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Quaternion Regularization of the Equations of the Perturbed Spatial Restricted Three-Body Problem: II

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Abstract—A quaternion method for the regularization of differential equations of the perturbed spatial restricted three-body problem is developed. It is closely related, from the methodological point of view, to the quaternion method for the regularization of the differential equations of the perturbed spatial three-body problem in Kustaanheimo–Stiefel variables that was earlier proposed by the author of this article.
Various local and global regular quaternion differential equations of the perturbed spatial restricted three-body problem (both circular and non-circular problem) i.e. equations that are regular in the vicinity of the first or second body of finite mass and equations that are regular at the same time both in the neighborhood of the first and second body of finite mass are obtained. The equations are systems of nonlinear nonstationary differential equations of the tenth or eleventh or nineteenth order with respect to the Kustaanheimo–Stiefel variables, their first derivatives, Kepler or total energies, or variables that are Jacobi integration constants in the case of the unperturbed spatial circular restricted three-body problem, as well as with respect to time and auxiliary time variable. The equations obtained allow one to construct different regular algorithms for integrating the differential equations of the perturbed spatial restricted three-body problem.
This study is an extension of [1, 2].

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1. THE INITIAL DIFFERENTIAL EQUATIONS OF THE PERTURBED SPATIAL RESTRICTED THREE-BODY PROBLEM. STATEMENT OF THE REGULARIZATION PROBLEM

Consider three material points M_0, M_1, and M_2 with the masses m_0, m_1, and m_2, respectively. They mutually attract each other according to the Newton's law of universal gravitation. The unrestricted three-body problem consists [3] in defining and studying all possible motions of material points M_0, M_1, M_2. The restricted three-body problem is the problem [3] of the motion of a material point $M_2 = M$ with zero mass $m_2 = 0$ (more precisely, with a mass $m_2 = m$ which is negligibly small compared to the masses m_0 and m_1). According to the Newton's law, this point is attracted by two other material points M_0, M_1 having nonzero masses m_0 and m_1.

The restricted three-body problem is [3] the limiting form of the unrestricted three-body problem. It has found wide application both in classical celestial mechanics (for example, the theory of the motion of the moon) and in the mechanics of space flight (for example, the problem of reaching the moon).

We introduce vectors $\mathbf{r}_0 = \overrightarrow{M_0M}$, $\mathbf{r}_1 = \overrightarrow{M_1M}$, $\mathbf{r}_{01} = \overrightarrow{M_0M_1}$, $\mathbf{r}_{10} = \overrightarrow{M_1M_0} = -\mathbf{r}_{01}$. The projections of the vectors $\mathbf{r}_0$ and $\mathbf{r}_1$ on the axis of the inertial coordinate system $O\xi\eta\zeta$ are respectively equal to $\xi_2 - \xi_0, \eta_2 - \eta_0, \zeta_2 - \zeta_0$ and $\xi_2 - \xi_1, \eta_2 - \eta_1, \zeta_2 - \zeta_1$, where ξ_0, η_0, ζ_0; ξ_1, η_1, ζ_1 and ξ_2, η_2, ζ_2 are the Cartesian coordinates of material points M_0, M_1, and $M_2 = M$ in the inertial coordinate system $O\xi\eta\zeta$.

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Using the introduced vectors $\mathbf{r}_0$ and $\mathbf{r}_1$ as the vector variables, we write the vector differential equations of the perturbed spatial restricted three-body problem in the following form [2]:

$$\frac{d^2\mathbf{r}_0}{dt^2} = -\frac{fm_0}{r_0^3}\,\mathbf{r}_0 - \frac{fm_1}{r_1^3}\,\mathbf{r}_1 - \frac{fm_1}{r_{01}^3}\,\mathbf{r}_{01} + \mathbf{p}, \quad (1.1)$$

$$\frac{d^2\mathbf{r}_1}{dt^2} = -\frac{fm_0}{r_0^3}\,\mathbf{r}_0 - \frac{fm_1}{r_1^3}\,\mathbf{r}_1 - \frac{fm_0}{r_{01}^3}\,\mathbf{r}_{10} + \mathbf{p}, \quad (1.2)$$

$$\mathbf{r}_{01} = \mathbf{r}_0 - \mathbf{r}_1, \quad \mathbf{r}_{10} = \mathbf{r}_1 - \mathbf{r}_0 = -\mathbf{r}_{01}, \quad r_0 = |\,\mathbf{r}_0\,|, \quad r_1 = |\,\mathbf{r}_1\,|, \quad r_{01} = |\,\mathbf{r}_{01}\,| = |\,\mathbf{r}_{10}\,|\,.$$

Here r_{01}, r_0, r_1 are the mutual distances between the points M_0 and M_1, M_0 and M, M_1 and M, respectively; f is the gravitational constant, $\mathbf{p}$ is the vector of perturbing acceleration of a material point M from the other forces acting on a point M that are not caused by forces of gravitational attraction acting from the points M_0 and M_1.

The differential equation (1.1) describes the movement of a point Min the coordinate system $M_0X_0Y_0Z_0$ having an origin at a point M_0 and the coordinate axes M_0X_0, M_0Y_0, M_0Z_0 parallel to the axes of the same name of the inertial coordinate system $O\xi\eta\zeta$, and the differential equation (1.2) defines the movement of this point in the coordinate system $M_1X_1Y_1Z_1$ having an origin at a point M_1 and coordinate axes M_1X_1, M_1Y_1, M_1Z_1 that are also parallel to the same inertial axes $O\xi, O\eta, O\zeta$.

Differential equation (1.1) can be considered independently of the differential equation (1.2), if we use the relation $r_1 = r_0 - r_{01}$ and take into account that the vector r_{01} satisfies the differential equation

$$\frac{d^2\mathbf{r}_{01}}{dt^2} = -\Big[\frac{f(m_0 + m_1)}{r_{01}^3}\Big]\mathbf{r}_{01} \quad (1.3)$$

of the unperturbed two-body problem (M_0 and M_1) that, as is known, is integrable. Therefore, we can assume that the vector r_{01} appearing in equation (1.1) is a known function of time: $\mathbf{r}_{01} = \mathbf{r}_{01}(t)$. Similarly, the differential equation (1.2) can be considered independently of the differential equation (1.1), if we use the relation $\mathbf{r}_0 = \mathbf{r}_1 - \mathbf{r}_{10}$ and take into account that the vector $\mathbf{r}_{10} = -\mathbf{r}_{01}$ is a known function of time.

Equations (1.1) and (1.2) can also be considered as a system of two differential equations with unknown vector variables $\mathbf{r}_0$ and $\mathbf{r}_1$.

Note that the equation (1.1) in coordinate form coincides (for $\mathbf{p} = 0$) with the equations of the restricted three-body problem (6.1) [4].

The vector equations (1.1) and (1.2) of the perturbed spatial restricted three-body problem contain singular points $r_0 = 0$, $r_1 = 0$, at which these equations degenerate. The problem of eliminating these singularities (both the separate exclusion of one of these singularities , and the simultaneous exclusion of both singularitie) is the subject of the regularization of the differential equations of the perturbed spatial restricted three-body problem. Note that the simultaneous fulfillment of the conditions $r_0 = 0$, $r_1 = 0$ is impossible in the most problems of celestial mechanics and astrodynamics. Nevertheless, it is of both theoretical and practical interest (from the point of view of constructing effective high-precision algorithms for the numerical integration of differential equations of the three-body problem. These algorithms are necessary for a high-precision prediction of the motion of celestial and cosmic bodies) to obtain such regular equations that do not degenerate under simultaneous fulfilling these conditions.

In this article, we develop a quaternion method for the regularization of differential equations for a perturbed spatial restricted three-body problem that is closely related from the methodological point of view to the quaternion method for regularizing differential equations for a perturbed spatial restricted three-body problem proposed by the author of this study in [5, 6] (see also [7–11]). In the first part of the study [2], the following points were considered: the original Newton equations of the perturbed spatial restricted three-body problem and the formulation of the problem for regularization of these equations; the energy relations and differential equations describing the change in the energies of the system in the perturbed spatial restricted three-body problem, as well as the first integrals of the differential equations of the unperturbed spatial restricted circular three-body problem (Jacobi integrals); the equations of the perturbed spatial restricted three-body problem written in rotating coordinate systems and the rotation quaternions (Euler (Rodrigues–Hamilton) parameters) used to describe the angular motion of these coordinate systems; differential equations for the moments of the quantities of motion in the three-body problem under study.

In the present article, which is an extension of [2], the local regular quaternion differential equations of the perturbed spatial restricted three-body problem are obtained. These equations utilize the Kepler energy or total energy of the system as an additional variable. In addition, the regular quaternion differential equations of the perturbed spatial restricted circular three-body problem have been derived. These equations utilize a quantity, that is a constant of the Jacobi motion in the unperturbed spatial restricted circular three-body problem, as an additional variable.

The obtained equations are systems of nonlinear nonstationary differential equations of the eleventh order with respect to the Kustaanheimo–Stiefel variables, their first derivatives, Kepler or total energy or variable, which is the Jacobi integration constant in the case of the unperturbed spatial restricted circular three-body problem, and also with respect to time and the auxiliary time variable .

The constructed sets of differential equations of the perturbed spatial restricted three-body problem allow us to construct a regular algorithm for integrating these equations, in which one of the systems of differential equations of the eleventh order is used to study the motion of a body M with a negligibly small mass in the vicinity of the body M_0 (when the distances r_0, r_1 between bodies M and M_0, M and M_1 satisfy the inequality $m_1 r_0^2 \leq m_0 r_1^2$), and another system of differential equations of the same order is used when studying the motion of a body M in the vicinity of the body M_1 (when the distance r_1 and r_0 satisfy the inequality $m_0 r_1^2 < m_1 r_0^2$) (In the above inequalities, m_0 and m_1 are the masses of bodies M_0 and M_1).

This article continues the study of the problem of constructing local regular quaternion differential equations of a perturbed spatial restricted three-body problem, and also deals with the problem of constructing global regular quaternion differential equations of a perturbed spatial restricted three-body problem; that is, the equations that are regular under simultaneous fulfillment of conditions $r_0 = 0$, $r_1 = 0$ or $r_0 \to 0$, $r_1 \to 0$. The construction of systems of differential equations used to solve these problems is based on the equations and ratios given in the first part of the study [2].

2. DIFFERENTIAL EQUATIONS OF THE PERTURBED SPATIAL RESTRICTED THREE-BODY PROBLEM WRITTEN IN ACCOMPANYING NONHOLONOMIC (AZIMUTHALLY FREE) COORDINATE TRIHEDRONS. INTRODUCING ROTATION QUATERNIONS AND KUSTAHANHEIMO–STIEFEL VARIABLES INTO THE EQUATIONS OF MOTION

We write the vector differential equations of the perturbed restricted three-body problem (1.1), (1.2) in the quaternion form:

$$\frac{d^2\mathbf{R}_0}{dt^2} = -\frac{fm_0}{r_0^3}\mathbf{R}_0 - \frac{fm_1}{r_1^3}\mathbf{R}_1 - \frac{fm_1}{r_{01}^3}\mathbf{R}_{01} + \mathbf{P}, \tag{2.1}$$

$$\frac{d^2\mathbf{R}_1}{dt^2} = -\frac{fm_0}{r_0^3}\mathbf{R}_0 - \frac{fm_1}{r_1^3}\mathbf{R}_1 - \frac{fm_0}{r_{01}^3}\mathbf{R}_{10} + \mathbf{P}, \tag{2.2}$$

$$\mathbf{R}_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}, \quad i = 0, 1, \tag{2.3}$$

$$\mathbf{R}_{01} = x_{01}\mathbf{i} + y_{01}\mathbf{j} + z_{01}\mathbf{k}, \quad \mathbf{R}_{10} = -\mathbf{R}_{01}, \quad \mathbf{P} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}. \tag{2.4}$$

Here $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the Hamiltonian vector imaginary units; differentiation is performed under the assumption that the orts of the hypercomplex space are unchanged $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$; $(x_i, y_i, z_i$ $(i = 0,\ 1)$ are the Cartesian coordinates of a point M in the coordinate system $M_iX_iY_iZ_i$ (the projection of the vector $\mathbf{r}_i$ on the axis of this coordinate system); x_{01}, y_{01}, z_{01} are the projections of the vector $\mathbf{r}_{01}$ on the axes of the same coordinate system (coordinates of a point M_1 in the coordinate system $M_0X_0Y_0Z_0$); $p_1, p_2, p_3$are the projections of the vector of disturbing acceleration $\mathbf{p}$ on the axis of the coordinate system $M_0X_0Y_0Z_0$ (they are equal to the corresponding projections of this vector on the axis of the inertial coordinate system, as well as on the axis of the coordinate system $M_1X_1Y_1Z_1$).

A quaternion $\mathbf{R}_i$ characterizes the position of a point$M$ in the coordinate system $M_iX_iY_iZ_i$.

Let us introduce into consideration two rotating coordinate systems $M_0X_0'Y_0'Z_0'$ and $M_1X_1'Y_1'Z_1'$ with the origins at the points M_0 and M_1, respectively. Further in these coordinate systems we will write down the equations of the spatial restricted three-body problem. The axes M_0X_0' and M_1X_1' of these coordinate systems are directed along the radii–vectors $\mathbf{r}_0$ and $\mathbf{r}_1$, respectively. We denote the vectors of absolute angular velocities of rotation of the coordinate systems $M_0X_0'Y_0'Z_0'$ and $M_1X_1'Y_1'Z_1'$ (in the inertial coordinate system $O\xi\eta\zeta$) by $\boldsymbol{\omega}_0$ and $\boldsymbol{\omega}_1$, and the projections of these vectors on the axes of the coordinate systems $M_0X_0'Y_0'Z_0'$ and $M_1X_1'Y_1'Z_1'$ by ω_{0i} and ω_{1i}, respectively. Note that when writing the equations of the spatial restricted three-body problem in such coordinate systems, the projections ω_{01} and ω_{11} of the angular velocity vectors $\boldsymbol{\omega}_0$ and $\boldsymbol{\omega}_1$ on the directions of the radius vectors $\mathbf{r}_0$ and $\mathbf{r}_1$ are the arbitrarily specified parameters.

To describe the orientation (angular position) of the coordinate system $M_0X_0'Y_0'Z_0'$ in the coordinate system $M_0X_0Y_0Z_0$ (and, therefore, in the inertial coordinate system $O\xi\eta\zeta$), we use the normalized rotation quaternion $\boldsymbol{\lambda}_0$, and to describe the orientation of the coordinate system $M_1X_1'Y_1'Z_1'$ in the coordinate system $M_1X_1Y_1Z_1$ (and, therefore, in the inertial coordinate system $O\xi\eta\zeta$) we use normalized rotation quaternion $\boldsymbol{\lambda}_1$:

$$\boldsymbol{\lambda}_i = \lambda_{i0} + \lambda_{i1}\mathbf{i} + \lambda_{i2}\mathbf{j} + \lambda_{i3}\mathbf{k}, \quad \| \boldsymbol{\lambda}_i \|^2 = \lambda_{i0}^2 + \lambda_{i1}^2 + \lambda_{i2}^2 + \lambda_{i3}^2 = 1, \quad i = 0,\, 1,$$

where λ_{ij} ($j = \overline{0,3}$) are the components of the quaternion $\boldsymbol{\lambda}_i$ (Rodrigues–Hamilton (Euler) parameters [12–16]) that characterize the orientation of the coordinate system $M_iX_i'Y_i'Z_i'$ in the inertial coordinate system.

We complete a definition of the motion of the coordinate system $M_iX_i'Y_i'Z_i'$ by setting an arbitrarily given projection ω_{i1} of absolute angular velocity vector $\boldsymbol{\omega}_i$ on the direction of the radius vector $\mathbf{r}_i$ (axis M_iX_i') to zero:

$$\omega_{i1} = 2(-\lambda_{i1}\dot{\lambda}_{i0} + \lambda_{i0}\dot{\lambda}_{i1} + \lambda_{i3}\dot{\lambda}_{i2} - \lambda_{i2}\dot{\lambda}_{i3}) = 0, \quad i = 0,\, 1. \tag{2.5}$$

Here one upper point and further two upper points mean the first and second derivative with time t, respectively.

The coordinate system $M_iX_i'Y_i'Z_i'$ in this case rotates with an absolute angular velocity $\boldsymbol{\omega}_i$ collinear to the vector $\mathbf{c}_i$ of the moment of velocity $\mathbf{v}_i$ of a point M in the coordinate system $M_iX_iY_iZ_i$ relative to the point M_i:

$$\boldsymbol{\omega}_i = r_i^{-2}\mathbf{c}_i, \quad \mathbf{c}_i = \mathbf{r}_i \times \dot{\mathbf{r}}_i = \mathbf{r}_i \times \mathbf{v}_i, \quad i = 0,\, 1. \tag{2.6}$$

Such a coordinate system is called a nonholonomic (azimuthally free) accompanying coordinate trihedron.

We note that the kinematic relation (2.6) was used in the first part of this work [2] to construct regular quaternion equations for the perturbed spatial restricted three-body problem in a different way.

The quaternion $\mathbf{R}_i$ defined by the relation (2.3) and characterizing the position of a point M in the coordinate system $M_iX_iY_iZ_i$ is associated with variables $\mathbf{r}_i$ and $\boldsymbol{\lambda}_i$ by the relation

$$\mathbf{R}_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k} = r_i\boldsymbol{\lambda}_i \circ \mathbf{i} \circ \bar{\boldsymbol{\lambda}}_i, \quad i = 0,\, 1. \tag{2.7}$$

Hereinafter, the symbol $\circ$ denotes quaternion multiplication, the top line specifies the conjugate quaternion, for example, $\bar{\boldsymbol{\lambda}}_0 = \lambda_{00} - \lambda_{01}\mathbf{i} - \lambda_{02}\mathbf{j} - \lambda_{03}\mathbf{k}$.

Let us denote the projections of the velocity vectors $\mathbf{v}_0 = dr_0/dt$ and $\mathbf{v}_1 = dr_1/dt$ of the point M in the coordinate systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$ on the axes of the coordinate systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$, respectively (these projections coincide with the projections of the vectors $\mathbf{v}_0$ and $\mathbf{v}_1$ on the axes of the inertial coordinate system) by v_{0k} and v_{1k}. We introduce velocity quaternions V_i composed of these projections:

$$\mathbf{V}_i = v_{i1}\mathbf{i} + v_{i2}\mathbf{j} + v_{i3}\mathbf{k} = \dot{x}_i\mathbf{i} + \dot{y}_i\mathbf{j} + \dot{z}_i\mathbf{k}, \quad i = 0,\, 1. \tag{2.8}$$

Differentiating relation (2.7) with respect to time and taking into account (2.8), we obtain the following expression for the velocity quaternion V_i in terms of the variables r_i and λ_i and their first derivatives with respect to time:

$$\mathbf{V}_i = \dot{\mathbf{R}}_i = \dot{r}_i\boldsymbol{\lambda}_i \circ \mathbf{i} \circ \bar{\boldsymbol{\lambda}}_i + 2r_i\boldsymbol{\lambda}_i \circ \mathbf{i} \circ \dot{\bar{\boldsymbol{\lambda}}}_i - 2r_i\mathrm{scal}(\boldsymbol{\lambda}_i \circ \mathbf{i} \circ \dot{\bar{\boldsymbol{\lambda}}}_i), \quad i = 0,\, 1. \tag{2.9}$$

Here, the equality $\mathbf{a} - \bar{\mathbf{a}} = 2(\mathbf{a} - \mathrm{scal}\,\mathbf{a})$ is taken into account, where $\mathbf{a}$ is an arbitrary quaternion, $\mathrm{scal}\,\mathbf{a}$ is the scalar part of the quaternion $\mathbf{a}$.

The scalar part of the quaternion $(\lambda_i \circ i \circ \dot{\bar{\lambda}}_i)$, by virtue of (2.5), is equal to zero:

$$\mathrm{scal}(\boldsymbol{\lambda}_i \circ \mathbf{i} \circ \dot{\bar{\boldsymbol{\lambda}}}_i) = -\lambda_{i1}\dot{\lambda}_{i0} + \lambda_{i0}\dot{\lambda}_{i1} + \lambda_{i3}\dot{\lambda}_{i2} - \lambda_{i2}\dot{\lambda}_{i3} = \frac{1}{2}\omega_{i1} = 0. \tag{2.10}$$

Therefore, from (2.9) and (2.10) we have

$$\mathbf{V}_i = \dot{\mathbf{R}}_i = \boldsymbol{\lambda}_i \circ \mathbf{i} \circ (\dot{r}_i \bar{\boldsymbol{\lambda}}_i + 2r_i \dot{\bar{\boldsymbol{\lambda}}}_i), \quad i = 0,\, 1. \tag{2.11}$$

Differentiating (2.11) with respect to time t , using the relation (2.10) and substituting the result of differentiation, as well as equality (2.7) into equations (2.1), (2.2), we obtain

$$\boldsymbol{\lambda}_0 \circ \mathbf{i} \circ [2r_0 \ddot{\bar{\boldsymbol{\lambda}}}_0 + 4\dot{r}_0 \dot{\bar{\boldsymbol{\lambda}}}_0 - 2r_0 \mathbf{i} \circ \bar{\boldsymbol{\lambda}}_0 \circ \dot{\boldsymbol{\lambda}}_0 \circ \mathbf{i} \circ \dot{\bar{\boldsymbol{\lambda}}}_0 + (\ddot{r}_0 + f m_0 r_0^{-2}) \bar{\boldsymbol{\lambda}}_0] = -f m_1 (r_1^{-3} \mathbf{R}_1 + r_{01}^{-3} \mathbf{R}_{01}) + \mathbf{P}, \tag{2.12}$$

$$\boldsymbol{\lambda}_1 \circ \mathbf{i} \circ [2r_1 \ddot{\bar{\boldsymbol{\lambda}}}_1 + 4\dot{r}_1 \dot{\bar{\boldsymbol{\lambda}}}_1 - 2r_1 \mathbf{i} \circ \bar{\boldsymbol{\lambda}}_1 \circ \dot{\boldsymbol{\lambda}}_1 \circ \mathbf{i} \circ \dot{\bar{\boldsymbol{\lambda}}}_1 + (\ddot{r}_1 + f m_1 r_1^{-2}) \bar{\boldsymbol{\lambda}}_1] = -f m_0 (r_0^{-3} \mathbf{R}_0 + r_{01}^{-3} \mathbf{R}_{10}) + \mathbf{P}. \tag{2.13}$$

Using the kinematic quaternion equation of the rotational motion of the coordinate system $M_i X_i' Y_i' Z_i'$ [13–16]

$$\frac{2d\boldsymbol{\lambda}_i}{dt} = \boldsymbol{\lambda}_i \circ \boldsymbol{\Omega}_i, \quad \boldsymbol{\Omega}_i = \omega_{i1}\mathbf{i} + \omega_{i2}\mathbf{j} + \omega_{i3}\mathbf{k}, \quad i = 0,\, 1$$

and equality (2.5), it can be shown that

$$\mathbf{i} \circ \bar{\boldsymbol{\lambda}}_i \circ \dot{\boldsymbol{\lambda}}_i \circ \mathbf{i} \circ \dot{\bar{\boldsymbol{\lambda}}}_i = (\dot{\boldsymbol{\lambda}}_i \circ \dot{\bar{\boldsymbol{\lambda}}}_i) \bar{\boldsymbol{\lambda}}_i, \quad \dot{\boldsymbol{\lambda}}_i \circ \dot{\bar{\boldsymbol{\lambda}}}_i = \sum_{j=0}^{3} \dot{\lambda}_{ij}^2 = \frac{1}{4}\omega_i^2, \quad \omega_i = |\boldsymbol{\omega}_i|, \quad i = 0,\, 1. \tag{2.14}$$

Taking into account (2.14) from (2.12) and (2.13), we obtain the equations

$$2r_0 \ddot{\bar{\boldsymbol{\lambda}}}_0 + 4\dot{r}_0 \dot{\bar{\boldsymbol{\lambda}}}_0 + (\ddot{r}_0 + f m_0 r_0^{-2} - \frac{1}{2} r_0 \omega_0^2) \bar{\boldsymbol{\lambda}}_0 = \mathbf{i} \circ \bar{\boldsymbol{\lambda}}_0 \circ [f m_1 (r_1^{-3}\, \mathbf{R}_1 + r_{01}^{-3}\, \mathbf{R}_{01}) - \mathbf{P}], \tag{2.15}$$

$$2r_1 \ddot{\bar{\boldsymbol{\lambda}}}_1 + 4\dot{r}_1 \dot{\bar{\boldsymbol{\lambda}}}_1 + (\ddot{r}_1 + f m_1 r_1^{-2} - \frac{1}{2} r_1 \omega_1^2) \bar{\boldsymbol{\lambda}}_1 = \mathbf{i} \circ \bar{\boldsymbol{\lambda}}_1 \circ [f m_0 (r_0^{-3}\, \mathbf{R}_0 + r_{01}^{-3}\, \mathbf{R}_{10}) - \mathbf{P}]. \tag{2.16}$$

In the quaternion equations (2.15) and (2.16) we pass from the Rodrigues–Hamilton parameters $\lambda_{ij}\,(i = 0, 1;\, j = 0, 1, 2, 3)$ to the Kustaanheimo–Stiefel variables u_{ij} [17–19] by using the formulas [5, 6, 10, 11]

$$\lambda_{i0} = r_i^{-1/2} u_{i0}, \quad \lambda_{ik} = -r_i^{-1/2} u_{ik}, \quad i = 0,\, 1; \quad k = 1,\, 2,\, 3. \tag{2.17}$$

The formulas (2.17) in the quaternion form are:

$$\bar{\boldsymbol{\lambda}}_i = r_i^{-1/2} \mathbf{u}_i, \quad \bar{\boldsymbol{\lambda}}_i = \lambda_{i0} - \lambda_{i1}\mathbf{i} - \lambda_{i2}\mathbf{j} - \lambda_{i3}\mathbf{k}, \quad \mathbf{u}_i = u_{i0} + u_{i1}\mathbf{i} + u_{i2}\mathbf{j} + u_{i3}\mathbf{k}, \quad i = 0,\, 1. \tag{2.18}$$

Substituting the relations (2.18) into equations (2.15) and (2.16), we obtain the quaternion differential equations of the perturbed spatial restricted three-body problem in Kustaanheimo–Stiefel variables:

$$\ddot{\mathbf{u}}_0 + r_0^{-1} \dot{r}_0 \dot{\mathbf{u}}_0 - \frac{1}{2} r_0^{-2} \Big(\frac{1}{2} \dot{r}_0^2 + \frac{1}{2} r_0^2 \omega_0^2 - f m_0 r_0^{-1} \Big) \mathbf{u}_0 = \frac{1}{2} r_0^{-1} \mathbf{i} \circ \mathbf{u}_0 \circ [f m_1 (r_1^{-3} \mathbf{R}_1 + r_{01}^{-3} \mathbf{R}_{01}) - \mathbf{P}], \tag{2.19}$$

$$\ddot{\mathbf{u}}_1 + r_1^{-1} \dot{r}_1 \dot{\mathbf{u}}_1 - \frac{1}{2} r_1^{-2} \Big(\frac{1}{2} \dot{r}_1^2 + \frac{1}{2} r_1^2 \omega_1^2 - f m_1 r_1^{-1} \Big) \mathbf{u}_1 = \frac{1}{2} r_1^{-1} \mathbf{i} \circ \mathbf{u}_1 \circ [f m_0 (r_0^{-3} \mathbf{R}_0 + r_{01}^{-3} \mathbf{R}_{10}) - \mathbf{P}]. \tag{2.20}$$

Here, the quaternions $\mathbf{R}_i$, $\mathbf{R}_{01}$, $\mathbf{R}_{10}$, and $\mathbf{P}$ that appear in the right-hand sides of equations (2.19), (2.20) are defined by relations (2.7), (2.4).

Note that in equations (2.19) and (2.20) obtained as a result of the transition from the Rodrigues–Hamilton parameters λ_{ij} to the Kustaanheimo–Stiefel variables u_{ij} in equations (2.15) and (2.16), the terms containing the second derivatives $\ddot{r}_i$ of the modules r_i of vectors $\mathbf{r}_i$ have been reduced; the Kepler energies of the motion of a point$M$ in the coordinate systems $M_0 X_0 Y_0 Z_0$ and $M_1 X_1 Y_1 Z_1$ (expressions in round brackets before the variables $\mathbf{u}_0$ and $\mathbf{u}_1$) have been distinguished in explicit form.

3. LOCAL AND GLOBAL REGULAR DIFFERENTIAL QUATERNION EQUATIONS OF THE PERTURBED SPATIAL RESTRICTED THREE BODY PROBLEM THAT USE KEPLER ENERGIES AS ADDITIONAL VARIABLES

In the equations (2.19), (2.20) we pass from the independent variable t to the new independent variables $\tau_i\,(i = 0,\, 1)$ using the formulas

$$dt = r_i d\tau_i, \quad \frac{d^2}{dt^2} = r_i^{-2} \frac{d^2}{d\tau_i^2} - r_i^{-3} \frac{dr_i}{d\tau_i} \frac{d}{d\tau_i}, \quad i = 0,\, 1. \tag{3.1}$$

We obtain

$$\frac{d^2\mathbf{u}_0}{d\tau_0^2} - \frac{1}{2}h_0^*\mathbf{u}_0 = \frac{1}{2}r_0\mathbf{i}\circ\mathbf{u}_0\circ[fm_1(r_1^{-3}\,\mathbf{R}_1 + r_{01}^{-3}\,\mathbf{R}_{01}) - \mathbf{P}], \tag{3.2}$$

$$\frac{d^2\mathbf{u}_1}{d\tau_1^2} - \frac{1}{2}h_1^*\mathbf{u}_1 = \frac{1}{2}r_1\mathbf{i}\circ\mathbf{u}_1\circ[fm_0(r_0^{-3}\,\mathbf{R}_0 + r_{01}^{-3}\mathbf{R}_{10}) - \mathbf{P}]. \tag{3.3}$$

Here $h_i^*(i = 0, 1)$ are the Kepler energies defined by the relations

$$h_i^* = \frac{1}{2}\dot{r}_i^2 + \frac{1}{2}r_i^2\omega_i^2 - fm_ir_i^{-1} = \frac{1}{2}v_i{}^2 - fm_ir_i^{-1}, \quad i = 0,\, 1. \tag{3.4}$$

It is seen that in this transition to the new independent variables τ_i, the summands of the left-hand sides of equations (3.2) and (3.3), that contain the first derivatives with respect to the independent variable τ_i from the quaternion variable u_i, are reduced, and each of equations (3.2) and (3.3) becomes regular with respect to distances r_0 and r_1 respectively , that is, they are local regular ones.

In the equations (3.2) and (3.3) we exclude quaternions R_1 and R_0 by using quaternion relations

$$\mathbf{R}_1 = \mathbf{R}_0 - \mathbf{R}_{01}, \quad \mathbf{R}_0 = \mathbf{R}_1 + \mathbf{R}_{01},$$
$$\mathbf{i}\circ\mathbf{u}_0\circ\mathbf{R}_1 = -r_0\mathbf{u}_0 - \mathbf{i}\circ\mathbf{u}_0\circ\mathbf{R}_{01}, \quad \mathbf{i}\circ\mathbf{u}_1\circ\mathbf{R}_0 = -r_1\mathbf{u}_1 + \mathbf{i}\circ\mathbf{u}_1\circ\mathbf{R}_{01},$$

we get

$$\frac{d^2\mathbf{u}_0}{d\tau_0^2} - \frac{1}{2}h_0^*\mathbf{u}_0 = -\frac{1}{2}fm_1r_0^2r_1^{-3}\mathbf{u}_0 + \frac{1}{2}r_0\mathbf{i}\circ\mathbf{u}_0\circ[fm_1(r_{01}^{-3} - r_1^{-3})\mathbf{R}_{01} - \mathbf{P}], \tag{3.5}$$

$$\frac{d^2\mathbf{u}_1}{d\tau_1^2} - \frac{1}{2}h_1^*\mathbf{u}_1 = -\frac{1}{2}fm_0r_1^2r_0^{-3}\mathbf{u}_1 + \frac{1}{2}r_1\mathbf{i}\circ\mathbf{u}_1\circ[fm_0(r_0^{-3} - r_{01}^{-3})\,\mathbf{R}_{01} - \mathbf{P}], \tag{3.6}$$

where quaternions $\mathbf{R}_{01}$ and $\mathbf{P}$ are defined by the ratios

$$\mathbf{R}_{01} = x_{01}\mathbf{i} + y_{01}\mathbf{j} + z_{01}\mathbf{k}, \quad \mathbf{P} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$$

The equations (3.5) and (3.6) coincide with local regular quaternion equations (7.10) and (7.11) of the perturbed spatial restricted three-body problem in the Kustaanheimo–Stiefel variables that were obtained in the first part of this work [2] in a different way.

The quantities h_0^* and h_1^* (Kepler energies), that appear in these equations and are defined by relations (3.4), are considered as additional variables. These variables satisfy the differential equations (7.14), (7.15) [2]:

$$\frac{dh_0^*}{dt} = -fm_1r_1^{-3}r_0\dot{r}_0 + fm_1(r_1^{-3} - r_{01}^{-3})(v_0\cdot r_{01}) + v_0\cdot p,$$
$$\frac{dh_1^*}{dt} = -fm_0r_0^{-3}r_1\dot{r}_1 + fm_0(r_{01}^{-3} - r_0^{-3})(v_1\cdot r_{01}) + v_1\cdot p,$$

which (after the transition to the new independent variables τ_i) take the form

$$\frac{dh_0^*}{d\tau_0} = -fm_1r_1^{-3}r_0\frac{dr_0}{d\tau_0} + fm_1(r_1^{-3} - r_{01}^{-3})\Big(\frac{d\mathbf{r}_0}{d\tau_0}\cdot\mathbf{r}_{01}\Big) + \frac{d\mathbf{r}_0}{d\tau_0}\cdot\mathbf{p}, \tag{3.7}$$

$$\frac{dh_1^*}{d\tau_1} = -fm_0r_0^{-3}r_1\frac{dr_1}{d\tau_1} + fm_0(r_{01}^{-3} - r_0^{-3})\Big(\frac{d\mathbf{r}_1}{d\tau_1}\cdot\mathbf{r}_{01}\Big) + \frac{d\mathbf{r}_1}{d\tau_1}\cdot\mathbf{p}. \tag{3.8}$$

In these equations and further, the center point is the symbol of the scalar product.

In the equations (3.5), (3.6) and (3.7), (3.8)

$$
\begin{aligned}
&r_i = u_{i0}^2 + u_{i1}^2 + u_{i2}^2 + u_{i3}^2, \quad r_{01}^2 = x_{01}^2 + y_{01}^2 + z_{01}^2,\\
&\frac{dr_i}{d\tau_i} = 2\Big(u_{i0}\frac{du_{i0}}{d\tau_i} + u_{i1}\frac{du_{i1}}{d\tau_i} + u_{i2}\frac{du_{i2}}{d\tau_i} + u_{i3}\frac{du_{i3}}{d\tau_i}\Big),\\
&\frac{d\mathbf{r}_i}{d\tau_i}\cdot \mathbf{r}_{01} = 2x_{01}\Big(u_{i0}\frac{du_{i0}}{d\tau_i} + u_{i1}\frac{du_{i1}}{d\tau_i} - u_{i2}\frac{du_{i2}}{d\tau_i} - u_{i3}\frac{du_{i3}}{d\tau_i}\Big)\\
&\qquad + 2y_{01}\Big(u_{i2}\frac{(du_{i1}}{d\tau_i} + u_{i1}\frac{du_{i2}}{d\tau_i} - u_{i3}\frac{du_{i0}}{d\tau_i} - u_{i0}\frac{du_{i3}}{d\tau_i}\Big)\\
&\qquad + 2z_{01}\Big(u_{i3}\frac{du_{i1}}{d\tau_i} + u_{i1}\frac{du_{i3}}{d\tau_i} + u_{i2}\frac{du_{i0}}{d\tau_i} + u_{i0}\frac{du_{i2}}{d\tau_i}\Big), \quad i = 0,\, 1.
\end{aligned}
\tag{3.9}
$$

The scalar product $(d\mathbf{r}_i/d\tau_i)\cdot\mathbf{p}$ has the form of the third relations from (3.9), in which, one need to take p_1, p_2, p_3 instead of x_{01}, y_{01}, z_{01}, respectively.

Differential equations (3.5) and (3.7) supplemented with the differential equations for time t and "fictitious" time τ_1:

$$
\frac{dt}{d\tau_0} = r_0, \quad \frac{d\tau_1}{d\tau_0} = r_0 r_1^{-1}, \tag{3.10}
$$

as well as by the ratios

$$
r_0 = u_{00}^2 + u_{01}^2 + u_{02}^2 + u_{03}^2, \quad r_1^2 = (x_{01} - x_0)^2 + (y_{01} - y_0)^2 + (z_{01} - z_0)^2, \tag{3.11}
$$

$$
x_0 = u_{00}^2 + u_{01}^2 - u_{02}^2 - u_{03}^2, \quad y_0 = 2(u_{01}u_{02} - u_{00}u_{03}), \quad z_0 = 2(u_{01}u_{03} + u_{00}u_{02}), \tag{3.12}
$$

form the differential equations of motion of a point M that are regular in a neighborhood of a point M_0. They are a system of nonlinear nonstationary differential equations of the eleventh order with respect to the Kustaanheimo—Stiefel variables u_{0j} $(j = 0,\, 1,\, 2,\, 3)$, their first derivatives $du_{0j}/d\tau_0$, the energy variable h_0^*, time t, and variable τ_1.

Differential equations (3.6) and (3.7) supplemented with the differential equations

$$
\frac{dt}{d\tau_1} = r_1, \quad \frac{d\tau_0}{d\tau_1} = r_1 r_0^{-1}, \tag{3.13}
$$

and by the ratios

$$
r_1 = u_{10}^2 + u_{11}^2 + u_{12}^2 + u_{13}^2, \quad r_0^2 = (x_{01} - x_1)^2 + (y_{01} - y_1)^2 + (z_{01} - z_1)^2, \tag{3.14}
$$

$$
x_1 = u_{10}^2 + u_{11}^2 - u_{12}^2 - u_{13}^2, \quad y_1 = 2(u_{11}u_{12} - u_{10}u_{13}), \quad z_1 = 2(u_{11}u_{13} + u_{10}u_{12}). \tag{3.15}
$$

form differential equations of motion of a point M that are regular in a neighborhood of a point M_1. They are a system of nonlinear nonstationary differential equations of the eleventh order with respect to the Kustaanheimo—Stiefel variables u_{1j} $(j = 0,\, 1,\, 2,\, 3)$, their first derivatives $du_{1j}/d\tau_1$, the energy variable h_1^*, time t, and variable τ_0.

These sets of differential equations of the perturbed spatial restricted three-body problem allow us to construct a regular algorithm for integrating these equations [2]. The equations (3.5), (3.7), (3.10)–(3.12) of this problem, supplemented with the relations (3.9) (for $i = 0$) , are used in this algorithm when studying the motion of a point M in a neighborhood of a point M_0 (when the distances r_0 and r_1 satisfy the inequality $m_1 {r_0}^2 \leq m_0 r_1^2$), and the equations (3.6), (3.8), (3.13)–(3.15) of this problem that are supplemented with the relations (3.9) (for $i = 1$) are used in studying movement of point M in a neighborhood of a point M_1 (when the distances r_1 and r_0 satisfy the inequality $m_0 r_1^2 < m_1 r_0^2$).

Remark 1. In the described algorithm for integrating the constructed regular differential equations of the perturbed spatial restricted three-body problem we assume that the projections $x_{01}, y_{01}, z_{01}$of the vector $\mathbf{r}_{01}$ on the axes of the inertial coordinate system (coordinates of a point M_1 in the coordinate system $M_0X_0Y_0Z_0$)that are included in this algorithm are known functions of time t. In particular, this is the case of the spatial restricted circular three body problem. In the general case, to find the projections x_{01}, y_{01}, z_{01} in the differential equations systems (3.5), (3.7) and (3.6), (3.8), it is necessary to additionally include the vector differential equation (1.3) having made a transition to a new independent variable τ_0 or τ_1 by using the second formula of (3.1). However, in the differential equation obtained from (1.3), a term $-(1/r_i)(dr_i/d\tau_i)(d\mathbf{r}_{01}/d\tau_i)$ suspicious for irregularity appears. It has a distance r_i in the denominator. Passing in this term from the “time” τ_i to the time t using the formula $dt = r_i d\tau_i$, we get

$$-\frac{1}{r_i}\frac{dr_i}{d\tau_i}\frac{d\mathbf{r}_{01}}{d\tau_i} = -r_i\frac{dr_i}{dt}\frac{d\mathbf{r}_{01}}{dt}.$$

From this relation, it can be seen that for $r_i = 0$ the indicated term does not tend to the infinity in time t, but is equal to zero.

Nevertheless, it seems that in the general case the vector differential equation (1.3) is appropriate to integrate separately over the time t with a variable step equal to the difference of time values t that is calculated at the previous and current moments of “time” τ_i (the current moment t is calculated as a result of integrating the first equation of (3.10) or (3.13) over the “time” τ_i).

Remark 2. To use the constructed regular differential equations of the perturbed spatial restricted three-body problem, it is necessary to determine their initial integration conditions, i.e. It is necessary to determine the initial values of the Kustahanheimo—Stiefel variables u_{ij} $(j = 0,\ 1,\ 2,\ 3)$ and their first derivatives $du_{ij}/d\tau_i$ via the given initial values of Cartesian coordinates x_i, y_i, z_i of a point M in the coordinate system $M_iX_iY_iZ_i$ and the initial values of the projections $\dot{x}_i,\ \dot{y}_i,\ \dot{z}_i$ of the velocity vector v_i of a point M in the coordinate system $M_iX_iY_iZ_i$ on the axis of the same coordinate system. The many-valued algorithms for solving this problem (problems with initial conditions) were proposed in [19, 5]. A single-valued algorithm for solving a problem with initial conditions was proposed by the author of the article in [20] (see also [9—11]). It is based on the relations (7.25)–(7.30) of the first part of this work [2].

The radius vector $\mathbf{r}_i$ characterizing the position of a point M in the coordinate system $M_iX_iY_iZ_i$, its modulus r_i, and the velocity vector $\mathbf{v}_i$ of a point M in this coordinate system are found in terms of variables u_i and $du_i/d\tau_i$ in accordance with quaternion formulas [6] (see also [9—11]):

$$\mathbf{R}_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k} = \bar{\mathbf{u}}_i \circ \mathbf{i} \circ \mathbf{u}_i, \quad r_i = \mathbf{u}_i \circ \bar{\mathbf{u}}_i = u_{i0}^2 + u_{i1}^2 + u_{i2}^2 + u_{i3}^2, \quad i = 0,\ 1, \tag{3.16}$$

$$\mathbf{V}_i = v_{i1}\mathbf{i} + v_{i2}\mathbf{j} + v_{i3}\mathbf{k} = \frac{d\mathbf{R}_i}{dt} = 2\bar{\mathbf{u}}_i \circ \mathbf{i} \circ \frac{d\mathbf{u}_i}{dt} = 2r_i^{-1}\bar{\mathbf{u}}_i \circ \mathbf{i} \circ \frac{d\mathbf{u}_i}{d\tau_i}, \quad i = 0,\ 1, \tag{3.17}$$

The formulas (3.16) and (3.17) in the scalar form are

$$\begin{aligned}
&x_i = u_{i0}^2 + u_{i1}^2 - u_{i2}^2 - u_{i3}^2, \quad y_i = 2(u_{i1}u_{i2} - u_{i0}u_{i3}), \quad z_i = 2(u_{i1}u_{i3} + u_{i0}u_{i2}), \quad i = 0,\ 1,\\
&v_{i1} = \dot{x}_i = 2(u_{i0}\dot{u}_{i0} + u_{i1}\dot{u}_{i1} - u_{i2}\dot{u}_{i2} - u_{i3}\dot{u}_{i3})\\
&\qquad = 2r_i^{-1}\Big(u_{i0}\frac{du_{i0}}{d\tau_i} + u_{i1}\frac{du_{i1}}{d\tau_i} - u_{i2}\frac{du_{i2}}{d\tau_i} - u_{i3}\frac{du_{i3}}{d\tau_i}\Big),\\
&v_{i2} = \dot{y}_i = 2(u_{i2}\dot{u}_{i1} + u_{i1}\dot{u}_{i2} - u_{i3}\dot{u}_{i0} - u_{i0}\dot{u}_{i3})\\
&\qquad = 2r_i^{-1}\Big(u_{i2}\frac{du_{i1}}{d\tau_i} + u_{i1}\frac{du_{i2}}{d\tau_i} - u_{i3}\frac{du_{i0}}{d\tau_i} - u_{i0}\frac{du_{i3}}{d\tau_i}\Big).\\
&v_{i3} = \dot{z}_i = 2(u_{i3}\dot{u}_{i1} + u_{i1}\dot{u}_{i3} + u_{i2}\dot{u}_{i0} + u_{i0}\dot{u}_{i2})\\
&\qquad = 2r_i^{-1}\Big(u_{i3}\frac{du_{i1}}{d\tau_i} + u_{i1}\frac{du_{i3}}{d\tau_i} + u_{i2}\frac{du_{i0}}{d\tau_i} + u_{i0}\frac{du_{i2}}{d\tau_i}\Big), \quad i = 0,\ 1,
\end{aligned}$$

These formulas allow one to find the Cartesian coordinates $x_i,\ y_i,\ z_i$ of a point M in the coordinate system $M_iX_iY_iZ_i$ and the projection of the velocity of a point M in the coordinate system $M_iX_iY_iZ_i$ on the axis of this coordinate system via the variables u_{ij} and their derivatives $\dot{u}_{ij}$ or $(du_{ij}/d\tau_i)$.

From the equations (3.5), (3.7) and (3.6), (3.8), local regular equations (8.1), (8.2) and (8.3), (8,4) [2] of the perturbed spatial restricted three-body problem can be obtained using the total energy h_0 and h_1 of motion of a point M in the coordinate systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$ as the additional variables. They are determined by the relations

$$h_0 = h_0^* - \frac{fm_1}{r_1} = \frac{1}{2}v_0^2 - \frac{fm_0}{r_0} - \frac{fm_1}{r_1}, \tag{3.18}$$

$$h_1 = h_1^* - \frac{fm_0}{r_0} = \frac{1}{2}v_1^2 - \frac{fm_0}{r_0} - \frac{fm_1}{r_1}, \tag{3.19}$$

$$v_0 = |\mathbf{v}_0|, \quad \mathbf{v}_0 = \frac{d\mathbf{r}_0}{dt}; \quad v_1 = |\mathbf{v}_1|, \quad \mathbf{v}_1 = \frac{d\mathbf{r}_1}{dt}.$$

where $\mathbf{v}_0$ and $\mathbf{v}_1$ are the velocity vectors of a point M in the coordinate systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$, respectively.

The usage of these equations to study the motion of a point M in a neighborhood of a point M_0 or M_1 can be carried out by the same methodology as using equations (3.5), (3.7), (3.10) and (3.6), (3.8), (3.13) containing Kepler energies h_0^* and h_1^*.

From equations (3.5), (3.7) and (3.6), (3.8) the local regular equations (8.5), (8.6) and (8.7), (8.8) [2] of the perturbed spatial restricted circular three-body problem can be obtained using as additional variables H_0 and H_1 (we call them Jacobi variables) defined by the relations (3.15) and (3.16) [2]:

$$\begin{aligned} H_0 &= h_0 + fm_1 r_{01}^{-3}(\mathbf{r}_0 \cdot \mathbf{r}_{01}) + n(y_0\dot{x}_0 - x_0\dot{y}_0) \\ &= \frac{1}{2}(\dot{x}_0{}^2 + \dot{y}_0{}^2 + \dot{z}_0{}^2) - \frac{fm_0}{r_0} - \frac{fm_1}{r_1} + \frac{fm_1}{r_{01}^3}(x_0x_{01} + y_0y_{01}) + n(y_0\dot{x}_0 - x_0\dot{y}_0), \end{aligned} \tag{3.20}$$

$$\begin{aligned} H_1 &= h_1 + fm_0 r_{01}^{-3}(\mathbf{r}_1 \cdot \mathbf{r}_{10}) + n(y_1\dot{x}_1 - x_1\dot{y}_1) \\ &= \frac{1}{2}(\dot{x}_1{}^2 + \dot{y}_1{}^2 + \dot{z}_1{}^2) - \frac{fm_0}{r_0} - \frac{fm_1}{r_1} + \frac{fm_0}{r_{01}^3}(x_1x_{10} + y_1y_{10}) + n(y_1\dot{x}_1 - x_1\dot{y}_1). \end{aligned} \tag{3.21}$$

The variables H_0 and H_1 satisfy the differential equations (3.13), (3.14) [2]:

$$\frac{dH_0}{dt} = \frac{d\mathbf{r}_0}{dt} \cdot \mathbf{p} + n(y_0p_x - x_0p_y), \tag{3.22}$$

$$\frac{dH_1}{dt} = \frac{d\mathbf{r}_1}{dt} \cdot \mathbf{p} + n(y_1p_x - x_1p_y), \quad p_x = p_1, \quad p_y = p_2, \tag{3.23}$$

and are the Jacobi costants of motion for the unperturbed spatial restricted circular three-body problem.

The reduced local regular quaternion equations of the perturbed spatial restricted three-body problem contain the oscillatory forms of quaternion differential equations in the Kustaanheimo–Stiefel variables. We obtain new forms of local and global regular quaternion equations of the perturbed spatial restricted three-body problem and global regular quaternion equations of the perturbed spatial restricted circular three-body problem. They contain normal forms of quaternion differential equations in Kustaanheimo–Stiefel variables.

We introduce new quaternion "velocity" variables

$$\mathbf{s}_i = \dot{\mathbf{u}}_i = \frac{d\mathbf{u}_i}{dt}, \quad i = 0, 1 \quad \mathbf{s}_i = s_{i0} + s_{i1}\mathbf{i} + s_{i2}\mathbf{j} + s_{i3}\mathbf{k}, \quad \dot{\mathbf{u}}_i = \dot{u}_{i0} + \dot{u}_{i1}\mathbf{i} + \dot{u}_{i2}\mathbf{j} + \dot{u}_{i3}\mathbf{k}. \tag{3.24}$$

Let us write the quaternion differential equations (2.19), (2.20) of the perturbed three-body spatial restricted problem in the Kustaanheimo–Stiefel variables by taking into account the notation (3.24) in the normal Cauchy form:

$$\frac{d\mathbf{u}_0}{dt} = \mathbf{s}_0, \tag{3.25}$$

$$\begin{aligned} \frac{d\mathbf{s}_0}{dt} + 2r_0^{-1}(\mathbf{u}_0 \cdot \mathbf{s}_0)\mathbf{s}_0 - \frac{1}{2}r_0^{-2}h_0^*\mathbf{u}_0 &= \frac{1}{2}r_0^{-1}\mathbf{i} \circ \mathbf{u}_0 \circ [fm_1(r_1^{-3}\,\mathbf{R}_1 + r_{01}^{-3}\,\mathbf{R}_{01}) - \mathbf{P}] \\ &= -\frac{1}{2}fm_1r_1^{-3}\mathbf{u}_0 + \frac{1}{2}r_0^{-1}\mathbf{i} \circ \mathbf{u}_0 \circ [fm_1(r_{01}^{-3} - r_1^{-3})\,\mathbf{R}_{01} - \mathbf{P}], \end{aligned} \tag{3.26}$$

$$\frac{d\mathbf{u}_1}{dt} = \mathbf{s}_1, \tag{3.27}$$

$$\begin{aligned} \frac{d\mathbf{s}_1}{dt} + 2r_1^{-1}(\mathbf{u}_1 \cdot \mathbf{s}_1)\mathbf{s}_1 - \frac{1}{2}r_1^{-2}h_1^*\mathbf{u}_1 &= \frac{1}{2}r_1^{-1}\mathbf{i} \circ \mathbf{u}_1 \circ [fm_0(r_0^{-3}\,\mathbf{R}_0 + r_{01}^{-3}\,\mathbf{R}_{10}) - \mathbf{P}] \\ &= -\frac{1}{2}fm_0r_0^{-3}\mathbf{u}_1 + \frac{1}{2}r_1^{-1}\mathbf{i} \circ \mathbf{u}_1 \circ [fm_0(r_0^{-3} - r_{01}^{-3})\,\mathbf{R}_{01} - \mathbf{P}]. \end{aligned} \tag{3.28}$$

Here the scalar products are

$$(\mathbf{u}_i \cdot \mathbf{s}_i) = u_{i0}s_{i0} + u_{i1}s_{i1} + u_{i2}s_{i2} + u_{i3}s_{i3}, \; s_{ij} = \dot{u}_{ij} = \frac{du_{ij}}{dt}, \quad i = 0, 1, \quad j = 0, 1, 2, 3. \tag{3.29}$$

Passing in equations (3.25), (3.26) and (3.27), (3.28) to new independent variables τ_0 and τ_1 by using the formulas $d\tau_0 = r_0^{-2}dt$ and $d\tau_1 = r_1^{-2}dt$ and supplementing the obtained equations with the differential equations for Kepler energies h_0^* and h_1^*, which follow in this case from the equations for these energies given before the equations (3.7) and (3.8), we obtain the following sets of normal local regular quaternion equations of the perturbed spatial restricted three-body problem:

$$\frac{d\mathbf{u}_0}{d\tau_0} = r_0^2\mathbf{s}_0, \tag{3.30}$$

$$\frac{d\mathbf{s}_0}{d\tau_0} + 2r_0(\mathbf{u}_0 \cdot \mathbf{s}_0)\mathbf{s}_0 - \frac{1}{2}h_0^*\mathbf{u}_0 = -\frac{1}{2}fm_1r_0^2r_1^{-3}\mathbf{u}_0 + \frac{1}{2}r_0\mathbf{i} \circ \mathbf{u}_0 \circ [fm_1(r_{01}^{-3} - r_1^{-3})\,\mathbf{R}_{01} - \mathbf{P}], \tag{3.31}$$

$$\frac{dh_0^*}{d\tau_0} = r_0^2[-fm_1r_1^{-3}r_0\frac{dr_0}{dt} + fm_1(r_1^{-3} - r_{01}^{-3})(\frac{d\mathbf{r}_0}{dt} \cdot \mathbf{r}_{01}) + \frac{d\mathbf{r}_0}{dt} \cdot \mathbf{p}], \tag{3.32}$$

$$\frac{dt}{d\tau_0} = r_0^2, \quad \frac{d\tau_1}{d\tau_0} = r_0^2r_1^{-2}, \tag{3.33}$$

$$\frac{d\mathbf{u}_1}{d\tau_1} = r_1^2\mathbf{s}_1, \tag{3.34}$$

$$\frac{d\mathbf{s}_1}{d\tau_1} + 2r_1(\mathbf{u}_1 \cdot \mathbf{s}_1)\mathbf{s}_1 - \frac{1}{2}h_1^*\mathbf{u}_1 = -\frac{1}{2}fm_0r_0^{-3}r_1^2\mathbf{u}_1 + \frac{1}{2}r_1\mathbf{i} \circ \mathbf{u}_1 \circ [fm_0(r_0^{-3} - r_{01}^{-3})\,\mathbf{R}_{01} - \mathbf{P}], \tag{3.35}$$

$$\frac{dh_1^*}{d\tau_1} = r_1^2[-fm_0r_0^{-3}r_1\frac{dr_1}{dt} + fm_0(r_{01}^{-3} - r_0^{-3})(\frac{d\mathbf{r}_1}{dt} \cdot \mathbf{r}_{01}) + \frac{d\mathbf{r}_1}{dt} \cdot \mathbf{p}], \tag{3.36}$$

$$\frac{dt}{d\tau_1} = r_1^2, \quad \frac{d\tau_0}{d\tau_1} = r_0^{-2}r_1^2, \tag{3.37}$$

$$\begin{aligned} r_i &= u_{i0}^2 + u_{i1}^2 + u_{i2}^2 + u_{i3}^2, \quad r_{01}^2 = x_{01}^2 + y_{01}^2 + z_{01}^2, \\ \frac{dr_i}{dt} &= 2(u_{i0}s_{i0} + u_{i1}s_{i1} + u_{i2}s_{i2} + u_{i3}s_{i3}), \\ \frac{d\mathbf{r}_i}{dt} \cdot \mathbf{r}_{01} &= 2x_{01}(u_{i0}s_{i0} + u_{i1}s_{i1} - u_{i2}s_{i2} - u_{i3}s_{i3}) + 2y_{01}(u_{i2}s_{i1} + u_{i1}s_{i2} \\ &\quad - u_{i3}s_{i0} - u_{i0}s_{i3}) + 2z_{01}(u_{i3}s_{i1} + u_{i1}s_{i3} + u_{i2}s_{i0} + u_{i0}s_{i2}), \\ \frac{d\mathbf{r}_i}{dt} \cdot \mathbf{p} &= 2p_1(u_{i0}s_{i0} + u_{i1}s_{i1} - u_{i2}s_{i2} - u_{i3}s_{i3}) + 2p_2(u_{i2}s_{i1} + u_{i1}s_{i2} \\ &\quad - u_{i3}s_{i0} - u_{i0}s_{i3}) + 2p_3(u_{i3}s_{i1} + u_{i1}s_{i3} + u_{i2}s_{i0} + u_{i0}s_{i2}), \quad i = 0,\, 1. \end{aligned} \tag{3.38}$$

The method of using normal forms of local regular equations (3.30)–(3.33) or (3.34)–(3.37), supplemented with the relations (3.29), (3.38), to study the motion of a point M in a neighborhood of a point M_0 or M_1 can be performed by the same methodology , as the use of oscillatory forms of local regular equations (3.5), (3.7), (3.10) and (3.6), (3.8), (3.13). The main difference between normal forms of local regular equations and oscillatory forms is the use of various independent variables. If in normal forms of local regular equations time transformations, that contain raised to the second power distances from a body of negligibly small mass to two bodies of finite mass, are used, then in oscillatory forms time transformations containing the first powers of these distances are used. Note also that the left - hand sides of equations (3.31) and (3.35) of normal forms are more complicated than the left-hand sides of equations (3.5) and (3.6) of oscillatory forms because they contain additional terms $2r_0(\mathbf{u}_0 \cdot s_0)s_0$ and $2r_1(\mathbf{u}_1 \cdot s_1)s_1$, and the right hand sides of these equations are the same. In addition, in the right-hand sides of equations (3.32) and (3.36) for Kepler energies that complement the normal forms of the equations, in contrast to equations (3.7) and (3.8) for Kepler energies that complement the oscillatory forms of the equations, the factors r_0^2 and r_1^2 are additionally included. Therefore, the reduced normal forms of local regular equations (3.30)–(3.33) and (3.34)–(3.37) of the perturbed spatial restricted three-body problem are simpler than the oscillatory forms of local regular equations (3.5), (3.7), (3.10) and (3.6), (3.8), (3.13) of this problem. At the same time, the use of normal forms of equations allows one to obtain normal forms of global regular equations of the perturbed spatial restricted three-body problem.

Passing in equations (3.25), (3.26) and (3.27), (3.28) to a new independent variable τ by using the formula $d\tau = r_0^{-2} r_1^{-2} dt$ and complementing the equations obtained with differential equations for Kepler energies h_0^* and h_1^* that follow from the equations for these energies and are given before the equations (3.7) and (3.8), we obtain other sets of local regular quaternion equations of the perturbed spatial restricted three-body problem (3.39)–(3.42) and (3.43)–(3.45), (3.42) in the normal Cauchy form:

$$\frac{du_0}{d\tau} = r_0^2 r_1^2 s_0, \tag{3.39}$$

$$\frac{ds_0}{d\tau} + 2r_0 r_1^2 (u_0 \cdot s_0) s_0 - \frac{1}{2} r_1^2 h_0^* u_0 \tag{3.40}$$
$$= -\frac{1}{2} f m_1 r_0^2 r_1^{-1} u_0 + \frac{1}{2} r_0 r_1^2 i \circ u_0 \circ [f m_1 (r_{01}^{-3} - r_1^{-3}) R_{01} - P],$$

$$\frac{dh_0^*}{d\tau} = r_0^2 \Big[- f m_1 r_1^{-1} r_0 \frac{dr_0}{dt} + f m_1 (r_1^{-1} - r_1^2 r_{01}^{-3}) \Big(\frac{dr_0}{dt} \cdot r_{01} \Big) + r_1^2 \frac{dr_0}{dt} \cdot p \Big], \tag{3.41}$$

$$\frac{dt}{d\tau} = r_0^2 r_1^2, \tag{3.42}$$

$$\frac{du_1}{d\tau} = r_0^2 r_1^2 s_1, \tag{3.43}$$

$$\frac{ds_1}{d\tau} + 2r_0^2 r_1 (u_1 \cdot s_1) s_1 - \frac{1}{2} r_0^2 h_1^* u_1 \tag{3.44}$$
$$= -\frac{1}{2} f m_0 r_0^{-1} r_1^2 u_1 + \frac{1}{2} r_0^2 r_1 i \circ u_1 \circ [f m_0 (r_0^{-3} - r_{01}^{-3}) R_{01} - P],$$

$$\frac{dh_1^*}{d\tau} = r_1^2 \Big[- f m_0 r_0^{-1} r_1 \frac{dr_1}{dt} + f m_0 (r_0^2 r_{01}^{-3} - r_0^{-1}) \Big(\frac{dr_1}{dt} \cdot r_{01} \Big) + r_0^2 \frac{dr_1}{dt} \cdot p \Big]. \tag{3.45}$$

The procedure of using normal forms of local regular equations (3.39)–(3.42) or (3.43)–(3.45), (3.42), supplemented with the relations (3.29), (3.38), to study the motion of a point M in a neighborhood of a point M_0 or M_1 can be carry out by the same methodology as the use of oscillatory forms of local regular equations (3.5), (3.7), (3.10) and (3.6), (3.8). (3.13) (as well as the use of normal forms of local regular equations (3.30)–(3.33) and (3.34)–(3.37)). Normal forms of local regular equations (3.39)–(3.42) and (3.43)–(3.45), (3.42) differ from the normal forms of local regular equations (3.30)–(3.33) and (3.34)–(3.37) in greater complexity and use more complex time transformations containing the product of the raised to the second power distances from the body of negligibly small mass to two bodies of finite mass. However, in normal forms of local regular equations (3.39)–(3.42) and (3.43)–(3.45), (3.42), in contrast to the normal forms of local regular equations (3.30)–(3.33) and (3.34)–(3.37) and oscillatory forms of local regular equations (3.5), (3.7), (3.10) and (3.6), (3.8). (3.13), we use not two different independent variables τ_0 and τ_1, but one independent variableτ, that is their advantage.

Let us pass in equations (3.25), (3.26) and (3.27), (3.28) to the new independent variable τ by using the formula $d\tau = r_0^{-3} r_1^{-3} dt$ and supplement the obtained equations with the differential equations for Kepler energies h_0^* and h_1^*, that follow from the equations given before the equations (3.7) and (3.8). As a result, we obtain the normal global regular quaternion equations of the perturbed spatial restricted three-body problem (3.46)–(3.49) and (3.50)–(3.52), (3.49):

$$\frac{d\mathbf{u}_0}{d\tau} = r_0^3 r_1^3 \mathbf{s}_0, \tag{3.46}$$

$$\frac{d\mathbf{s}_0}{d\tau} + 2r_0^2 r_1^3 (\mathbf{u}_0 \cdot \mathbf{s}_0) \mathbf{s}_0 - \frac{1}{2} r_0 r_1^3 h_0^* \mathbf{u}_0 \tag{3.47}$$
$$- \frac{1}{2} f m_1 r_0^3 \mathbf{u}_0 + \frac{1}{2} r_0^2 \mathbf{i} \circ \mathbf{u}_0 \circ [f m_1 (r_{01}^{-3} r_1^3 - 1) \mathbf{R}_{01} - r_1^3 \mathbf{P}],$$

$$\frac{dh_0^*}{d\tau} = r_0^3\Big[-fm_1r_0\frac{dr_0}{dt} + fm_1(1 - r_1^3r_{01}^{-3})\Big(\frac{d\mathbf{r}_0}{dt}\cdot\mathbf{r}_{01}\Big) + r_1^3\frac{d\mathbf{r}_0}{dt}\cdot\mathbf{p}\Big], \quad (3.48)$$

$$\frac{dt}{d\tau} = r_0^3r_1^3, \quad (3.49)$$

$$\frac{d\mathbf{u}_1}{d\tau} = r_0^3r_1^3\mathbf{s}_1, \quad (3.50)$$

$$\frac{d\mathbf{s}_1}{d\tau} + 2r_0^3r_1^2(\mathbf{u}_1\cdot\mathbf{s}_1)\mathbf{s}_1 - \frac{1}{2}r_0^3r_1h_1^*\mathbf{u}_1 \quad (3.51)$$
$$= -\frac{1}{2}fm_0r_1^3\mathbf{u}_1 + \frac{1}{2}r_1^2\mathbf{i}\circ\mathbf{u}_1\circ[fm_0(1 - r_{01}^{-3}r_0^3)\mathbf{R}_{01} - r_0^3\mathbf{P}],$$

$$\frac{dh_1^*}{d\tau} = r_1^3\Big[-fm_0r_1\frac{dr_1}{dt} + fm_0(r_0^3r_{01}^{-3} - 1)\Big(\frac{d\mathbf{r}_1}{dt}\cdot\mathbf{r}_{01}\Big) + r_0^3\frac{d\mathbf{r}_1}{dt}\cdot\mathbf{p}\Big]. \quad (3.52)$$

The equations (3.46)–(3.49), supplemented with the relations (3.29), (3.38) and (3.11), (3.12), and also the equations (3.50)–(3.52), (3.49), supplemented with the same relations (3.29), (3.38) and (3.11), (3.12), are two different normal forms of global regular equations of the perturbed spatial restricted three-body problem. They form systems of nonlinear differential equations of tenth order with respect to the Kustaanheimo–Stiefel variables u_{ij} ($j = 0, 1, 2, 3$) (component of a quaternion variable $\mathbf{u}_i$), their first derivatives with respect to time $s_{ij} = du_{ij}/dt$ (component of a quaternion variable $s_i = d\mathbf{u}_i/dt$), Kepler energy h_i^* and time t ($i = 0, 1$). Any of these systems can be separately used as a closed system of differential equations for regular study of the motion of a point M both in a neighborhood of a point M_0 (when distances r_0 and r_1 satisfy the inequality $m_1r_0^2 \le m_0r_1^2$) and to study the motion of a point M in a neighborhood of a point M_1 (when distances r_0 and r_1 satisfy the inequality $m_0r_1^2 < m_1r_0^2$).

Note that in [21] (R. Roman, I. Szucs–Csillik, 2014) a generalization of Levi–Civita regularization in the unperturbed plane restricted three-body problem was given. For the regularization of the equations of the problem, complex Levi–Civita variables and a time transformation $dt = r_0^3r_1^3d\tau$ analogous to (3.49), that contains the product of raised to the third power distances from a body of negligibly small mass to two bodies of attraction with finite mass, were used.

Note also that in an unbounded three-body problem to make a regularization, another time transformation $dt = r_0r_1d\tau$ is often used [22, 23]. It contains the product of the distances from a body of negligibly small mass to two bodies of finite mass. In [22] (S. J. Aarseth, K. Zare, 1974) it is noted that for the purposes of regularization, a time transformation $dt = r_0^3r_1^3d\tau$ containing raised to the third power distances can also be used. However, the authors of [22] point out that when using this time transformation, regularization is not actually achieved, because the new time becomes infinite when two bodies collide. In fairness it must be said that this effect of the infinite growth of the new time also takes place when using time transformations that contain smaller powers of specified distances, including the case when using time transformation $dt = r_0r_1d\tau$ containing the first power of distances. However, this growth takes place to a far less degree. Therefore, first of all, it is of interest to construct such local and global regular equations of the perturbed spatial restricted three-body problem, in which the time transformations containing powers (less than third)of distances from a body of negligibly small mass to two bodies of attraction of finite masses are used.

4. GLOBAL REGULAR QUATERNION DIFFERENTIAL EQUATIONS OF THE PERTURBED SPATIAL RESTRICTED THREE BODY PROBLEM THAT USE THE FULL ENERGY OR JACOBI VARIABLES AS THE ADDITIONAL VARIABLES

4.1. The Normal Form of the Global Regular Equations of the Perturbed Spatial Restricted Three-Body Problem that Use the Total Energy

In the normal local regular equations (3.39)–(3.42) and (3.43)–(3.45), (3.42) of the perturbed spatial restricted three-body problem, we replace the Kepler energies h_0^* and h_1^* by the total energies h_0 and h_1 of motions of the point M in the coordinate systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$, respectively. Total energies are defined by the relations (3.18) and (3.19) and satisfy the differential equations (3.2) and (3.3) [2]:

$$\frac{dh_0}{dt} = -fm_1[r_{01}^{-3}(\mathbf{v}_0\cdot\mathbf{r}_{01}) + r_1^{-3}(\mathbf{v}_{01}\cdot\mathbf{r}_1)] + \mathbf{v}_0\cdot\mathbf{p}, \quad (4.1)$$

$$\frac{dh_1}{dt} = -fm_0[r_{01}^{-3}(\mathbf{v}_1 \cdot \mathbf{r}_{10}) + r_0^{-3}(\mathbf{v}_{10} \cdot \mathbf{r}_0)] + \mathbf{v}_1 \cdot \mathbf{p}, \tag{4.2}$$

$$\mathbf{v}_0 = \frac{d\mathbf{r}_0}{dt}, \quad \mathbf{v}_{01} = \frac{d\mathbf{r}_{01}}{dt}; \quad \mathbf{v}_1 = \frac{d\mathbf{r}_1}{dt}, \quad \mathbf{v}_{10} = \frac{d\mathbf{r}_{10}}{dt}.$$

For this replacement of energies, we transform the right-hand sides of equations (3.40) and (3.44) using the equations

$$r_0\mathbf{i} \circ \mathbf{u}_0 \circ \mathbf{R}_1 = -r_1^2\mathbf{u}_0 + \mathbf{u}_0 \circ \mathbf{R}_{01} \circ \mathbf{R}_1, \quad r_1\mathbf{i} \circ \mathbf{u}_1 \circ \mathbf{R}_0 = -r_0^2\mathbf{u}_1 - \mathbf{u}_1 \circ \mathbf{R}_{01} \circ \mathbf{R}_0.$$

As a result of such transformations from equations (3.39)–(3.45) and (4.1), (4.2) we obtain the following global regular differential equations of the perturbed spatial restricted three-body problem in the normal Cauchy form:

$$\frac{d\mathbf{u}_0}{d\tau} = r_0^2 r_1^2 \mathbf{s}_0 \tag{4.3}$$

$$\frac{d\mathbf{s}_0}{d\tau} + 2r_0 r_1^2(\mathbf{u}_0 \cdot \mathbf{s}_0)\mathbf{s}_0 - \frac{1}{2}r_1^2 h_0 \mathbf{u}_0 \tag{4.4}$$

$$= \frac{1}{2}fm_1 r_1^{-1}\mathbf{u}_0 \circ \mathbf{R}_{01} \circ \mathbf{R}_1 + \frac{1}{2}r_0 r_1^2 \mathbf{i} \circ \mathbf{u}_0 \circ (fm_1 r_{01}^{-3}\mathbf{R}_{01} - \mathbf{P}),$$

$$\frac{dh_0}{d\tau} = r_0^2 r_1^2[-fm_1[r_{01}^{-3}(\mathbf{v}_0 \cdot \mathbf{r}_{01}) + r_1^{-3}(\mathbf{v}_{01} \cdot \mathbf{r}_1)] + \mathbf{v}_0 \cdot \mathbf{p}] \tag{4.5}$$

$$= -fm_1[r_0^2 r_1^2 r_{01}^{-3}(\mathbf{v}_0 \cdot \mathbf{r}_{01}) + r_0^2 r_1^{-1}(\mathbf{v}_{01} \cdot \mathbf{r}_1)] + r_0^2 r_1^2 \mathbf{v}_0 \cdot \mathbf{p},$$

$$\frac{d\mathbf{u}_1}{d\tau} = r_1^2 r_0^2 \mathbf{s}_1 \tag{4.6}$$

$$\frac{d\mathbf{s}_1}{d\tau} + 2r_1 r_0^2(\mathbf{u}_1 \cdot \mathbf{s}_1)\mathbf{s}_1 - \frac{1}{2}r_0^2 h_1 \mathbf{u}_1 \tag{4.7}$$

$$= \frac{1}{2}fm_0 r_0^{-1}\mathbf{u}_1 \circ \mathbf{R}_{10} \circ \mathbf{R}_0 + \frac{1}{2}r_1 r_0^2 \mathbf{i} \circ \mathbf{u}_1 \circ (fm_0 r_{10}^{-3}\mathbf{R}_{10} - \mathbf{P}),$$

$$\frac{dh_1}{d\tau} = r_0^2 r_1^2[-fm_0[r_{01}^{-3}(\mathbf{v}_1 \cdot \mathbf{r}_{10}) + r_0^{-3}(\mathbf{v}_{10} \cdot \mathbf{r}_0)] + \mathbf{v}_1 \cdot \mathbf{p}] \tag{4.8}$$

$$= -fm_0[r_0^2 r_1^2 r_{01}^{-3}(\mathbf{v}_1 \cdot \mathbf{r}_{10}) + r_0^{-1} r_1^2(\mathbf{v}_{10} \cdot \mathbf{r}_0)] + r_0^2 r_1^2 \mathbf{v}_1 \cdot \mathbf{p}.$$

$$\frac{dt}{d\tau} = r_0^2 r_1^2. \tag{4.9}$$

Here $r_{10} = r_{01}$, the quantities $\mathbf{R}_i$ and r_i are determined by the relations (3.16):

$$\mathbf{R}_{01} = x_{01}\mathbf{i} + y_{01}\mathbf{j} + z_{01}\mathbf{k}, \quad \mathbf{R}_{10} = -\mathbf{R}_{01}, \quad \mathbf{P} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k},$$

$$(\mathbf{u}_i \cdot \mathbf{s}_i) = u_{i0}s_{i0} + u_{i1}s_{i1} + u_{i2}s_{i2} + u_{i3}s_{i3}, \quad s_{ij} = \dot{u}_{ij} = \frac{du_{ij}}{dt}, \quad i = 0, 1, \quad j = 0, 1, 2, 3,$$

$$\begin{aligned} \mathbf{v}_i \cdot \mathbf{r}_{01} = -\mathbf{v}_i \cdot \mathbf{r}_{10} = \frac{d\mathbf{r}_i}{dt} \cdot \mathbf{r}_{01} &= 2x_{01}(u_{i0}s_{i0} + u_{i1}s_{i1} - u_{i2}s_{i2} - u_{i3}s_{i3}) \\ &+ 2y_{01}(u_{i2}s_{i1} + u_{i1}s_{i2} - u_{i3}s_{i0} - u_{i0}s_{i3}) + 2z_{01}(u_{i3}s_{i1} + u_{i1}s_{i3} + u_{i2}s_{i0} + u_{i0}s_{i2}) \\ \mathbf{v}_{01} \cdot \mathbf{r}_i = -\mathbf{v}_{10} \cdot \mathbf{r}_i &= \dot{x}_{01}(u_{i0}^2 + u_{i1}^2 - u_{i2}^2 - u_{i3}^2) \\ &+ 2\dot{y}_{01}(u_{i1}u_{i2} - u_{i0}u_{i3}) + 2\dot{z}_{01}(u_{i1}u_{i3} + u_{i0}u_{i2}), \quad i = 0, 1, \\ \frac{d\mathbf{r}_i}{dt} \cdot \mathbf{p} &= 2p_1(u_{i0}s_{i0} + u_{i1}s_{i1} - u_{i2}s_{i2} - u_{i3}s_{i3}) + 2p_2(u_{i2}s_{i1} \\ &+ u_{i1}s_{i2} - u_{i3}s_{i0} - u_{i0}s_{i3}) + 2p_3(u_{i3}s_{i1} + u_{i1}s_{i3} + u_{i2}s_{i0} + u_{i0}s_{i2}), \quad i = 0, 1. \end{aligned} \tag{4.10}$$

In the case of a perturbed restricted circular three-body problem, the derivatives $\dot{x}_{01}$, $\dot{y}_{01}$, $\dot{z}_{01}$ occurring in (4.10) are determined by the relations (3.8), (3.9) [2]:

$$\dot{x}_{01} = -an\sin(nt) = -ny_{01}, \quad \dot{y}_{01} = an\cos(nt) = nx_{01}, \quad \dot{z}_{01} = 0,$$

$$n^2 = \frac{f(m_0 + m_1)}{a^3}, \quad a = |\, \mathbf{r}_{01} \,| \,.$$

The quaternions $r_1^{-1}\mathbf{R}_1$ and $r_0^{-1}\mathbf{R}_0$ appearing in equations (4.4) and (4.7) correspond to the vectors $r_1^{-1}\mathbf{r}_1$ and $r_0^{-1}\mathbf{r}_0$ occurring in equations (4.5) and (4.8). The norms and moduli of these quaternions and vectors are equal to unit. Therefore, the equations (4.3)–(4.9) of the perturbed spatial restricted three-body problem are global and regular. They form a system of normal nonlinear differential equations of the 5^{th} order with respect to four quaternion variables $\mathbf{u}_0$, $\mathbf{s}_0$, $\mathbf{u}_1$, $\mathbf{s}_1$ and three scalar variables: total energies h_0, h_1, and time t. This quaternion system of equations is equivalent to a scalar system of nonlinear differential equations of 19^{th} order with respect to the Kustahanheimo–Stiefel variables u_{ij} $(j = 0, 1, 2, 3)$ (component of a quaternion variable $\mathbf{u}_i$), their first derivatives with respect to time $s_{ij} = du_{ij}/dt$ (component of a quaternion variable $s_i = d\mathbf{u}_i/dt$), total energies h_i and time t $(i = 0, 1)$.

4.2. The Normal and Oscillatory Forms of Global Regular Quaternion Equations of the Perturbed Spatial Restricted Circular Three-Body Problem that Use the Jacobi Variables

Using equations (4.3)–(4.9), we obtain the global regular equations of the perturbed spatial restricted circular three-body problem, in which the Jacobi variables H_0 and H_1 defined by relations (3.20) and (3.21) are used instead of the total energies h_0 and h_1. From the relations (3.20) and (3.21) we express the total energies h_0 and h_1 via the Jacobi variables H_0 and H_1 and substitute the obtained relations

$$\begin{aligned} h_0 &= H_0 - fm_1 r_{01}^{-3}(\mathbf{r}_0 \cdot \mathbf{r}_{01}) - n(y_0\dot{x}_0 - x_0\dot{y}_0), \\ h_1 &= H_1 - fm_0 r_{01}^{-3}(\mathbf{r}_1 \cdot \mathbf{r}_{10}) - n(y_1\dot{x}_1 - x_1\dot{y}_1) \end{aligned} \tag{4.11}$$

into the equations (4.4) and (4.7). In the differential equations (3.22) and (3.23) for the variables H_0 and H_1 we pass from time t to new independent variable τ in accordance with equation (4.9). We use these obtained equations instead of equations (4.5) and (4.8).

As a result, we obtain the following global regular equations of the perturbed spatial restricted circular three-body problem:

$$\frac{d\mathbf{u}_0}{d\tau} = r_0^2 r_1^2 \mathbf{s}_0, \tag{4.12}$$

$$\begin{aligned} &\frac{d\mathbf{s}_0}{d\tau} + 2r_0 r_1^2(\mathbf{u}_0 \cdot \mathbf{s}_0)\mathbf{s}_0 - \frac{1}{2}r_1^2[H_0 - fm_1 r_{01}^{-3}(\mathbf{r}_0 \cdot \mathbf{r}_{01}) - n(y_0\dot{x}_0 - x_0\dot{y}_0)]\mathbf{u}_0 \\ &\qquad = \frac{1}{2}fm_1 r_1^{-1}\mathbf{u}_0 \circ \mathbf{R}_{01} \circ \mathbf{R}_1 + \frac{1}{2}r_0 r_1^2 \mathbf{i} \circ \mathbf{u}_0 \circ (fm_1 r_{01}^{-3}\,\mathbf{R}_{01} - \mathbf{P}) \end{aligned} \tag{4.13}$$

$$\frac{dH_0}{d\tau} = \frac{d\mathbf{r}_0}{d\tau} \cdot \mathbf{p} + nr_0^2 r_1^2 (y_0 p_1 - x_0 p_2), \tag{4.14}$$

$$\frac{d\mathbf{u}_1}{d\tau} = r_1^2 r_0^2 \mathbf{s}_1, \tag{4.15}$$

$$\begin{aligned} &\frac{d\mathbf{s}_1}{d\tau} + 2r_1 r_0^2(\mathbf{u}_1 \cdot \mathbf{s}_1)\mathbf{s}_1 - \frac{1}{2}r_0^2[h_1 - fm_0 r_{01}^{-3}(\mathbf{r}_1 \cdot \mathbf{r}_{10}) - n(y_1\dot{x}_1 - x_1\dot{y}_1)]\mathbf{u}_1 \\ &\qquad = \frac{1}{2}fm_0 r_0^{-1}\mathbf{u}_1 \circ \mathbf{R}_{10} \circ \mathbf{R}_0 + \frac{1}{2}r_1 r_0^2 \mathbf{i} \circ \mathbf{u}_1 \circ (fm_0 r_{10}^{-3}\,\mathbf{R}_{10} - \mathbf{P}), \end{aligned} \tag{4.16}$$

$$\frac{dH_1}{d\tau} = \frac{d\mathbf{r}_1}{d\tau} \cdot \mathbf{p} + nr_0^2 r_1^2 (y_1 p_1 - x_1 p_2), \tag{4.17}$$

$$\frac{dt}{d\tau} = r_0^2 r_1^2, \tag{4.18}$$

$$\mathbf{r}_0 \cdot \mathbf{r}_{01} = a[\cos(nt)\,x_0 + \sin(nt)\,y_0], \quad \mathbf{r}_1 \cdot \mathbf{r}_{10} = -a[\cos(nt)\,x_1 + \sin(nt)\,y_1],$$

$$x_i = u_{i0}^2 + u_{i1}^2 - u_{i2}^2 - u_{i3}^2, \quad y_i = 2(u_{i1}u_{i2} - u_{i0}u_{i3}), \quad i = 0, 1,$$

$$-(y_i\dot{x}_i - x_i\dot{y}_i) = c_{iz} = 2r_i[u_{i3}s_{i0} - u_{i2}s_{i1} + u_{i1}s_{i2} - u_{i0}s_{i3}], \quad i = 0, 1.$$

other values have been explained above.

The resulting system of regular quaternion equations (4.12)–(4.18) is equivalent to a scalar system of nonlinear differential equations of 19^{th} order with respect to the Kustahanheimo–Stiefel variables u_{ij} $(j = 0, 1, 2, 3)$ (component of a quaternion variable $\mathbf{u}_i$), their first derivatives with respect to time $s_{ij} = du_{ij}/dt$ (component of a quaternion variable $s_i = d\mathbf{u}_i/dt$), Jacobi variables H_i, and time t $(i = 0, 1)$.

In the case of an unperturbed spatial restricted circular three-body problem (for $p_1 = p_2 = p_3 = 0$) the variables H_i become the Jacobi costants of motion: $H_i(\tau) = H_i(0)$ =const. Therefore, the differential equations (4.14) and (4.17) are out of consideration and the unperturbed spatial restricted circular three-body problem is described by global regular equations (4.12), (4.13), (4.15), (4.16), (4.18), which are equivalent to the scalar system of nonlinear differential equations of the 17^{th} order with respect to the Kustahanheimo–Stiefel variables u_{ij} $(j = 0, 1, 2, 3)$, their first derivatives with respect to time $s_{ij} = du_{ij}/dt$ and time t $(i = 0, 1)$.

In the quaternion differential equations of the perturbed spatial restricted three-body problem in the Kustaanheimo–Stiefel variables (2.19) and (2.20) we pass to the new independent variable τ using the formulas

$$dt = r_0 r_1 d\tau, \quad \frac{d^2}{dt^2} = (r_0 r_1)^{-2}\frac{d^2}{d\tau^2} - (r_0 r_1)^{-3}\frac{d(r_0 r_1)}{d\tau}\frac{d}{d\tau}, \quad i = 0, 1, \tag{4.19}$$

and taking into account the equalities

$$r_0 \mathbf{i} \circ \mathbf{u}_0 \circ \mathbf{R}_1 = -r_1^2 \mathbf{u}_0 + \mathbf{u}_0 \circ \mathbf{R}_{01} \circ \mathbf{R}_1, \;\; r_1 \mathbf{i} \circ \mathbf{u}_1 \circ \mathbf{R}_0 = -r_0^2 \mathbf{u}_1 + \mathbf{u}_1 \circ \mathbf{R}_{10} \circ \mathbf{R}_1.$$

As a result, we obtain the equations

$$\frac{d^2\mathbf{u}_0}{d\tau^2} - \frac{1}{2} r_1^2 h_0 \mathbf{u}_0 = \frac{1}{r_1}\frac{dr_1}{d\tau}\frac{d\mathbf{u}_0}{d\tau} + \frac{1}{2}\frac{fm_1}{r_1}\mathbf{u}_0 \circ \mathbf{R}_{01} \circ \mathbf{R}_1 + \frac{1}{2} r_0 r_1^2 \mathbf{i} \circ \mathbf{u}_0 \circ \Big[\frac{fm_1}{r_{01}^3}\mathbf{R}_{01} - \mathbf{P}\Big], \tag{4.20}$$

$$\frac{d^2\mathbf{u}_1}{d\tau^2} - \frac{1}{2} r_0^2 h_1 \mathbf{u}_1 = \frac{1}{r_0}\frac{dr_0}{d\tau}\frac{d\mathbf{u}_1}{d\tau} + \frac{1}{2}\frac{fm_0}{r_0}\mathbf{u}_1 \circ \mathbf{R}_{10} \circ \mathbf{R}_0 + \frac{1}{2} r_1 r_0^2 \mathbf{i} \circ \mathbf{u}_1 \circ \Big[\frac{fm_0}{r_{10}^3}\mathbf{R}_{10} - \mathbf{P}\Big]. \tag{4.21}$$

Here $h_i (i = 0, 1)$ are the total energies defined by relations (3.18), (3.19) and satisfying the equations (4.1), (4.2).

We show that the terms $(1/r_1)(dr_1/d\tau)(d\mathbf{u}_0/d\tau)$ and $(1/r_0)(dr_0/d\tau)(d\mathbf{u}_1/d\tau)$ of equations (4.20) and (4.21) are finite values for $r_1 \to 0$ and $r_0 \to 0$, consequently, under the condition that the velocities $\mathbf{v}_0$ and $\mathbf{v}_1$ of the point M in the coordinate systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$ are finite. In accordance with (3.17) and (4.19) we have

$$\frac{1}{r_1}\frac{dr_1}{d\tau}\frac{d\mathbf{u}_0}{d\tau} = -\frac{1}{2}\frac{dr_1}{d\tau}\mathbf{i} \circ \mathbf{u}_0 \circ \mathbf{V}_0 = r_0^2 r_1 \frac{dr_1}{dt}\frac{d\mathbf{u}_0}{dt},$$

$$\frac{1}{r_0}\frac{dr_0}{d\tau}\frac{d\mathbf{u}_1}{d\tau} = -\frac{1}{2}\frac{dr_0}{d\tau}\mathbf{i} \circ \mathbf{u}_1 \circ \mathbf{V}_1 = r_1^2 r_0 \frac{dr_0}{dt}\frac{d\mathbf{u}_1}{dt}.$$

From these relations it can be seen that the indicated terms are finite quantities under the condition that the velocities $\mathbf{v}_0$ and $\mathbf{v}_1$ have finite values and for $r_1 = 0$ and $r_0 = 0$ these terms do not tend to the infinity in time t, but are equal to zero.

Using equations (4.20), (4.21), we obtain the oscillatory global regular equations of the perturbed spatial restricted circular three-body problem, in which the Jacobi variables H_0 and H_1 defined by the relations (3.20) and (3.21) are used instead of the total energies h_0 and h_1. To this end, we substitute relations (4.11), which establish the connections of the energies h_0 and h_1 with the Jacobi variables H_0 and H_1 into the equations (4.20), (4.21). Let us supplement the obtained equations with differential equations (3.22) and (3.23) for Jacobi variables H_0 and H_1, having previously passed to the new independent variable τ using the formula $dt = r_0 r_1 d\tau$. As a result, we obtain the following global regular quaternion equations of the perturbed spatial restricted circular three-body problem:

$$\begin{aligned}\frac{d^2\mathbf{u}_0}{d\tau^2} &- \frac{1}{2} r_1^2 \Big[H_0 - fm_1 r_{01}^{-3}(\mathbf{r}_0 \cdot \mathbf{r}_{01}) - n(y_0\dot{x}_0 - x_0\dot{y}_0)\Big]\mathbf{u}_0 \\ &= \frac{1}{r_1}\frac{dr_1}{d\tau}\frac{d\mathbf{u}_0}{d\tau} + \frac{1}{2}\frac{fm_1}{r_1}\mathbf{u}_0 \circ \mathbf{R}_{01} \circ \mathbf{R}_1 + \frac{1}{2} r_0 r_1^2 \mathbf{i} \circ \mathbf{u}_0 \circ \Big[\frac{fm_1}{r_{01}^3}\mathbf{R}_{01} - \mathbf{P}\Big],\end{aligned} \tag{4.22}$$

$$\frac{dH_0}{d\tau} = \frac{d\mathbf{r}_0}{d\tau} \cdot \mathbf{p} + n r_0 r_1 (y_0 p_1 - x_0 p_2), \tag{4.23}$$

$$\frac{d^2\mathbf{u}_1}{d\tau^2} - \frac{1}{2} r_0^2 \Big[H_1 - fm_0 r_{01}^{-3}(\mathbf{r}_1 \cdot \mathbf{r}_{10}) - n(y_1\dot{x}_1 - x_1\dot{y}_1)\Big]\mathbf{u}_1 \tag{4.24}$$

$$= \frac{1}{r_0}\frac{dr_0}{d\tau}\frac{d\mathbf{u}_1}{d\tau} + \frac{1}{2}\frac{fm_0}{r_0}\mathbf{u}_1 \circ \mathbf{R}_{10} \circ \mathbf{R}_0 + \frac{1}{2}r_1 r_0^2 \mathbf{i} \circ \mathbf{u}_1 \circ \Big[\frac{fm_0}{r_{10}^3}\mathbf{R}_{10} - \mathbf{P}\Big],$$

$$\frac{dH_1}{d\tau} = \frac{d\mathbf{r}_1}{d\tau} \cdot \mathbf{p} + nr_0 r_1 (y_1 p_1 - x_1 p_2), \tag{4.25}$$

$$\frac{dt}{d\tau} = r_0 r_1, \tag{4.26}$$

$$\mathbf{R}_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k} = \bar{\mathbf{u}}_i \circ \mathbf{i} \circ \mathbf{u}_i, \quad r_i = \mathbf{u}_i \circ \bar{\mathbf{u}}_i = u_{i0}^2 + u_{i1}^2 + u_{i2}^2 + u_{i3}^2, \quad i = 0,\, 1,$$

$$\mathbf{r}_0 \cdot \mathbf{r}_{01} = a[\cos(nt)\, x_0 + \sin(nt)\, y_0], \quad \mathbf{r}_1 \cdot \mathbf{r}_{10} = -a[\cos(nt)\, x_1 + \sin(nt)\, y_1],$$

$$x_i = u_{i0}^2 + u_{i1}^2 - u_{i2}^2 - u_{i3}^2, \quad y_i = 2(u_{i1}u_{i2} - u_{i0}u_{i3}), \quad i = 0,\, 1,$$

$$-(y_0\dot{x}_0 - x_0\dot{y}_0) = c_{0z} = 2r_1^{-1}\Big[u_{03}\frac{du_{00}}{d\tau} - u_{02}\frac{du_{01}}{d\tau} + u_{01}\frac{du_{02}}{d\tau} - u_{00}\frac{du_{03}}{d\tau}\Big],$$

$$-(y_1\dot{x}_1 - x_1\dot{y}_1) = c_{1z} = 2r_0^{-1}\Big[u_{13}\frac{du_{10}}{d\tau} - u_{12}\frac{du_{11}}{d\tau} + u_{11}\frac{du_{12}}{d\tau} - u_{10}\frac{du_{13}}{d\tau}\Big],$$

$$\frac{d\mathbf{r}_i}{d\tau} \cdot \mathbf{p} = 2p_1\Big[u_{i0}\frac{du_{i0}}{d\tau_i} + u_{i1}\frac{du_{i1}}{d\tau_i} - u_{i2}\frac{du_{i2}}{d\tau_i} - u_{i3}\frac{du_{i3}}{d\tau_i}\Big]$$
$$+ 2p_2\Big[u_{i2}\frac{du_{i1}}{d\tau_i} + u_{i1}\frac{du_{i2}}{d\tau_i} - u_{i3}\frac{du_{i0}}{d\tau_i} - u_{i0}\frac{du_{i3}}{d\tau_i}\Big]$$
$$+ 2p_3\Big[u_{i3}\frac{du_{i1}}{d\tau_i} + u_{i1}\frac{du_{i3}}{d\tau_i} + u_{i2}\frac{du_{i0}}{d\tau_i} + u_{i0}\frac{du_{i2}}{d\tau_i}\Big], \quad i = 0,\, 1.$$

The quaternions $r_1^{-1}\mathbf{R}_1$ and $r_0^{-1}\mathbf{R}_0$, appearing in equations (4.22) and (4.24), have norms equal to unit. The terms $(1/r_1)(dr_1/d\tau)(d\mathbf{u}_0/d\tau)$ and $(1/r_0)(dr_0/d\tau)(d\mathbf{u}_1/d\tau)$ of these equations are, as shown above, the finite values for $r_1 \to 0$ and $r_0 \to 0$ under the condition that the velocities $\mathbf{v}_0$ and $\mathbf{v}_1$ are finite. Moreover, for $r_1 = 0$ and $r_0 = 0$, these terms do not tend to the infinity in time$t$, but are equal to zero. Therefore, the equations (4.22)–(4.26) are global regular ones.

The resulting system (4.22)–(4.26) of global regular quaternion equations of a perturbed spatial restricted circular three-body problem is equivalent to a scalar system of nonlinear differential equations of the 19^{th} order with respect to the Kustahanheimo–Stiefel variables $u_{ij}\,(j = 0,\, 1,\, 2,\, 3)$ (component of the quaternion variable $\mathbf{u}_i$), their first derivatives $du_{ij}/d\tau$ with respect to a new independent variable τ (component of the quaternion variable $d\mathbf{u}_i/d\tau$), the Jacobi variables H_i and time t ($i = 0,\, 1$). The advantage of these global regular equations is that they use a time transformation that contains the product of the first power of the distance from a body of negligibly small mass to two bodies of attraction of finite masses.

In the case of an unperturbed spatial restricted circular three-body problem, the perturbing accelerations p_1, p_2, p_3 are equal to zero and the variables H_i become the Jacobi costants of motion: $H_i(\tau) = H_i(0) = \text{const}$. Therefore, the differential equations (4.23) and (4.25) drop out of consideration and the unperturbed spatial restricted circular three-body problem is described by global regular equations(4.22), (4.24), (4.26) that are equivalent to a scalar system of nonlinear differential equations of 17^{th} order with respect to the Kustahanheimo–Stiefel variables $u_{ij}\,(j = 0, 1, 2, 3)$, their first derivatives $du_{ij}/d\tau$ and time $t\,(i = 0, 1)$.

In conclusion, we note that the choice of one or another constructed regular quaternion differential equations of the spatial restricted three-body problem (both non-circular and circular) can be made on the basis of numerical integration of these equations and comparing the accuracy of the obtained solutions. Therefore, one of the directions for the further study of the problem of regularization of the equations of the spatial restricted three-body problem using the proposed quaternion approach is connected with the construction and study of numerical solutions of the obtained regular quaternion equations of this problem.

We also note that each of the systems of local regular quaternion equations for the perturbed three-body spatial problem in the Kustaanheimo–Stiefel variables (3.5), (3.7), (3.10)–(3.12) and (3.6), (3.8), (3.13)–(3.15) is close to a linear system of differential equations when considering the movement of a body under study M in the vicinity of an attracting body M_0 or M_1. Indeed, in the absence of perturbing accelerations p_1, p_2, p_3 and body M_1 (for $m_1 = 0$), the Kepler energy $h_0^* =$const and quaternion differential equation (3.5) becomes equivalent to the equation of motion for a single-frequency four-dimensional harmonic oscillator, and the differential equation (3.7) for Kepler energy h_0^* drops out of consideration. Similarly, in the absence of perturbing accelerations p_1, p_2, p_3 and body M_0 (for $m_0 = 0$) the Kepler energy $h_1^* =$ const and quaternion differential equation (3.6) becomes equivalent to the equation of motion for a single-frequency four-dimensional harmonic oscillator, and the differential equation (3.8) for Kepler energy h_1^* drops out of consideration. Therefore, in the case of small perturbing accelerations p_1, p_2, p_3, the movement of the body under study M in the vicinity of the attracting body M_0 or $M_1$is described by the above mentioned equations , which are close to linear differential equations, since the terms of these equations that describe the effect of the body M_1 or M_0 on the body under study M in these cases are also small.

If the mass of the body M_1 or M_0 is equal to zero, each of the systems of regular quaternion equations (3.5), (3.7), (3.10)–(3.12) and (3.6), (3.8), (3.13)–(3.15) coincides with the known regular quaternion equations of the perturbed spatial two-body problem in the Kustahanheimo–Stiefel variables that were proposed by the author of this study in [6] (see also articles [9–11]). These regular quaternion equations of the two-body problem in scalar form coincide with the well-known regular equations of the perturbed spatial two-body problem obtained by P. Kustaanheimo and E. Stiefel [19] (coincide with the regular equations of the two-body spatial problem in variables, which are now called Kustaanheimo–Stiffel variables). E. Stiefel, G. Sheyfele, T. V. Bordovitsyna and other scientists [19, 24, 25] state that using the regular equations of the perturbed two-body problem in the Kustaanheimo–Stiefel variables makes it possible to increase the accuracy of the numerical solution of a number of problems in celestial mechanics and astrodynamics (for example, the problems of the motion of an artificial satellite of the Earth in orbits with large eccentricities) from three to five orders of magnitude compared to solutions obtained using classical (Newton) equations in the Cartesian coordinate system.

Taking into account the influence of the body M_1 on the movement of the body under study M with a negligibly small mass in the vicinity of the body M_0 or, conversely, taking into account the influence of body M_0 on the movement of the body under stud M with a negligibly small mass in the vicinity of the body M_1, in the above mentioned local regular quaternion equations of the spatial restricted three-body problem that have the same order as the perturbations taken into account in the well-known equations of the two-body problem in the Kustaanheimo–Stiefel variables. Therefore, in view of this qualitative analogy between the well-known regular equations of the spatial two-body problem in the Kustaanheimo–Stiefel variables and the regular equations of the spatial restricted three-body problem (3.5), (3.7), (3.10)–(3.12) and (3.6), (3.8), (3.13)–(3.15), it is arguable that the numerical integration of these local regular equations of the perturbed three-body problem in the Kustahanheimo–Stiefel variables is more effective than the numerical integration of the three-body problem equations in rectangular coordinates from the point of view of integration accuracy. However, the degree of effectiveness of these and other quaternion equations for numerical integration of the spatial restricted three-body problem that have been proposed in this article should be studied further.

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