $O(\varepsilon^2)$ by the following formulas:

$$p_1 = (4+\mu m)f_{12}, \quad q_1 = 2\mu\gamma f_{12}, \quad p_2 = 2\mu\gamma f_{21},$$
$$q_2 = -(4+\mu m)f_{21}; \quad f_{ij} = \frac{2F_i + UF_j}{(4+m^2)(4-U^2)}, \tag{2.16}$$
$$p_3 = \frac{-2F_1}{4+m^2}, \quad q_3 = \frac{-mF_1}{4+m^2}, \quad p_4 = \frac{-mF_2}{4+m^2}, \quad q_4 = \frac{2F_2}{4+m^2}.$$

The function $\mathbf{S}$ depends on the variables ψ, θ, and U only. Therefore, equations (2.12) take the following form:

$$\dot{y}_k = -\frac{\partial S_k}{\partial\theta}(y_1+S_1) - \frac{\partial S_k}{\partial\psi}\frac{y_2+S_2}{\sin\theta} - \frac{\partial S_k}{\partial U}\frac{\mu\gamma}{(1+\varepsilon)}W_3 + L_k(\mathbf{y}) + G_k(\mathbf{y}+\mathbf{S}); \quad k = 1,2,3,4,$$
$$\dot{W}_3 = (y_2+S_2)(y_3+S_3) - (y_1+S_1)(y_4+S_4) - \frac{\mu(1+\gamma+\varepsilon)}{1+\varepsilon}W_3, \tag{2.17}$$
$$\dot{U} = \frac{\mu\gamma}{1+\varepsilon}W_3, \quad \dot{\theta} = y_1 + S_1, \quad \dot{\psi} = \frac{y_2+S_2}{\sin\theta}.$$

A detailed analysis of the resulting system showed that variables y_2 and y_3 are the bounded functions of ε and variables y_1, y_4, and W_3 are the bounded functions of ε^2 in the slow evolution mode. In this case, for the first five equations, the average values of the right-hand sides calculated by virtue of the motion equations coincide with the accuracy of $O(\varepsilon^3)$ with the averages over the explicitly incoming time. Considering this, as well as the formulas derived from expressions (2.15) (the time average is indicated by angle brackets, and the prime is the derivatives with respect to the variable θ)

$$\left\langle \frac{\partial S_k}{\partial\theta}S_1 \right\rangle = \frac{p_k' p_1 + q_k' q_1}{2} + O(\varepsilon^3) \quad \text{and} \quad \left\langle \frac{\partial S_k}{\partial\psi}S_2 \right\rangle = p_2 q_k - q_2 p_k + O(\varepsilon^3), \tag{2.18}$$

we obtain the following equations for the average values $\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4, \bar{W}_3$ of the variables y_1, y_2, y_3, y_4, W_3:

$$\dot{\bar{y}}_1 = -(1+\varepsilon)U\bar{y}_2 + \mu\gamma\bar{y}_3 - F_1 + O(\varepsilon^2), \quad \dot{\bar{y}}_3 = \varepsilon U\bar{y}_2 - m\bar{y}_3 + F_1 + O(\varepsilon^2),$$
$$\dot{\bar{y}}_2 = U\bar{y}_1 + \mu\gamma\bar{y}_4 - \cot\theta(p_2p_1 + q_2q_1)/2 - (p_2'p_1 + q_2'q_1)/2,$$
$$\dot{\bar{y}}_4 = -m\bar{y}_4 - \cot\theta(2\bar{y}_2\bar{y}_3 + p_2p_3 + q_2q_3)/2 - (p_4'p_1 + q_4'q_1)/2 - (p_2q_4 - q_2p_4)/\sin\theta, \tag{2.19}$$
$$\dot{\bar{W}}_3 = -m\bar{W}_3 + \bar{y}_2\bar{y}_3 + (p_2p_3 + q_2q_3 - p_1p_4 - q_1q_4)/2.$$

In the last three equations of system (2.19), the right-hand sides are written with an accuracy of $O(\varepsilon^3)$.

Let us find the values of variables $\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4, \bar{W}_3$ in the slow evolution mode from Eqs. (2.19) by setting the time derivatives of these variables to be zero. Accounting for relations (2.7), (2.15), and (2.16), the average values of the variables y_2 and y_3 are determined with an accuracy of $O(\varepsilon^2)$ from the first two equations of the system by the following formulas:

$$\bar{y}_2 = \frac{-F_1}{(1+\gamma+\varepsilon)U} + O(\varepsilon^2) \quad \text{and} \quad \bar{y}_3 = \frac{F_1}{\mu(1+\gamma+\varepsilon)} + O(\varepsilon^2). \tag{2.20}$$

The average values of the variables y_1 and y_4 are found from the third and fourth equations of system (2.19). Based on formulas (2.16) and (2.20), we obtain with accuracy of up to $O(\varepsilon^3)$ the following:

$$p_2'p_1 + q_2'q_1 = 0, \quad p_2p_1 + q_2q_1 = 0, \quad \bar{y}_2\bar{y}_3 = -\mu F_1^2/(m^2U), \quad p_2p_3 + q_2q_3 = \mu f_{21}F_1,$$
$$p_4'p_1 + q_4'q_1 = -\mu f_{12}F_2\cot\theta, \quad p_1p_4 + q_1q_4 = -\mu f_{12}F_2, \quad p_2q_4 - q_2p_4 = -\mu f_{21}F_2. \tag{2.21}$$

Accounting for relations (2.21) and (2.7), the average value of the variable y_1 is expressed as follows:

$$\bar{y}_1 = \frac{\mu\gamma F_2^2}{2(1+\gamma)U\sin\theta}\left(\frac{U(\cos^2\theta - 1)\cos\theta - 2(2+U\cos\theta)}{(4+m^2)(4-U^2)} - \frac{2\cos^3\theta}{m^2U}\right) + O(\varepsilon^3), \tag{2.22}$$

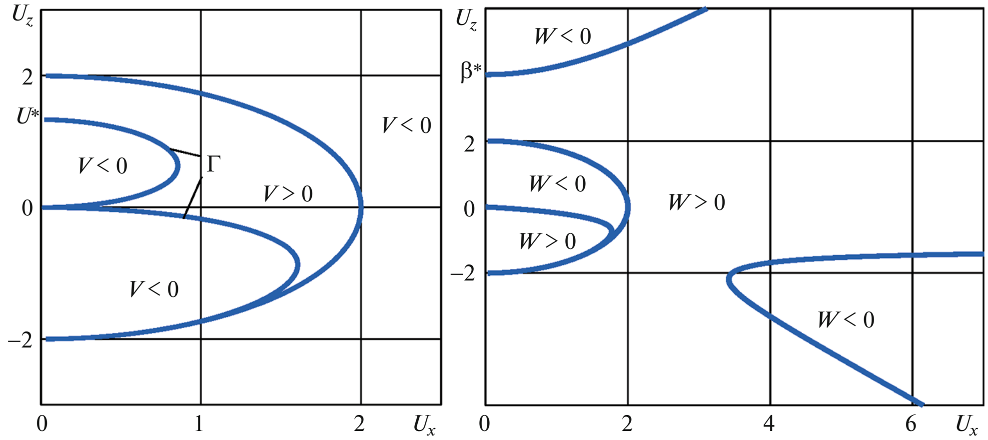

Fig. 3.

and it is the bounded function of ε^2. The expression for $\bar{y}_4$ is unnecessary below and so we do not write it out. We only note that it is also a bounded function of ε^2.

The value of $\bar{W}_3$ is determined from the fifth equation of system (2.19). Given the formulas (2.20), (2.21), (2.22) and (2.7), we obtain the following:

$$\bar{W}_3 = \frac{F_2^2}{2(1+\gamma)}\left(\frac{U(1+\cos^2\theta)+4\cos\theta}{(4+m^2)(4-U^2)} - \frac{2\cos^2\theta}{m^2 U}\right) + O(\varepsilon^3). \tag{2.23}$$

From the eighth equation of the system (2.17) and taking into account relations (2.20), the average value of the satellite precession rate is determined with accuracy of to $O(\varepsilon^2)$ using the following equation:

$$\dot{\psi} = \frac{\bar{y}_2}{\sin\theta} = -\frac{F_1}{(\alpha+\gamma)U\sin\theta} = -\frac{3\varepsilon\cos\theta}{2(\alpha+\gamma)U} = \frac{3(A-C)\cos\theta}{2CU} \tag{2.24}$$

that completely coincides with the expression for the precession rate of a satellite modeled as a single solid [11].

From the sixth and seventh equations of the system (2.17), the average values of the time derivatives of the nutation angle and the satellite angular velocity are determined by the formulas:

$$\dot{\theta} = \bar{y}_1 + \bar{S}_1 \quad \text{and} \quad \dot{U} = \mu\gamma\bar{W}_3/(1+\varepsilon).$$

It can be shown (the corresponding calculations are not given due to their bulkiness) that the averaged-over-period value of the function S_1 calculated based on the motion equations is expressed by terms of the third order with respect to ε, i.e., $\bar{S}_1 = O(\varepsilon^3)$. Therefore, taking into account relations (2.22) and (2.23), we obtain, with accuracy of up to $O(\varepsilon^3)$, the following expressions for average values of $\dot{\theta}$ and $\dot{U}$:

$$\begin{aligned} \dot{\theta} &= \frac{\mu\gamma F_2^2}{2(1+\gamma)U\sin\theta}\left(\frac{U(\cos^2\theta-3)\cos\theta-4}{(4+m^2)(4-U^2)} - \frac{2\cos^3\theta}{m^2U}\right), \\ \dot{U} &= \frac{\mu\gamma F_2^2}{2(1+\gamma)}\left(\frac{U(1+\cos^2\theta)+4\cos\theta}{(4+m^2)(4-U^2)} - \frac{2\cos^2\theta}{m^2U}\right). \end{aligned} \tag{2.25}$$

The equations form a closed system of evolutionary equations of the satellite's rotational motion with respect to the variables θ and U. As follows from these equations, the evolution rate over variables θ and U is proportional to ε^2 while the satellite precession rate (2.24) is proportional to ε.

As follows from the second equation (2.25), the derivative $\dot{U}$ changes its sign at the points $U = 2$ and points that satisfy the following equation:

$$U^2[(2+M)\cos^2\theta + M] + 4MU\cos\theta - 8\cos^2\theta = 0, \quad M = m^2/(4+m^2). \tag{2.26}$$

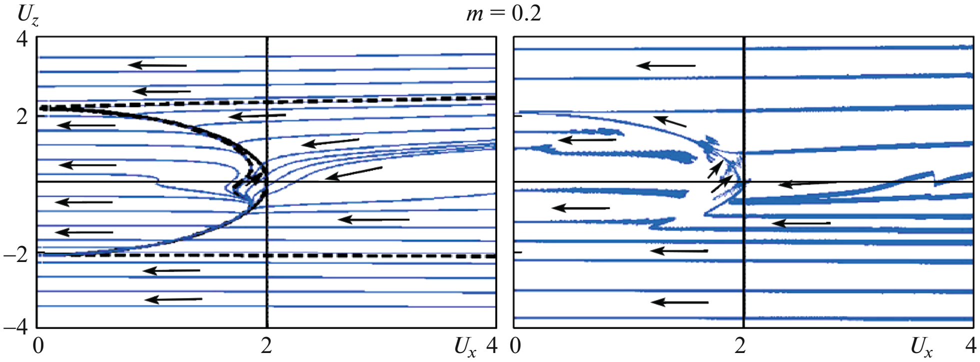

Fig. 4.

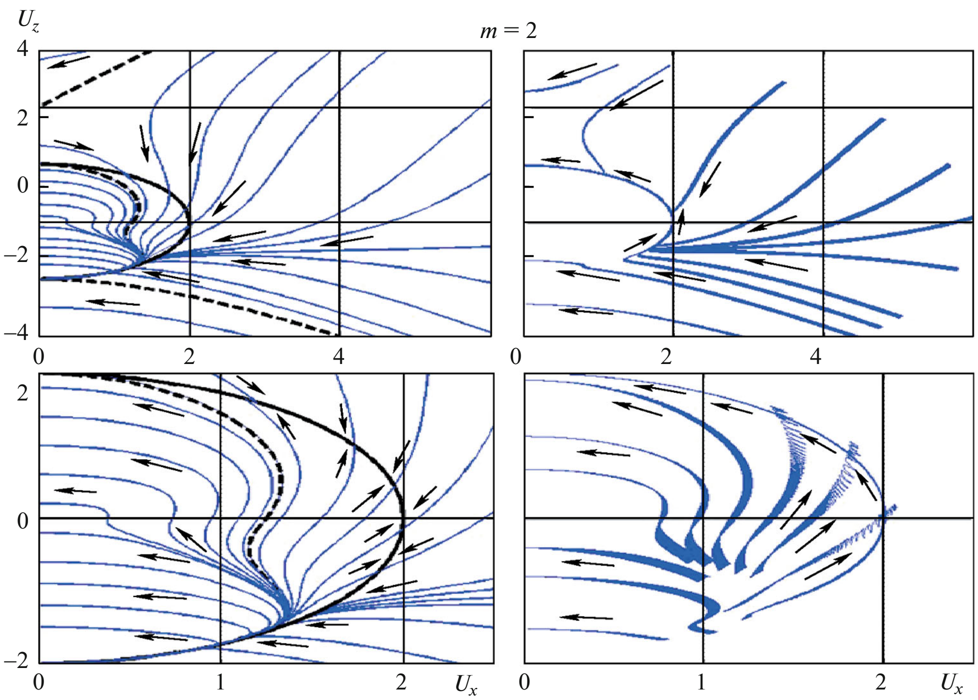

Fig. 5.

Figure 3 on the left shows the curve Γ defined by this equation and the regions of positive and negative values of the derivative $V = \dot{U}$ in the plane of variables U_x and U_z, where $U_x = U\sin\theta$ and $U_z = U\cos\theta$ are the angular velocity projections on the orbit plane and on the normal to the orbit plane. At $U > 2$, as well as inside the circle $U < 2$, the satellite angular velocity decreases in two regions limited by a curve Γ while it increases in the rest of the region. The value of U^* is determined by the following formula:

$$U^* = \frac{2}{1+M} = \frac{4+m^2}{2+m^2}. \tag{2.27}$$

Figure 3 on the right shows the regions of positive and negative values of the derivative $W = \dot{\theta}$ determined using the first equation (2.25). Here,

$$\beta^* = \frac{2}{1-M} = \frac{4+m^2}{2}. \tag{2.28}$$

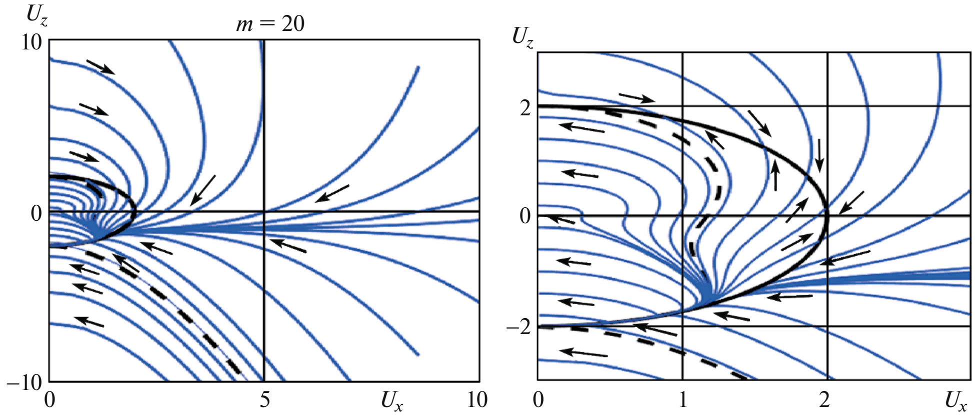

Fig. 6.

The evolution Eqs. (2.25) have the same stationary solutions $\theta = 0, \pi$ and $U = \beta = \text{const}$ as the exact equations (1.2). The conditions of stability/instability of these solutions for the evolutionary equations are determined by the sign of the derivative $\dot{\theta}$ in the vicinity of "direct" ($\theta = 0$) and "inverse" ($\theta = \pi$) stationary rotations. Based on results of the analysis of this derivative shown in Fig. 3, direct stationary rotations ($U_x = 0$ and $U_z > 0$) are asymptotically stable in the ranges of $U \in (0, 2)$ and $U \in (\beta^*, \infty)$ and unstable in the range of $U \in (2, \beta^*)$. Excepting, perhaps, points $U = 2$, all inverse stationary rotations ($U_x = 0$ and $U_z < 0$) are asymptotically stable. These conclusions completely coincide with the results of the stability analysis of stationary rotations of a satellite close to spherically symmetric obtained in Section 1.

The time can be eliminated from Eqs. (2.25), and a single equation can be obtained:

$$\frac{d\theta}{dU} = \frac{U^2 \cos\theta[(2+M)\cos^2\theta - 3M] - 4MU - 8\cos^3\theta}{\{U^2[(2+M)\cos^2\theta + M] + 4MU\cos\theta - 8\cos^2\theta\}U\sin\theta} \tag{2.29}$$

that describes the evolution trajectories of the satellite's rotational motion in variables U and θ.

Below are the results of the analysis of the phase trajectories of the satellite rotational motion evolution in terms of variables U_x, U_z for various values of the parameter m and various initial conditions. The left parts of Figs. 4 and 5 show the phase trajectories obtained using the evolution Eq. (2.29) while the right parts show the trajectories obtained by numerical integration of the exact Eqs. (1.1) for a dynamically symmetric satellite at a parameter value of $a = 1.1$. The arrows indicate the evolution direction. Dashed lines are separatrices separating trajectories that fall on a circle $U = 2$ from other trajectories. The upper separatrix starts at the point $U_x = 0, U_z = \beta^*$, where the value of β^* is determined by formula (2.28), while the lower separatrix starts at the point $U_x = 0$, $U_z = -2$.

The presented phase portraits show the complete coincidence of the satellite evolution trajectories obtained using evolution Eq. (2.29), on the one hand, and exact Eqs. (1.1), on the other hand.

As can be seen in the figures, there is a region of initial conditions bounded by separatrices (denoted as G_2), for which any trajectory eventually falls on a circle $U = 2$. Moreover, for the majority of such trajectories, the further (final) evolution stage is counterclockwise rotation along the arc of a circle $U = 2$, i.e., 2 : 1-resonance rotation (the satellite angular velocity is twice the angular velocity of the orbital basis), at which the satellite angular velocity remains constant while the rotation axis is turned in the normal direction to the orbit plane. At the end of such motions, a stationary rotation around the normal to the orbit plane is established at an angular velocity equal to twice the angular velocity of the orbital basis.

At small m-values compared with unity (Fig. 4), the separatrix asymptotes are located at a small angle to the axis U_x, and the region G_2 occupies a relatively small part of the half-plane of possible initial conditions. Therefore, only for a small fraction of the initial conditions, the 2 : 1-resonance mode is realized at the final stage. For the remaining initial conditions, phase trajectories are close to horizontal straight lines (U_z decreases much more slowly than U_x).

For m-values comparable to unity (Fig. 5), the dimensions of the region G_2 are comparable to the dimensions of the regions of the remaining initial conditions, and the fraction of the phase trajectories, the final stage of which is the 2 : 1-resonant mode, is comparable to the fraction of all other phase trajectories.

At $m \gg 1$ (Fig. 6), the final stage will be a 2 : 1-resonant mode for the vast majority of initial conditions from the region $U > 2$.

Phase trajectories starting above the upper separatrix are characterized by a monotonic decrease of the angle θ to zero, while phase trajectories below the lower separatrix are characterized by a monotonic increase of the angle θ to π. This behavior of the phase trajectories confirms earlier conclusions about the nature of the stability of the corresponding direct and inverse stationary rotations of the satellite.

The angle θ changes nonmonotonically on phase trajectories from a region G_2. In this case, a part of the phase trajectories starting in the upper half-plane ($U_z(0) > 0$) intersect the axis $U_z = 0$, but they all end with the arc of a circle $U = 2$ (2 : 1-resonant mode). Thus, the final results of the evolutionary process with initial phase trajectories from the upper half-plane of the region G_2 are direct stationary rotations at an angular velocity equal to twice the angular velocity of the orbital basis. The majority of the phase trajectories from the lower half-plane of the region G_2 have the same end. Only trajectories close to the lower separatrix are not captured in the 2 : 1-resonant mode. These phase trajectories penetrate the resonant circle $U = 2$ and end with reverse stationary rotations at an angular velocity of $U < 2$.

Note that the region G_2 includes a part of the circle $U < 2$. The satellite angular velocity increases on phase trajectories from this part of the circle.

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