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Resonant and Bifurcation Oscillations of the Rod with Regard to the Resistance Forces and Relaxation Properties of the Medium

A. V. Eremin*, **V. V. Zhukov**, **V. A. Kudinov**, and **I. V. Kudinov**
Samara State Technical University,
ul. Molodogvardeyskaya 244, Samara, 443100 Russia
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Abstract—A mathematical model of elastic oscillations of a rod under the influence of an external harmonic load, taking into account the relaxation properties and forces of the medium resistance, has been developed. The derivation of the differential equation of the model is based on taking into account the time dependence of the stresses and strains in the formula of Hooke's law, which, when presented in this way, coincides with the formula of the complicated Maxwell and Kelvin–Voigt models. The study of the model using numerical method showed that when the frequency of the natural oscillations of the rod coincides with the frequency of the external load oscillations (if the resistance of the medium and its relaxation properties are not taken into account), the amplitude of the oscillations (resonance) increases unlimited in time. When taking into account the resistance and relaxation properties of the medium at resonant frequencies, the amplitude of oscillations stabilizes on a value depending on the values of the resistance and relaxation coefficients. At frequencies close to resonant, bifurcation oscillations (beats) are observed, at which there is a periodic increase and decrease of the amplitude of oscillations. At frequencies substantially different from resonant ones, in the case of taking into account resistance forces and relaxation properties of materials, bifurcation oscillations are not observed. In this case, the amplitude of oscillations is stabilized in time at a value depending on the amplitude of oscillations of the external load, the resistance coefficient and the relaxation coefficients.

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Keywords: *wave equation, numerical solution, stress and strain relaxation, external load, medium resistance, resonance and bifurcation oscillations, beats.*

1. INTRODUCTION

In technique there are often cases when the process of the own longitudinal oscillations of a rod is accompanied by oscillations of an external load applied at its free end. In this case, of particular practical interest is the study of changes in the amplitudes of resonant oscillations as well as oscillations in nonresonant frequencies [1–6]. In the latter case, depending on the ratio of the natural oscillation frequencies of the rod and the oscillations of the external load, significantly different variants of the oscillations can be observed. And, in particular, bifurcation oscillations (beats) can occur, when the amplitude periodically increases from zero to maximum, as well as oscillations in which each point of the rod participates in two oscillatory processes — in high-amplitude low-frequency and low-amplitude high-frequency. The determination of the ratios of the frequencies at which the indicated variants of oscillations are realized is of both scientific and practical interest.

2. FORMULATION OF THE PROBLEM

The basis of derivation of the differential equations of longitudinal oscillations of rods, springs, strings is Hooke's law

$$\sigma = E\,\varepsilon, \tag{2.1}$$

and the equation of motion (equilibrium)

$$\rho \frac{d^2U}{dt^2} = \frac{d\sigma}{dx}, \tag{2.2}$$

*e-mail: totig@yandex.ru

where σ is normal stress; E is the modulus of elasticity; U is displacement; x is the coordinate; ρ is density; t is time; $\varepsilon = dU/dx$ is deformation.

On the basis of (2.1), (2.2) the classical hyperbolic (wave) equation is derived [2–4, 7]

$$\frac{\partial^2 U(x,t)}{\partial t^2} = c^2 \frac{\partial^2 U(x,t)}{\partial x^2}, \tag{2.3}$$

where $c = \sqrt{E/\rho}$ is the propagation velocity of the longitudinal perturbation, m/s.

Equation (2.3) describes continuous oscillations, which is associated with the absence of terms in it that take into account the internal resistance of the medium. In order to take it into account, let us assume that the resistance force F_C is proportional to the intensity of the change in displacement over time, characterized by a certain coefficient r, kg/s.

$$F_C = -r \frac{\partial U}{\partial t}, \tag{2.4}$$

where the minus sign means that the resistance force has the opposite direction to the movement. The equation of motion (2.2) in this case takes the form

$$\rho \frac{\partial^2 U}{\partial t^2} = \frac{\partial \sigma}{\partial x} - \frac{r}{\Delta V} \frac{\partial U}{\partial t}, \tag{2.5}$$

where ΔV is the volume of the elementary segment.

Note that the last term on the right-hand side of (2.5) is the resistance force per volume unit.

Substituting (2.1) into (2.5), we find the wave equation for the oscillations of the rod, taking into account the forces of resistance to the process of changing its shape

$$\frac{\partial^2 U(x,t)}{\partial t^2} = c^2 \frac{\partial^2 U(x,t)}{\partial x^2} - \gamma \frac{\partial U(x,t)}{\partial t}, \tag{2.6}$$

where $\gamma = r/(\rho \Delta V)$ is the resistance coefficient, 1/s.

In deriving equation (2.6), Hooke's law was used, in which there is no causal relationship between phenomena. The cause (the acting force) here is the deformation ε, and the consequence is the stress σ. The absence of a time variable in this formula indicates that a consequence with a change in the cause occurs instantaneously. However, the propagation speeds of the potentials of any physical fields cannot take infinite values. In the real body, the process of their change occurs with some delay in time according to the relaxation properties of the material.

To take into account the relaxation properties, the equation (2.1) of the Hooke's law is represented as

$$\sigma + \tau_1 \frac{\partial \sigma}{\partial t} = E\Big(\frac{\partial U}{\partial x} + \tau_2 \frac{\partial^2 U}{\partial x \partial t}\Big), \tag{2.7}$$

where τ_1, τ_2 are stress relaxation and strain coefficients, s.

The relation (2.7) completely coincides with the standard models of a viscoelastic body, known as the Maxwell and Kelvin–Voigt models [8–10]. Note that relation (2.7) corresponds to the complicated models of Maxwell and Kelvin–Voigt, in which a third element is added — a spring located parallel to the series-connected Hooke and Newton bodies (in the Maxwell model) and placed in series with the parallel-connected Hooke and Newton bodies (in the Kelvin–Voigt model).

The combined formula of the complicated Maxwell and Kelvin–Voigt models has the form [8–13]

$$\sigma + Q \frac{\partial \sigma}{\partial t} = E\Big(\varepsilon + Q \frac{\partial \varepsilon}{\partial t}\Big), \tag{2.8}$$

where the constant Q, which has the dimension of time, in [9, 11] is characterized as a delay time or relaxation time (period). It is obvious that equation (2.7), if we assume $\tau_1 = \tau_2 = Q$, completely coincides with equation (2.8).

Equation (2.7) can also be obtained from the generalized system of Onsager differential equations proposed by A. V. Lykov [12]

$$J_i = L_i^{(r)} \frac{\partial J_i}{\partial t} + \sum_{k=1}^{N} \Big(L_{ik} X_k + L'_{ik} \frac{\partial X_k}{\partial t}\Big) \tag{2.9}$$

where J_i is the flow of substance (heat, mass, momentum, etc.); X_k are driving forces; $L_i^{(r)}$, L_{ik}, L'_{ik} are constants.

If we assume $L_i^{(r)} = -\tau_1$; $L_{ik}^{(r)} = E$; $L'_{ik} = E\tau_2$; $J_i = \sigma$; $X_k = \varepsilon$, then equation (2.9) is reduced to equation (2.7). The consistency of equations (2.7), (2.9) suggests that equation (2.7) takes into account the cross effects associated with the simultaneous consideration of temporal and spatial nonequilibrium and their mutual influence in the nonlocal process of momentum transfer.

To derive a differential equation that takes into account the change in time of stresses and strains in Hooke's law equation, we express the stress from (2.7) and substitute the obtained equations into the equation of motion (2.5). After some transformations, taking into account (2.2), we find

$$\tau_1 \frac{\partial^3 U}{\partial t^3} + \frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2} + c^2 \tau_2 \frac{\partial^3 U}{\partial x^2 \partial t} - \gamma \frac{\partial U}{\partial t}. \tag{2.10}$$

Equation (2.10) represents the wave equation of the longitudinal oscillations of the rod, which takes into account the relaxation properties of the material and its resistance to the process of changing the shape. Obviously, for $\tau_1 = \tau_2 = \gamma = 0$, equation (2.10) is reduced to the equation of undamped oscillations (2.3).

Let us find the solution of the boundary value problem of oscillations of a rod, one end of which is rigidly fixed (for $x = \delta$), and on the second, the force F is applied per unit area, varying according to the cosine-law (for $x = 0$)

$$\frac{F}{ES} = \frac{\partial U(0,\, t)}{\partial x} = \cos(\omega t), \tag{2.11}$$

where S is the cross-sectional area of the rod, m^2; $\omega = 2\pi\nu$ is the circular frequency, $1/c$.

At the initial moment of time, the rod is deformed according to a linear law, according to which its free end has maximal displacement. The initial velocities of the elements of the rod also vary according to a linear law, according to which its free end also has maximal initial velocity.

The mathematical formulation of the problem in this case takes the form

$$\tau_1 \frac{\partial^3 U(x,t)}{\partial t^3} + \frac{\partial^2 U(x,t)}{\partial t^2} = c^2 \Big[\frac{\partial^2 U(x,t)}{\partial x^2} + \tau_2 \frac{\partial^3 U(x,t)}{\partial x^2 \partial t} \Big] - \gamma \frac{\partial U(x,t)}{\partial t} \quad (t > 0;\ 0 < x < \delta), \tag{2.12}$$

$$U(x,\, 0) = b(\delta - x), \tag{2.13}$$

$$\frac{\partial U(x,\, 0)}{\partial t} = \nu(\delta - x), \tag{2.14}$$

$$\frac{\partial^2 U(x,\, 0)}{\partial t^2} = 0, \tag{2.15}$$

$$\frac{\partial U(0,t)}{\partial x} = \cos(\omega t), \tag{2.16}$$

$$U(\delta, t) = 0, \tag{2.17}$$

where δ is the length of the rod, m; b is the coefficient taking into account the initial displacements of the rod elements; ν is the coefficient taking into account the initial velocity, $1/c$.

Let us introduce the following dimensionless variables and parameters

$$\Theta = \frac{U}{U_0}; \quad \xi = \frac{x}{\delta}; \quad \mathrm{Fo} = \frac{ct}{\delta}; \quad F_1 = \frac{c\tau_1}{\delta}; \quad F_2 = \frac{c\tau_2}{\delta}; \quad F_3 = \frac{\delta\gamma}{c}, \tag{2.18}$$

where Θ is the dimensionless displacement; ξ is the dimensionless coordinate; Fo is the Fourier number (dimensionless time); $U_0 = b\delta$; F_1, F_2 are dimensionless relaxation coefficients; F_3 is the dimensionless coefficient of medium resistance.

In view of (2.18), problem (2.12)–(2.17) takes the form

$$F_3 \frac{\partial \Theta(\xi, \mathrm{Fo})}{\partial \mathrm{Fo}} + F_1 \frac{\partial^3 \Theta(\xi, \mathrm{Fo})}{\partial \mathrm{Fo}^3} + \frac{\partial^2 \Theta(\xi, \mathrm{Fo})}{\partial \mathrm{Fo}^2} = \frac{\partial^2 \Theta(\xi, \mathrm{Fo})}{\partial \xi^2} + F_2 \frac{\partial^3 \Theta(\xi, \mathrm{Fo})}{\partial \xi^2 \partial \mathrm{Fo}} \ (\mathrm{Fo} > 0;\ 0 < \xi < 1), \tag{2.19}$$

$$\Theta\,(\xi, 0) = 1 - \xi, \tag{2.20}$$

$$\frac{\partial \Theta(\xi, 0)}{\partial \mathrm{Fo}} = B(1 - \xi), \tag{2.21}$$

$$\frac{\partial^2\Theta(\xi,0)}{\partial \mathrm{Fo}^2} = 0, \tag{2.22}$$

$$\frac{\partial\Theta(0,\mathrm{Fo})}{\partial\xi} = F_4\cos(F_5\mathrm{Fo}), \tag{2.23}$$

$$\Theta(1,\mathrm{Fo}) = 0, \tag{2.24}$$

where $F_4 = \delta/U_0$; $B = \delta^2\nu/(eU_0)$ is the dimensionless coefficient; $F_5 = \omega\delta/e$ is the dimensionless frequency of forced oscillations (external load oscillations).

3. SOLUTION OF THE PROBLEM

To solve the problem (2.19)–(2.24) by finite difference method a spatial grid is introduced in the considered area with steps $\Delta\xi = 0.005$, $\Delta\mathrm{Fo} = 0.005$ in the variables ξ and Fo, respectively, so that

$$\xi_k = k\,\Delta\xi, \quad k = 0,\dots,K, \quad \mathrm{Fo}_i = i\,\Delta\mathrm{Fo}, \quad i = 0,\,\dots,I, \tag{3.1}$$

where $K = 200$, $I = 50000$ are the number of steps in the coordinates ξ, Fo.

The grid functions $\Theta_k^i = \Theta(\xi_k,\,\mathrm{Fo}_i)$ are introduced on the grid (3.1). Using the adopted approximation scheme for differential operators, the problem (2.19)–(2.24) is written as

$$F_3\frac{\Theta_k^{i+1}-\Theta_k^i}{\Delta\mathrm{Fo}} + F_1\frac{\Theta_k^{i+1}-3\Theta_k^i+3\Theta_k^{i-1}-\Theta_k^{i-2}}{\Delta\mathrm{Fo}^3} + \frac{\Theta_k^{i-1}-2\Theta_k^i+\Theta_k^{i+1}}{\Delta\mathrm{Fo}^2}$$
$$= \frac{\Theta_{k-1}^i-2\Theta_k^i+\Theta_{k+1}^i}{\Delta\xi^2} + F_2\Big(\frac{\Theta_{k-1}^i-2\Theta_k^i+\Theta_{k+1}^i}{\Delta\xi^2\Delta\mathrm{Fo}} - \frac{\Theta_{k-1}^{i-1}-2\Theta_k^{i-1}+\Theta_{k+1}^{i-1}}{\Delta\xi^2\Delta\mathrm{Fo}}\Big),$$

$$\Theta_k^0 = 1-\xi_k, \quad \frac{\Theta_k^1-\Theta_k^0}{\Delta\mathrm{Fo}} = B(1-\xi_k), \quad \frac{\Theta_k^0-2\Theta_k^1+\Theta_k^2}{\Delta\xi} = 0,$$

$$\frac{\Theta_1^i-\Theta_0^i}{\Delta\xi} = F_4\cos(F_5\mathrm{Fo}_i), \quad \Theta_K^i = 0.$$

4. ANALYSIS OF RESULTS

The results of the studies performed are shown in Figs. 1–10. In the case when the relaxation properties of materials and the resistance forces of the medium ($F_1 = F_2 = F_3 = 0$) are not taken into account, as well as in case of absence of an external load ($F_4 = 0$) and with zero initial velocities ($B = 0$), continuous oscillations of the rod are observed with the frequency of its own oscillations, the dimensionless value of which is 1.575.

At non-zero initial speeds ($B \neq 0$) ($F_1 = F_2 = F_3 = F_4 = 0$), oscillations occur at an increased and constant amplitude in time, depending on the coefficient B, in a continuous process of oscillations.

For any non-zero values of the resistance coefficient ($F_3 \neq 0$), regardless of the magnitudes of the initial velocities ($B \neq 0$), without taking into account the relaxation properties of the material ($F_1 = F_2 = 0$), the oscillations are damped (Fig. 1).

When $F_1 = F_2 = F_3 = B = 0$; $F_4 = 0.5$; $F_5 = 1.575$, the dimensionless frequency of forced oscillations $F_5 = 0$, arising under the action of an external load of the form (2.23), coincides with the natural frequency of the rod. When this happens, there is an unlimited increase in the amplitude of oscillations over time, that is, resonant oscillations are observed (Fig. 2).

If, under the conditions of the previous version, the rod element speeds are not zero ($q = 0$), then, depending on the value of w, the oscillation amplitude at the initial time interval decreases to a certain minimum value and with a further increase in time, it increases indefinitely in a continuous process of oscillations (resonant oscillations) (Fig. 3).

At small values of the relaxation coefficients ($F_1 = F_2 = 0.1$) and at $F_3 = 0.3$; $F_4 = 0.1$; $B = 0$; $F_5 = 1.575$ (oscillations at resonant frequencies) in the initial part of time, the amplitude of oscillations decreases, reaching a certain minimum. With an increase in time, the amplitude increases, stabilizing at some constant value in time (Fig. 4).

An example of rod oscillations at resonant frequencies for large values of relaxation coefficients ($F_1 = F_2 = 2$) and for $F_3 = 1$; $F_4 = 0.1$; $F_5 = 1.575$; $B = 0$ is shown in Fig. 5. From the analysis

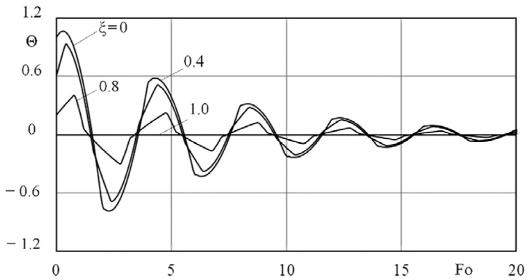

Fig. 1.

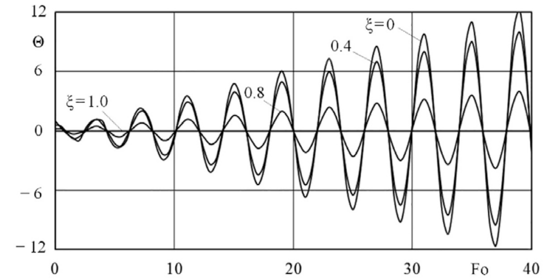

Fig. 2.

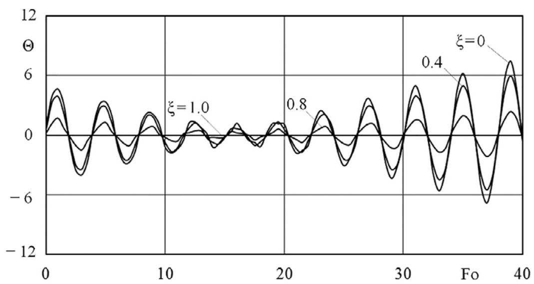

Fig. 3.

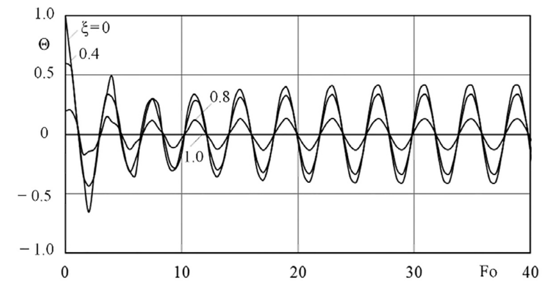

Fig. 4.

of the results, it follows that under the action of an external load, the amplitude increases periodically and decreases in a continuous process of oscillations.

Calculations were also performed in non-resonant frequencies of external load oscillations. In this case, two options were considered - at frequencies close to the resonant ones and at significantly different ones. And, in particular, an example of a rod oscillation at frequencies close to resonant ($F_5 = 1.5$) is shown in Figs. 6 and 7 ($F_1 = F_2 = 10$; $F_3 = 0.3$; $F_4 = 0.1$; $B = 0$). From an analysis of the results, it follows that in this case there is a bifurcation change in amplitude, at which significant periodic increases in amplitude are accompanied by equally significant drops in it. These types of oscillations are also called beats [12].

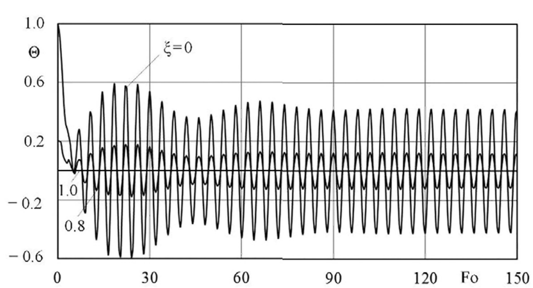

Fig. 5.

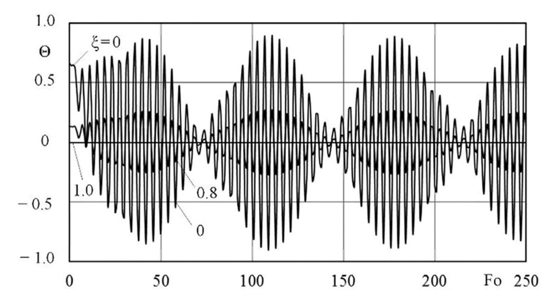

Fig. 6.

An example of a rod oscillation at frequencies far from resonant is shown in Fig. 8. An analysis of the results obtained allows us to conclude that each point of the rod has two oscillation amplitudes with a significantly different frequency. That is, each point of the rod participates in two oscillatory processes, in one of which high-frequency and low-amplitude oscillations are performed, and in the other — low-frequency and high-amplitude oscillations. At some large values of the resistance coefficient F_3 in both resonant and non-resonant frequencies, a critical damping of oscillations is observed, at which, despite the influence of an external load, the rod withdrawn from the equilibrium state returns in the absence of oscillations of its internal points (Figs. 9 and 10).

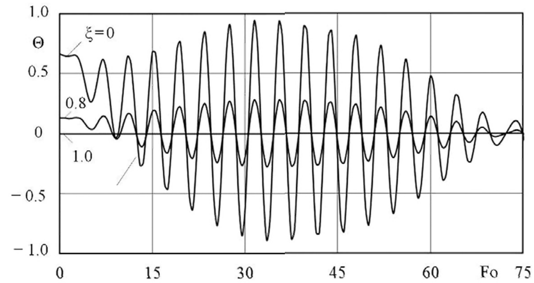

Fig. 7.

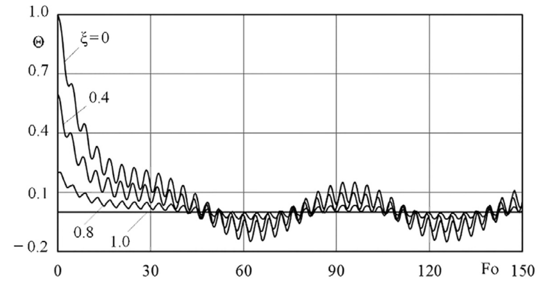

Fig. 8.

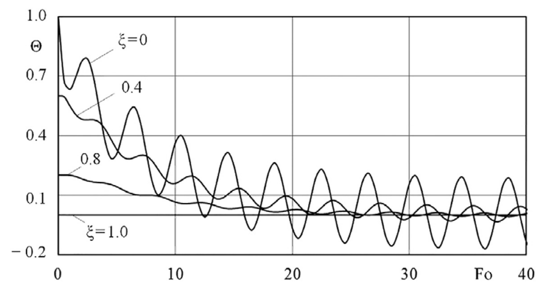

Fig. 9.

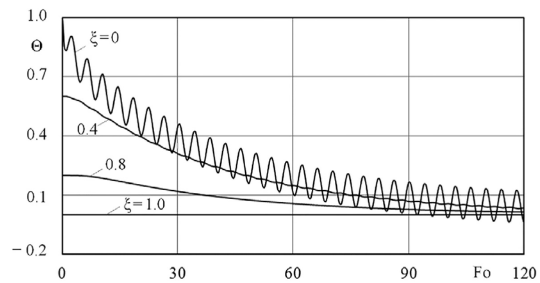

Fig. 10.

5. CONCLUSION

On the basis of taking into account the relaxation terms in the formula of Hooke's law, which include time derivatives of stress and strain in a product with relaxation coefficients, as well as taking into account the internal resistance of the material of the rod to the process of changing its shape, the wave equation for oscillations of the elastic rod is derived.

A numerical method was used to study the longitudinal oscillations of a rod with rigid fixing of one of its ends and setting the external harmonic load on the other, taking into account the initial velocity of various points of the rod varying over the spatial variable with the initial deformation of the rod according to a linear law.

If we ignore the forces of resistance of the material of the rod and its relaxation properties at resonant frequencies, the oscillation amplitude increases unlimited in time. When taking into account the relaxation properties and the resistance of the medium, a periodic change in the amplitude is observed in a continuous oscillation process.

At frequencies close to resonant, bifurcation oscillations (beats) are observed, accompanied by significant changes in the amplitude of oscillations in the non-attenuating processes of the oscillations of the rod.

At frequencies far from resonant, each point of the rod participates in two oscillatory processes. In one of them high-frequency and low-amplitude vibrations are made, and in the other low-frequency and high-amplitude vibrations.

At large values of the resistance coefficient, a critical damping of the oscillations is observed, in which the rod withdrawn from the equilibrium state returns to its original (undeformed) state when there is practically no oscillatory processes at all its internal points, except for the point where the external load is specified.

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