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Solution of Eigenvalue Problems for Linear Hamiltonian Systems with a Nonlinear Dependence on the Spectral Parameter

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Abstract—A method for solving self-adjoint eigenproblems for linear Hamiltonian systems with equation, coefficient, and boundary conditions nonlinearly dependent on the spectral parameter is presented. The suggested approach is based on the iterative Newton procedure with spectral correction. The fast convergence of the method is demonstrated, and two-sided estimates of the eigenvalue sought are obtained. The results of the test application of the outlined algorithm are presented for the problem of the transverse natural oscillations of nonhomogeneous rods with a density defect, using the Euler–Bernoulli, Rayleigh, and Timoshenko models.

Keywords: Sturm–Liouville problem, linear Hamiltonian system, boundary value problem, eigenvalues, eigenfunctions, spectral correction

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1. INTRODUCTION

Eigenvalue (EV) problems arise in applications, mainly as a result of the separation of variables of partial differential equations. In the simplest case of a homogeneous oscillating system of the string type, the boundary value problem takes the form of a second-order equation with constant coefficients and self-adjoint boundary conditions: this is the classical Sturm–Liouville problem (SLP). When we study mechanical systems described by more complicated models and admit vector notation, or when the parameter of separation of variables enters the equations nonlinearly, the formulation in the form of the linear Hamiltonian system can appear more convenient, due to its generality. Here, in contrast to the terms extensively used in mechanics [1, 2], EV is understood as the parameter on which the system matrix depends (including nonlinear dependence). Equations of this type are studied developing both Sturm–theory [3–7] and the constructive methods to solve boundary value problems for eigenvalues and eigenfunctions [8, 9].

The suggested method differs from the methods mentioned above because the interval length in which the boundary value problem is posed, corresponding to the length of a string, rod, fluid vessel, etc., is considered as an independent variable. The difference between initial length l and actual value l_* corresponding to the given approximation λ_* of EV λ, is taken as a residual function. This approach allows a constructive search for EV λ, varying value l and correcting λ_* by value $\partial\lambda(l)/\partial l$ (see below). This method (the method of accelerated convergence [10]) was developed for the classical and generalized SL problems [11, 12], for the vector SL problem [13], for the transverse oscillations of the Euler–Bernoulli rod [14, 15], and for the vector SL problem with a nonlinear dependence on the spectral parameter [16–18], and it has also been applied in a variety of theoretical-experimental studies [19–22]. The results described below generalize the mentioned cases and allow employing the method of accelerated convergence to the oscillation problems for mechanical systems of the more general form, for instance, to the model of transverse oscillations of the Timoshenko rod, to the models of the pipelines oscillations [23, 24], etc., including the problems with the (dynamic) boundary conditions depending on the spectral parameter [12, 25, 26].

2. PROBLEM FORMULATION

Consider the following boundary value problem for eigenvalues and eigenfunctions of the linear Hamiltonian system, self-adjoint at fixed λ [3–6, 8, 9]: we need to find λ within the domain of admitted values $\Lambda \subset \mathbb{R}$ such that nontrivial solution $y(x)$ to the problem

$$Jy(x)' = A(x,\lambda)y(x), \quad 0 < x < l, \quad \lambda \in \Lambda; \quad y(x) = (y_1(x),\ldots,y_m(x))^* \in (C_1[0,l])^m, \tag{2.1}$$

$$J = \begin{pmatrix} O_n & -I_n \\ I_n & O_n \end{pmatrix}, \quad A = A^* : (0,l)\times\Lambda \to \mathbb{R}^{m\times m}$$

with the boundary conditions

$$B_0(\lambda)y(0) = B_l(\lambda)y(l) = 0; \quad B_{0,l}(\lambda): \Lambda \to \mathbb{R}^{n\times m}, \quad m = 2n, \quad \operatorname{rank}B_0 = \operatorname{rank}B_l = n \tag{2.2}$$

exists. Here, J is the simplectic matrix, O_n and I_n are the null and unit matrices of dimension n, $y(x)$ is the continuously differentiable vector function depending on scalar variable x. The system matrix is symmetric, and the matrices of the boundary conditions satisfy the adjointness conditions

$$B_0 J B_0^* = 0, \quad B_l J B_l^* = 0, \tag{2.3}$$

where asterisk denotes the transposition.

Elements $a_{ij}(x,\lambda)$ of matrix A are considered to be continuously differentiable with respect to x and λ functions; the derivative of the matrix with respect to parameter λ does not vanish (see below); the elements of matrices $B_{0,l}$ are also continuously differentiable with respect to λ; and EVs $\lambda \in \Lambda$ are considered smoothly dependent on interval length l in some range $l \pm \delta$, $\delta > 0$, with the same being true for the inverse function $l(\lambda)$. Further, in the cases where no confusion is possible, the dependence of matrix A on x,λ or on x is dropped. It is assumed that the solution to the problem, the eigenpair λ, y, exists and the spectrum of the problem is discrete. For brevity, we limit ourselves by the case of simple EVs. Thus, we are studying a classical non-singular oscillating system, whereas mechanical systems with nonstandard behavior in the spectrum (for instance, inhomogeneous media to which the averaging procedure is applied [27] and which have the segments of continuous spectrum in some cases) are ignored, at least with reference to these segments of the spectrum.

3. DIFFERENTIAL PROPERTIES OF THE SOLUTION

As stated above, length l is usually considered to be fixed. In the present approach, the mechanical system is assumed to have the following property: when the length of this system (or some parameter corresponding to the length) is changed by a small value, the system frequencies also change by a small value. This assumption allows an algorithm to be developed that takes the rate of this change into consideration.

Considering the free parameter length l and differentiating the problem (2.1) and (2.2) with respect to parameter l, we obtain

$$\begin{gathered} Jz' = \lambda' A_\lambda' y + Az; \quad \lambda' B_{0,\lambda}' y(0) + B_0 z(0) = 0, \quad \lambda' B_{l,\lambda}' y(l) + B_l y'(l) + B_l z(l) = 0, \\ z = \frac{\partial y}{\partial l}, \quad \lambda' = \frac{d\lambda}{dl}, \quad A_\lambda' = \frac{\partial A}{\partial \lambda}, \quad B_{0,\lambda}' = \frac{\partial B_0}{\partial \lambda}, \quad B_{l,\lambda}' = \frac{\partial B_l}{\partial \lambda}. \end{gathered} \tag{3.1}$$

We consider the scalar product of the studied equation and solution y in $L_2(0,l)$ (in the following, $\langle f,g\rangle_l = \int_0^l (f,g)_x dx$ where $(f,g)_x$ is the scalar product of the function values at point x in $\mathbb{R}^m$ or $\mathbb{R}^n$, $\|f\|_l^2 = \langle f,f\rangle_l$). Integrating by parts, we obtain

$$-\lambda'\langle A_\lambda' y, y\rangle_l - (z,Jy)_l + (z,Jy)_0 = 0. \tag{3.2}$$

We rewrite the boundary conditions and adjointness conditions, Eqs. (2.2), (2.3), and (3.1), in the form

$$x = 0, l\colon B_x = (\psi_x, \chi_x), \quad \psi_x, \chi_x \in \mathbb{R}^{n\times n}, \quad \chi_x\psi_x^* = \psi_x\chi_x^*, \quad \psi_x u(x) + \chi_x v(x) = 0,$$
$$\lambda'\psi'_{0,\lambda}u(0) + \lambda'\chi'_{0,\lambda}v(0) + \psi_0\mu(0) + \chi_0\nu(0) = 0,$$
$$\lambda'\psi'_{l,\lambda}u(l) + \lambda\chi'_{l,\lambda}v(l) + \psi_l u'(l) + \chi_l v'(l) + \psi_l\mu(l) + \chi_l\nu(l) = 0,$$

where

$$u = (y_1, \ldots, y_n)^*, \quad v = (y_{n+1}, \ldots, y_m)^*, \quad \mu = \frac{\partial u}{\partial l}, \quad \nu = \frac{\partial v}{\partial l}.$$

Without loss in generality, we consider that $\operatorname{rank}\psi_0 = n$. In the other case, we consider the canonical change of coordinates leading to the required property: $y = C_0\tilde{y}$, where C_0 is the symplectic orthogonal matrix, $C_0^*JC_0 = J$. Then

$$\tilde{B}_0 = (\tilde{\psi}_0, \tilde{\chi}_0) = B_0C_0, \quad \tilde{\psi}_0^{-1}\tilde{\chi}_0 = \tilde{\chi}_0^*(\tilde{\psi}_0^{-1})^*,$$
$$(z, Jy)_0 = (C_0\tilde{z}, JC_0\tilde{y})_0 = (\tilde{z}, C_0^*JC_0\tilde{y})_0 = (\tilde{z}, J\tilde{y})_0$$

and, dropping the tilde, we obtain

$$(z, Jy)_0 = -(\mu, v)_0 + (\nu, u)_0$$
$$= (\psi_0^{-1}\chi_0\nu + \lambda'\psi_0^{-1}(\psi'_{0,\lambda}u + \chi'_{0,\lambda}v), v)_0 - (\nu, \psi_0^{-1}\chi_0 v)_0$$
$$= \lambda'(\psi_0^{-1}(\psi'_{0,\lambda}u + \chi'_{0,\lambda}v), v)_0 + (\psi_0^{-1}\chi_0\nu, v)_0 - (\chi_0^*(\psi_0^{-1})^*\nu, v)_0.$$

We denote $D_0 = (\tilde{\psi}_0^{-1})^*$ and $q_0 = \tilde{v}(0)$. Then the expression for $(z, Jy)_0$ in the original coordinates takes the following form

$$(z, Jy)_0 = \lambda'((\psi'_{0,\lambda}u + \chi'_{0,\lambda}v), D_0q_0)_0 = \lambda'(B'_{0,\lambda}y, D_0q_0)_0. \tag{3.3}$$

Similarly, assuming $\operatorname{rank}\psi_l = n$ and applying, if needed, the change of coordinates, we obtain

$$\psi_l^{-1}\chi_l = \chi_l^*(\psi_l^{-1})^*,$$
$$(z, Jy)_l = \lambda'(\psi_l^{-1}(\psi'_{l,\lambda}u + \chi'_{l,\lambda}v), v)_l$$
$$+ (\psi_l^{-1}\chi_l v', v)_l + (\psi_l^{-1}\chi_l\nu, v)_l + (u', v)_l + (\nu, u)_l$$
$$= \lambda'(\psi_l^{-1}(\psi'_{l,\lambda}u + \chi'_{l,\lambda}v), v)_l + (\psi_l^{-1}\chi_l v', v)_l + (u', v)_l$$
$$= \lambda'(\psi_l^{-1}B'_{l,\lambda}y, v)_l - (y', Jy)_l = \lambda'(\psi_l^{-1}B'_{l,\lambda}y, v)_l + (Ay, y)_l.$$

Thus, by using the denotations $D_l = (\psi_l^{-1})^*$ and $q_0 = v(l)$ similar to (3.3), from Eq. (3.2) we obtain

$$\lambda'(l) = -\frac{(Ay, y)_l}{N(l)}, \quad N(l) = \langle A'_\lambda y, y\rangle_l + (B'_{x,\lambda}y, D_xq_x)_x\big|_{x=0}^{x=l}. \tag{3.4}$$

Value $N(l)$ is considered nonvanishing.

If the boundary conditions do not depend on spectral parameter λ, then the formula (3.4) simplifies to the following:

$$\lambda'(l) = -\frac{(Ay, y)_l}{\langle A'_\lambda y, y\rangle_l}. \tag{3.5}$$

If the first-order system (2.1) is the reformulated generalized vector SL problem [16–18] (for simplicity, for the boundary conditions independent from the spectral parameter)

$$(P(x,\lambda)u')' + R(x,\lambda)u = 0, \quad 0 < x < l, \quad \lambda \in \Lambda,$$
$$u = (u_1, \ldots, u_n)^*, \quad R = R^*, \quad P = P^* > 0,$$
$$x = 0, l\colon \quad F_xu(x) = G_xPu'(x), \quad F_xG_x^* = G_xF_x^*, \quad \operatorname{rank}(F_x, G_x) = n,$$

then in the problem (2.1)

$$y = \begin{pmatrix} u \\ Pu' \end{pmatrix}, \quad A = \begin{pmatrix} R & 0 \\ 0 & P^{-1} \end{pmatrix}; \quad x = 0, \quad l\colon B_x = \begin{pmatrix} F_x & -G_x \end{pmatrix},$$

and the expression (3.5) takes the form [18]:

$$\lambda'(l) = -\frac{(Py', y')_l + (Ry, y)_l}{-\langle P'_\lambda y', y'\rangle_l + \langle R'_\lambda y, y\rangle_l}.$$

The system (2.1) corresponds to the linear Hamiltonian system (here $x = t$):

$$\begin{gathered} \dot q_i = \frac{\partial H}{\partial p_i}, \quad \dot p_i = -\frac{\partial H}{\partial q_i}, \quad q_i = y_i, \quad p_i = y_{n+i}, \quad i = 1,\dots,n; \\ 2H(p,q,t,\lambda) = (p^*\ q^*)A(t,\lambda)\begin{pmatrix} p \\ q \end{pmatrix}, \quad p = (p_1,\dots,p_n)^*, \quad q = (q_1,\dots,q_n)^*. \end{gathered} \tag{3.6}$$

In terms of the quadratic Hamiltonian (3.6), the expression for the derivative (3.5) becomes

$$\lambda'(l) = -H(p,q,l,\lambda)\left[\int_0^l H'_\lambda(p,q,t,\lambda)dt\right]^{-1}.$$

Note that the pseudonorm $N(l)$ in formulas (3.4) (for some problems that are linear in parameter, for instance SL problems, this value coincides with the classical norm) contains an integral quadratic in y term $\langle A'_\lambda y, y\rangle_l$ and can be determined by the solution of the differential equation

$$Jz' = Az - A'_\lambda y, \quad 0 < x < l \tag{3.7}$$

with the initial condition $z(0) = y(0)$. Therefore,

$$N(l) = (z, Jy)_l + (B'_{x,\lambda} y, D_x q_x)_x\big|_{x=0}^{x=l}.$$

Indeed, multiplying Eq. (3.7) by y in L_2, we obtain

$$\begin{aligned} 0 &= \langle Jz', y\rangle_l - \langle Az, y\rangle_l + \langle A'_\lambda y, y\rangle_l = -\langle z', Jy\rangle_l - \langle Az, y\rangle_l + \langle A'_\lambda y, y\rangle_l \\ &= \langle z, Jy'\rangle_l - \langle z, Ay\rangle_l - (zJy)_x\big|_{x=0}^{x=l} + \langle A'_\lambda y, y\rangle_l = -(z, Jy)_l + \langle A'_\lambda y, y\rangle_l. \end{aligned}$$

Note also that, as follows from the derivation of expression (3.4), this formula holds true for complex-valued functions and for complex EVs where the scalar products and Hermitian matrix A are adequately determined. However, in this case, the application of the computational procedure described below may remove EV from the domain of admitted values and lead to a non-self-adjoint boundary value problem. Therefore, in the following, for simplicity, we examine only real eigenvalues and eigenfunctions. Here, the non-self-adjointness of the original problem is not a limitation of the proposed approach. In this case we should consider the adjoint system $y'_a = -A^* y_a$

$$\begin{pmatrix} O_m & -I_m \\ I_m & O_m \end{pmatrix}\begin{pmatrix} y' \\ y'_a \end{pmatrix} = \begin{pmatrix} O_m & A^* \\ A & O_m \end{pmatrix}\begin{pmatrix} y \\ y_a \end{pmatrix}$$

together with the original system $y = Ay$ with the added adjoint boundary conditions.

4. APPROXIMATION OF EIGENVALUES

In the problem formulation, we assumed the smoothness of dependence $\lambda(l)$, which corresponds to the property of a small variation in the natural frequency at a small variation in the length parameter of the system. This property is well-known and has been widely observed in many mechanical systems. Consequently, we may conclude that for some $|\varepsilon| \ll 1$, $\varepsilon = (l - \xi)/l$ the expansion is valid:

$$\lambda(l) = \lambda(\xi) + \varepsilon l\lambda'(\xi) + O(\varepsilon^2). \tag{4.1}$$

Using equalities (3.4), we obtain

$$\lambda(l) \approx \lambda(\xi) - \varepsilon l(A(\lambda(\xi))y, y)(\xi)/N(\xi),$$

$$N(\xi) = (A'_\lambda(\lambda(\xi))y, y)_\xi + (B'_{l,\lambda}(\lambda(\xi))y, D_l q_l)(\xi) - (B'_{0,\lambda}(\lambda(\xi))y, D_0 q_0)(0),$$

where y is the solution of the system (2.1) for $\lambda = \lambda(\xi)$, that satisfies the boundary conditions at points $x = 0, \xi$:

$$Jy(x)' = A(x, \lambda(\xi))y(x), \quad 0 < x < \xi, \quad \lambda(\xi) \in \Lambda,$$
$$B_0(\lambda(\xi))y(0) = B_l(\lambda(\xi))y(\xi) = 0.$$

Now, we construct such solution and determine point ξ: let $Y(x)$ be the solution to the matrix equation

$$JY' = A(\lambda(\xi))Y, \quad Y(0) = I_m.$$

Then because of continuous dependency $\lambda(l)$, there exists a constant m-vector c such that the function $y(x) = Y(x)c$ satisfies the left and right boundary conditions at some, adjoint for $x = 0$, point ξ:

$$B_0 y(0) = B_0 Y(0)c = B_0 c = 0, \quad B_l y(\xi) = B_l Y(\xi)c = 0.$$

Hence, at this point ξ, the determinant of the $m \times m$ matrix $B(x) = \begin{pmatrix} B_0 \\ B_l Y(x) \end{pmatrix}$ is equal to zero:

$$\Delta(\xi) = \det B(\xi) = 0.$$

Thus, by the numerical and analytical determination of value ξ and by calculating the vector c as a solution of the homogeneous equation $B(\xi)c = 0$, we may construct the eigenfunction sought $y(x) = Y(x)c$ that corresponds to EV $\lambda(\xi)$, and here

$$\lambda(l) - (\lambda(\xi) - \varepsilon l(Ay, y)_\xi / N(\xi)) = O(\varepsilon^2).$$

5. ITERATIVE PROCEDURE AND TWO-SIDED ESTIMATE

Following from the above discussion, we define the iterative procedure:

1. For a given approximation $\lambda^{(i)}$, numerically or analytically, construct a solution $Y_{(i)}$ to the matrix equation given by

$$JY'_{(i)} = AY_{(i)}, \quad Y_{(i)}(0) = I_m. \tag{5.1}$$

2. Find $\xi_{(i)}$ as an appropriate root (nearest to l or kth if the number k of EV is known and if for the studied system the corresponding oscillation Sturm theorems are satisfied):

$$\Delta_{(i)}(x) = 0, \quad \Delta_{(i)} = \det B_{(i)}, \quad B_{(i)}(x) = \begin{pmatrix} B_0 \\ B_l Y_{(i)}(x) \end{pmatrix}.$$

3. Construct the eigenfunction $y^{(i)}(x) = Y_{(i)}(x)c_{(i)}$, where $c_{(i)}$ is the solution of the homogeneous linear system $B(\xi_{(i)})c_{(i)} = 0$.

4. Compute the next approximation of EV:

$$\lambda^{(i+1)} = \lambda^{(i)} - \varepsilon_{(i)} l(A(\lambda^{(i)})y^{(i)}, y^{(i)})(\xi_{(i)})/N(\xi_{(i)}); \quad \varepsilon_{(i)} = (l - \xi_{(i)})/l.$$

In many special cases, there is no need to calculate an entire fundamental system of solution to m-order Eq. (5.1). For instance, for $B_0 = (I_n, O_n)$, $B_l = (O_n, I_n)$, it is sufficient to consider the initial conditions in the form:

$$Y(0) = \begin{pmatrix} O_n & O_n \\ O_n & I_n \end{pmatrix}.$$

Hence, if $B_l Y(x) = (B_1(x), B_2(x))$, then the matrix B should be determined in the form $B(x) = B_2(x)$. Consequently, $\Delta(x) = \det B_2(x)$.

Note also that steps 1–3 of the algorithm, although they are laborious, may be replaced for the corresponding modification by the process of transition of the boundary conditions to an unknown point $\xi_{(i)}$ [9, 28] and require a separate investigation.

For the practical application of this method, in the described approach the criterion of solution accuracy will be the residual value $\varepsilon_{(i)}l$: when the value of degree of accuracy of the Cauchy problem (5.1) is achieved, the approximated boundary value problem becomes close to the original, which means that the same is true for the eigenvalues and eigenfunctions.

However, the method allows a more direct estimation of the found EV. From the expression (3.4), it follows clearly that a two-sided estimate can be obtained: if λ' keeps its sign in some neighborhood δ_l of admitted values $(l, \lambda(l))$ and for some i, j, and points $(\xi_{(i)}, \lambda^{(i)})$ and $(\xi_{(j)}, \lambda^{(j)})$ lie in this neighborhood, then, for the positive value

$$(Ay^{(i)}, y^{(i)})_{\xi_{(i)}}/N(\xi_{(i)}) \tag{5.2}$$

and $\xi_{(i)} < l\,(> l)$, we have $\lambda^{(i)} > \lambda\,(< \lambda)$, correspondingly. For the negative value of (5.2), the signs of the last four inequalities are changed to the opposite ones. Thus, the sought EV is enclosed between $\lambda^{(i)}$ and $\lambda^{(j)}$ for some i and j.

When the boundary conditions are independent from the spectral parameter, the rule for obtaining the two-sided estimate may likewise be related to the property of having a fixed sign of matrices A and A'_λ. Then, this rule, based on the signs of the value (5.2) considered above, is now based on the consideration of the sign of the value

$$(Ay^{(i)}, y^{(i)})_{\xi_{(i)}}/\langle A'_\lambda y^{(i)}, y^{(i)}\rangle_{\xi_{(i)}}.$$

The obtained two-sided estimate theoretically allows finding EV with an arbitrary degree of accuracy, which in practice however, cannot exceed the degree of accuracy of the calculation of point ξ, that is, the degree of accuracy of the solution to the initial problem (5.1).

Note also that, despite the assumption that EVs are simple, the outlined procedure may be also used in the case of multiple eigenvalues, at least where there are no bifurcation points in the considered domain. Then, the system $B(\xi_{(i)})c_{(i)} = 0$ has several linearly independent solutions $c^1_{(i)}, c^2_{(i)}, \ldots$, and the eigenfunctions are defined as $Y_{(i)}c^1_{(i)}, Y_{(i)}c^2_{(i)}, \ldots$. However, the justification of convergence of the procedure becomes here substantially more complicated and requires additional investigation.

6. CONVERGENCE OF ITERATIVE PROCEDURE

Abstracting ourselves from assumption (4.1), we draft the proof for the following theorem:

Theorem. When the assumptions of Sect. 2 are satisfied for a sufficiently close initial guess $\lambda^{(0)}$, the application of the step of the procedure of Sect. 5 leads to the following estimate uniform in $\varepsilon^{(0)}$, $|\varepsilon^{(0)}| \ll 1$:

$$\|y - y^{(1)}\|_{C_1} \le C_y(\varepsilon^{(0)})^2, \quad |\lambda - \lambda^{(1)}| \le C_\lambda(\varepsilon^{(0)})^2.$$

The proof is follows the perturbation method and the method of successive approximations, like the method of accelerated convergence for the scalar and generalized vector SL problems [11, 17, 29], but with some changes. We note a number of technical distinctions. Consider an estimate $\lambda^{(0)}$ of the sought EV λ for the problem (2.1). According to the assumption made in Sect. 2, λ smoothly depends on l, and, hence, for some $\xi_{(0)}$ the equation

$$Jy^{(0)\prime} = A(x, \lambda^{(0)})y^{(0)}, \quad 0 < x < \xi_{(0)}$$

is true, and the boundary conditions are met

$$B_0(\lambda^{(0)})y^{(0)}(0) = B_l(\lambda^{(0)})y^{(0)}(\xi_{(0)}) = 0.$$

Function $y^{(0)}$ is constructed in the standard manner with the algorithm of Sect. 5: if $Y_{(0)}$ is a solution to the matrix equation

$$JY'_{(0)} = A(\lambda^{(0)})Y_{(0)}, \quad Y_{(0)} = I_m,$$

and $c_{(0)}$ is some vector and the solution of the system of linear equations (see Sec. 5) $B_{(0)}c_{(0)} = 0$, then $y^{(0)}(x) = Y_{(0)}(x)c_{(0)}$.

Considering that the value $\lambda^{(0)}$ is sufficiently close to the sought value λ, we introduce the small parameter

$$\varepsilon_{(0)} = (l - \xi_{(0)})/l, \quad |\varepsilon_{(0)}| \ll 1$$

and denote

$$s = x\xi_{(0)}/l, \quad z(s) = y(ls/\xi_{(0)}).$$

Then, original problem (2.1) is represented in the form:

$$(1 - \varepsilon_{(0)})Jz'(s) = A\left(\frac{s}{1 - \varepsilon_{(0)}}, \lambda\right)z(s), \quad 0 < s < \xi_{(0)},$$
$$B_0(\lambda)z(0) = B_l(\lambda)z(\xi_{(0)}) = 0.$$

Applying the expansion in powers of $\varepsilon_{(0)}$

$$\lambda = \lambda^{(0)} + \varepsilon_{(0)}\lambda^{(0,1)} + \ldots, \quad z = z_{(0)} + \varepsilon_{(0)}z_{(0,1)} + \ldots,$$
$$A\left(\frac{s}{1 - \varepsilon_{(0)}}, \lambda\right) = A(s, \lambda^{(0)}) + \varepsilon_{(0)}(sA'(s, \lambda^{(0)}) + \lambda^{(0,1)}A'_\lambda(s, \lambda^{(0)})) + \ldots,$$
$$x = 0, \quad l\colon B_x(\lambda) = B_x(\lambda^{(0)}) + \varepsilon_{(0)}\lambda^{(0,1)}B'_{x,\lambda}(\lambda^{(0)}) + \ldots,$$

for the zero power we have the boundary value problem:

$$Jz'_{(0)}(s) = A(s, \lambda^{(0)})z_{(0)}(s), \quad 0 < s < \xi_{(0)},$$
$$B_0(\lambda^{(0)})z_{(0)}(0) = B_l(\lambda^{(0)})z_{(0)}(\xi_{(0)}) = 0. \tag{6.1}$$

Hence, $z_{(0)} = y_{(0)}$. In the following, we consider that $\|z_{(0)}\|_{\xi_{(0)}} = 1$.

To simplify, we represent the desired eigenvalue and eigenfunction in the form of perturbation of the known solution $\lambda^{(0)}$, $z_{(0)}$ of the problem (6.1):

$$\lambda = \lambda^{(0)} + \varepsilon_{(0)}\lambda^{(1)}(\varepsilon_{(0)}), \quad z(s) = z_{(0)}(s) + \varepsilon_{(0)}z_{(1)}(s, \varepsilon_{(0)}).$$

Then, to determine this perturbation, we derive the inhomogeneous boundary value problem

$$Jz'_{(1)} = A(\lambda^{(0)})z_{(1)} + G(z_{(0)}, \lambda^{(0)}, \lambda^{(1)}) + F(z_{(1)}, \lambda^{(0)}, \lambda^{(1)}, \varepsilon_{(0)}),$$
$$G = Jz'_{(0)} + sA'(\lambda^{(0)})z_{(0)} + \lambda^{(1)}A'_\lambda(\lambda^{(0)})z_{(0)}, \tag{6.2}$$
$$x = 0, \quad \xi_{(0)}\colon \quad \lambda^{(1)}B'_{x,\lambda}(\lambda^{(0)})z_{(0)}(x) + B_x z_{(1)}(x) = 0,$$

where function F ($|F| \le C\varepsilon_{(0)}$, $F|_{\varepsilon_{(0)}=0} = 0$) is Lipschitzian with respect to $\lambda^{(1)}$ and $z_{(1)}$.

Not taking small term F into account, the result is a problem whose equation we multiply by $z_{(0)}$ in $L_2(0, \xi_{(0)})$. On the left-hand side, we obtain

$$\langle Jz'_{(0,1)}, z_{(0)}\rangle_{\xi_{(0)}} = -(z_{(0,1)}, Jz_{(0)})_{\xi_{(0)}} + (z_{(0,1)}, Jz_{(0)})_0 + \langle z_{(0,1)}, Jz'_{(0)}\rangle_{\xi_{(0)}},$$

where, similarly to Sect. 3, using the boundary conditions, we reduce the expressions at the end points to the form:

$$\lambda^{(0,1)}(B'_{0,\lambda}y, D_0q_0)_0 - \lambda^{(0,1)}(B'_{l,\lambda}y, D_lq_l)_{\xi_{(0)}}.$$

We also reorganize the right-hand side with integration by parts into the relation

$$\langle z_{(0,1)}, Az_{(0)}\rangle_{\xi_{(0)}} + \langle Az_{(0)}, z_{(0)}\rangle_{\xi_{(0)}} + \lambda^{(0,1)}\langle A'_\lambda z_{(0)}, z_{(0)}\rangle_{\xi_{(0)}} + \langle sA'z_{(0)}, z_{(0)}\rangle_{\xi_{(0)}},$$

where

$$\begin{aligned}\langle sA'z_{(0)}, z_{(0)}\rangle_{\xi_{(0)}} &= \langle s(Az_{(0)})', z_{(0)}\rangle_{\xi_{(0)}} - \langle sz'_{(0)}, Jz'_{(0)}\rangle_{\xi_{(0)}}\\ &= -\langle Az_{(0)}, z_{(0)}\rangle_{\xi_{(0)}} + \xi_{(0)}(Az_{(0)}, z_{(0)})_{\xi_{(0)}}.\end{aligned}$$

Thus, the expression for the perturbation is obtained (it is the orthogonality condition between the solution to the homogeneous equation and the solution to the inhomogeneous one)

$$\lambda^{(0,1)} = -\frac{\xi_{(0)}(Az_{(0)}, z_{(0)})_{\xi_{(0)}}}{\langle A'_\lambda z_{(0)}, z_{(0)}\rangle_{\xi_{(0)}} + (B'_{l,\lambda}z_{(0)}, D_l q_l)_x\Big|_{x=0}^{x=\xi_{(0)}}} \tag{6.3}$$

analogous to the expression (3.4) with an accuracy up to ε^2. We substitute the found value of the correction into the problem without perturbation G, Eq. (6.2), and, after that, similar to function $z_{(0)}$, we construct the function $z_{(0,1)}$ by using the solution of the Cauchy problems and by satisfying the inhomogeneous boundary conditions. Using the method of successive approximations, we substitute the found $\lambda^{(0,1)}$, $z_{(0,1)}$ into function F (6.2), and in the next step we have the following boundary value problem:

$$\begin{gathered}Jz'_{(0,2)} = A(\lambda^{(0)})z_{(0,2)} + G(z_{(0)}, \lambda^{(0)}, \lambda^{(0,2)}) + F(z_{(0,1)}, \lambda^{(0)}, \lambda^{(0,1)}, \varepsilon_{(0)}),\\ x = 0, \quad \xi_{(0)}\colon \lambda^{(0,2)}B'_{x,\lambda}(\lambda^{(0)})z_{(0)}(x) + B_x z_{(0,2)}(x) = 0.\end{gathered} \tag{6.4}$$

The expression for the EV correction (orthogonality condition) takes the following form:

$$\lambda^{(0,2)} = \lambda^{(0,1)} - \langle F(z_{(0,1)}, \lambda^{(0)}, \lambda^{(0,1)}, \varepsilon_{(0)}), z_{(0)}\rangle_{\xi_{(0)}}/N(\xi_{(0)}),$$

where the denominator N is determined similarly to the denominator in expression (6.3) for $\lambda^{(0,1)}$. As above, we substitute the found value $\lambda^{(0,2)}$ into problem (6.4) and determine $z_{(0,2)}$ and so on. The squeeze theorems allow us to state that the presented iterative procedure converges to the unique solution $\lambda^{(1)}$, $z_{(1)}$. Further, the uniformity of the estimate is demonstrated in a similar way to the simpler case of the linear scalar problem [11], with the integral representation of a solution to (6.2).

Applying the iterative procedure from Sect. 5 allows the quadratic estimates of the following type at the kth step to be obtained:

$$|\lambda - \lambda^{(k+1)}| \le C_\lambda(\varepsilon_{(0)})^{2^k}, \quad \|y - y^{(k+1)}\|_{C_1} \le C_y(\varepsilon_{(0)})^{2^k}.$$

Note that, in the case of multiple EV, perturbation techniques can also be employed; however, expansion is needed in the fractional powers of small parameters [30].

7. TEST EXAMPLE

The proposed general technique was tested for special cases of problems having an analytical solution; for instance, it was examined for the problem of the natural oscillations of a pipeline [24] and in the scalar and vector SL problems [16–18], including problems with boundary conditions dependent on the spectral parameter [12]. Keeping in mind the application of the suggested method to the study of natural oscillations of rod systems, to illustrate the general concept and the convergence rate, we limit ourselves to a comparison with the results of numerical experiments that are well-known in the literature for fourth-order equations with variable coefficients without parameters in the boundary condition. Here, we use the implementation of the above-discussed algorithm in Maple package. To integrate Eq. (5.1), we used the *dsolve* procedure and the fourth-fifth order Runge–Kutta method with the default accuracy parameters (henceforth, an absolute error of 10^{-7} and a relative error of 10^{-6} are meant) and with increased accuracy parameters (with an absolute error of 10^{-16} and a relative error of 10^{-16}). To determine the conjugate point

(the solution to equation $\Delta(x) = 0$), we utilized the *fsolve* procedure. We used 20 significant digits for notionally low accuracy computations and 40 significant digits for increased accuracy computations.

Consider the boundary value problem

$$u^{iv} - (2\alpha x^2 u')' + (\alpha^2 x^4 - 2\alpha - \lambda)u = 0, \quad 0 < x < 5, \quad \alpha = 10^{-2}, \\ u(0) = u(5) = 0, \quad u''(0) = u''(5) = 0, \tag{7.1}$$

which can be interpreted as the equation of the natural transverse oscillations of a rod with variable tension in an inhomogeneous medium.

We rewrite it in form (2.1):

$$y = (u, -pu'', -(pu'')' + su', u')^*, \quad A = \begin{pmatrix} A_{11} & O_2 \\ O_2 & A_{22} \end{pmatrix}, \quad A_{11} = \begin{pmatrix} -(q-\lambda) & 0 \\ 0 & p^{-1} \end{pmatrix}, \quad A_{22} = \begin{pmatrix} 0 & 1 \\ 1 & -s \end{pmatrix},$$

$$p = 1, \quad s = 2\alpha x^2, \quad q = \alpha^2 x^4 - 2\alpha, \quad B_{0,l} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{7.2}$$

The numerical solution to boundary value problem (7.1) is known obtained through different methods [31–35], including the use of the packages SLEUTH [36] and SLEING2 [37]. The presented data differ by in the sixth digit [31–35]. According to the criterion taken in the present technique (residual value εl), we take the values of the first and second EVs as the closest to the true EVs [34]:

$$\lambda_1 = 0.21505086437, \quad \lambda_2 = 2.75480993468.$$

In the current implementation with increased accuracy, they correspond to the following residual values:

$$\varepsilon l = 2.3 \times 10^{-12}, \quad \varepsilon l = -1.5 \times 10^{-12}.$$

In the general case of a system that is nonlinear in its parameter, the initial guess for the iterative procedure can be obtained with a standard shooting procedure. In the special case of the rod systems, variational principles of the Rayleigh type [38] may be used to provide the upper estimates. For problem (7.1), we have

$$\lambda + 2\alpha \le (\langle v'', v'' \rangle_5 + 2\alpha \langle x^2 v', v' \rangle_5 + \alpha^2 \langle x^4 u, u \rangle_5) \langle v, v \rangle_5^{-1}. \tag{7.3}$$

We choose the following functions satisfying the boundary conditions as approximations of the first and second eigenfunctions:

$$v_1 = \sin \frac{\pi x}{5}, \quad v_2 = \sin \frac{2\pi x}{5}$$

and have the sufficiently accurate estimates for the first and second EV, $\lambda_1^{(0)} = 0.21878$ and $\lambda_2^{(0)} = 2.75784$. These correspond to the residual values $\varepsilon_{0,1} l = 0.03$ and $\varepsilon_{0,2} l = 1.5 \times 10^{-3}$. The application of the iterative procedure gives values $\lambda_1^{(1)}$ and $\lambda_2^{(1)}$ with relative errors of 2.7×10^{-4} and 8.4×10^{-7} for residual values $\varepsilon_{1,1} l = -4.8 \times 10^{-4}$ and $\varepsilon_{1,2} l = -1.2 \times 10^{-6}$ after the first step and $\lambda_1^{(2)} = \lambda_1 - 1.4 \times 10^{-8}$ (relative error of 6.7×10^{-8}, residual value $\varepsilon_{1,1} l = -1.2 \times 10^{-7}$) and $\lambda_2^{(2)} = \lambda_2$ (i.e., equal to the exact value) after the second step. Thus, to create an accurate initial estimate, we employed fast convergence to a value very close to the exact one.

A different, simple way to estimate EV can be suggested if the solution is known to a problem that is near in some sense. Its EV can be also used as an initial guess, and to increase the accuracy of approximation, perturbation techniques can be used that are similar to the techniques discussed in Sect. 6.

Let $\lambda_{(0)}$, y_0 be the solution that generates the boundary value problem given by

$$Jy_{0\cdot} = A_0(\lambda_{(0)}) y_0, \quad B_0 y_0(0) = B_l y_0(l) = 0$$

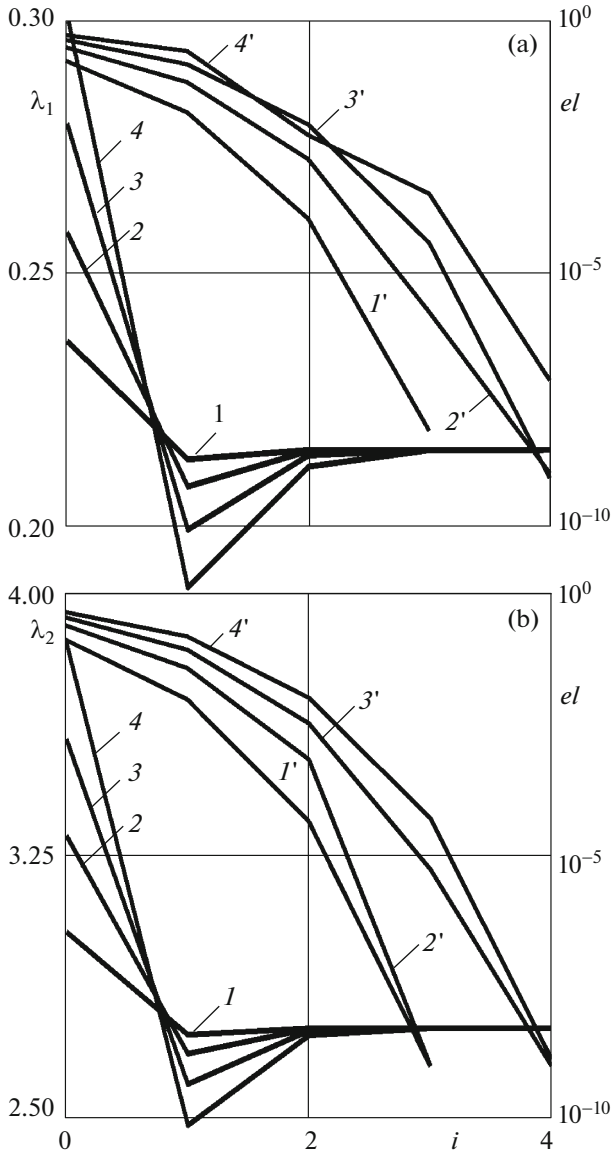

Fig. 1.

(for simplicity we suppose that the boundary conditions do not depend on the parameter). Taking into account that the system matrix of the studied system (2.1) can be represented in the form

$$A = A_0 + \mu K, \quad \mu \ll 1,$$

we have the following boundary value problem:

$$Jy' = A_0(\lambda)y + \mu K(\lambda)y, \quad B_0y(0) = B_ly(l) = 0.$$

We suppose that the sought solution, λ, y, is a perturbation of the known one, $\lambda_{(0)}, y_0$; that is,

$$\lambda = \lambda_{(0)} + \mu\lambda_{(1)} + O(\mu^2), \quad y = y_0 + \mu y_1 + O(\mu^2),$$

and obtain for the correction value [24]

$$\lambda_{(1)} = -\langle K(\lambda_{(0)})y_0, y_0\rangle_l / \langle A'_{0,\lambda}(\lambda_{(0)})y_0, y_0\rangle_l.$$

Then, as an initial guess we can take $\lambda^{(0)} = \lambda_{(0)} + \mu\lambda_{(1)}$.

For the problem (7.1), as a generating problem, we consider

$$u_0^{iv} - \lambda_{(0)}u_0 = 0, \quad 0 < x < 5, \quad \mu = \alpha,$$
$$u_0(0) = u_0(5) = 0, \quad u_0''(0) = u_0''(5) = 0.$$

Hence,

$$\lambda_{(0)} = \left(\frac{\pi n}{5}\right)^4, \quad u_0 = \sin\frac{\pi n}{5}, \quad n = 1, 2.$$

Matrices A_0 and K and vector y_0 can apparently be determined. It is easy to check analytically that for the present problem, the estimate of the value $\lambda_{(0)} + \mu\lambda_{(1)}$ coincides with the estimate derived from relation (7.3) for all functions v chosen above.

For a vivid demonstration of the quadratic convergence rate of the algorithm, we increase the error of the initial guess and observe the results at low accuracy. In the following, as an initial guess $\lambda_1^{(0)}$, $\lambda_2^{(0)}$, we take values which differ from the exact one by 10, 20, 30, and 40%. In Fig. 1, residual values $\varepsilon^{(i)}l$ and values $\lambda_1^{(i)}$ and $\lambda_2^{(i)}$ are shown after the ith iteration of the method with the integration at default accuracy. The calculations were performed up to achieving the value lower than 10^{-6} by a relative error of EV. Curve *1* corresponds to the variation of $\lambda_1^{(i)}$ in Fig. 1a and to the variation of $\lambda_2^{(i)}$ in Fig. 1b; curves *1'* correspond to the variation of $\varepsilon_{(i)}l$ for $\lambda_1^{(i)}$ (Fig. 1a) and $\lambda_2^{(i)}$ (lower part) at 10% of initial error; curves *2, 2'*, *3, 3'*, and *4, 4'* correspond to the 20, 30, and 40% of initial error, respectively. Let us note (see Sec. 4) that, at every step of the algorithm, we compute $\varepsilon_j l$ for current approximation $\lambda^{(j)}$ and subsequent approximation $\lambda^{(j+1)}$; for instance, $\varepsilon_0 l$ corresponds to the initial guess $\lambda^{(0)}$, and after that, we compute the correction and $\lambda^{(1)}$. Thus, the calculation of the last residual, $\varepsilon^{(i)}l$, is strictly speaking redundant. The calculations for the increased accuracy of integration showed similar results in the convergence rate.

As the presented results show, in solving the Cauchy problem (with the use of the shooting procedure), several iterations are sufficient, even with a relatively moderate accuracy to achieve a relative error of about 10^{-6}–10^{-7}.

Note that in the case when the condition of the smoothness of function $\lambda(l)$ does not fulfill, the numerical procedure may not converge or may converge to the different EV (see the analytical example in [17]).

8. SMALL TRANSVERSE OSCILLATIONS OF INHOMOGENEOUS ROD

As a benchmark example of the capability of the application of the described technique, we consider the calculation of the lowest eigenvalue for three widely used engineer models of natural transverse oscillations of the elastic rod: the Bernoulli–Euler model, the Rayleigh model, and the Timoshenko model with the hinged boundary conditions [39].

In the Bernoulli–Euler model, the problem considering the natural oscillations of the inhomogeneous rod taking into account the Rayleigh correction (the second term) has the form

$$(EIu'') + \omega^2(\rho Iu') - \rho S\omega^2 u = 0, \quad 0 < x < l,$$
$$u(0) = u(l) = u''(0) = u''(l) = 0,$$

where u is the displacement, E is the Young modulus, I is the moment of inertia of the cross section area, S is the cross section area, ρ is the linear density, ω is the frequency of natural oscillations, and l is the rod length.

In the Timoshenko model, the oscillations are described by the system

$$(EI\psi')' + \kappa GS(u' - \psi) + \rho I\omega^2\psi = 0,$$
$$(\kappa GS(u' - \psi))' + \rho S\omega^2 u = 0, \quad 0 < x < l,$$
$$u(0) = u(l) = \psi'(0) = \psi'(l) = 0,$$

where G is the shear modulus, κ is the Timoshenko coefficient of the cross-sectional expansion (for the circle rod it is $\kappa = 6(1+\nu)^2/(7+12\nu+5\nu^2)$, and ν is the Poisson coefficient).

We transit to dimensionless quantities and introduce the following denotations:

$$\sigma = \frac{EI}{E_0 l^4}, \quad \eta = \frac{\rho I}{\rho_0 l^4}, \quad \lambda = \rho_0 \frac{\omega^2 l^2}{E_0}, \quad \chi = \frac{\kappa GS}{E_0 l^2}, \quad \zeta = \frac{\rho S}{\rho_0 l^2}.$$

From this point on, zero subscript denotes the characteristic value of a physical quantity. We introduce the vector functions

$$y_E = (u, -\sigma u'', -(\sigma u'')', u')^*, \quad y_R = (u, -\sigma u'', -(\sigma u'')' - \lambda\eta u', u')^*, \quad y_T = (u, \sigma\psi', \chi(u' - \psi), -\psi)^*$$

and the matrix functions

$$A_M = \begin{pmatrix} A_{11} & O_2 \\ O_2 & A_{22,M} \end{pmatrix}, \quad M = E, R, T$$
$$A_{11} = \begin{pmatrix} \lambda\zeta & 0 \\ 0 & \sigma^{-1} \end{pmatrix}, \quad A_{22,E} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_{22,R} = \begin{pmatrix} 0 & 1 \\ 1 & \lambda\eta \end{pmatrix}, \quad A_{22,T} = \begin{pmatrix} 0 & -1 \\ -1 & \lambda\eta \end{pmatrix},$$

where subscripts E, R, and T correspond to the Euler–Bernoulli, Rayleigh, and Timoshenko model, respectively. Thus, the equations for the rod oscillations are written in a unified form (2.1) with the matrices of the boundary conditions, $B_{0,l}$ (7.2).

In problems of nondestructive testing, the behavior of the frequencies of rod systems with various defects are of major interest. In the present approach, we illustrate the possibility of the efficient calculation of the lowest frequencies. Consider the oscillations of a circle rod with a density defect [21]. For simplicity, the other physical parameters will be considered to be constant.

The case of longitudinal oscillations of a free rod when modeling the density defect with function was considered in [21]:

$$\rho(x) = \rho_0 + \rho_0 \frac{af}{a^2 + (x-d)^2}, \quad 0 \le x \le 1, \quad 0 \le d \le 1.$$

Using the algorithm implementation outlined in Sect. 7, we computed the first two lower frequencies of the transverse oscillations of the hinged rod. These calculations were performed up to achieving the relative error of EV lower than 10^{-6} for the parameter values $a = 0.01$, $0.05 \le d \le 0.5$ (due to the symmetry around the center of the segment of the density function definition, the EV is also symmetric), $f = f_1, f_2$, $f_1 = 0.003$, and $f_2 = 0.005$, for the rod length $l = 20r_0$ (short rod) and $l = 100r_0$ (long rod), where r_0 is the radius of the rod cross section. Under the chosen parameter values f, the defect of density reaches 30% for f_1 and 50% for f_2. To start the iteration procedure, as an initial estimate we took the frequency of the homogeneous rod found analytically (that is, a solution to the "close" problem). Then, we passed along parameter d (we used the determined value of the frequency as the starting guess for the next parameter value).

In Fig. 2, we present EV curves for the short and long rod, computed with a relative error lower than 10^{-6}, dependent on the position of the defect center (that is, on parameter d). Figure 2a corresponds to the natural oscillations in the Euler model (solid lines); Fig. 2b shows the oscillations with the Rayleigh correction; and Fig. 2c is the Timoshenko model. Solid lines *1* in Figs. 2a–2c reproduce the behavior of the first EV scaled to the EV of the homogeneous rod, $\nu_1 = 4\lambda_1 20^2/\pi^4$ for $f = f_1$; curves *2* reproduce ν_1 for $f = f_2$; curves *3* reproduce the behavior of $\nu_2 = 4\lambda_2 20^2/(2^4\pi^4)$ for $f = f_1$; and curves *4* reproduce ν_2 for $f = f_2$. In Figs. 2a (dashed lines) and 2d we use a similar enumeration of the curves; however, in Fig. 2a,

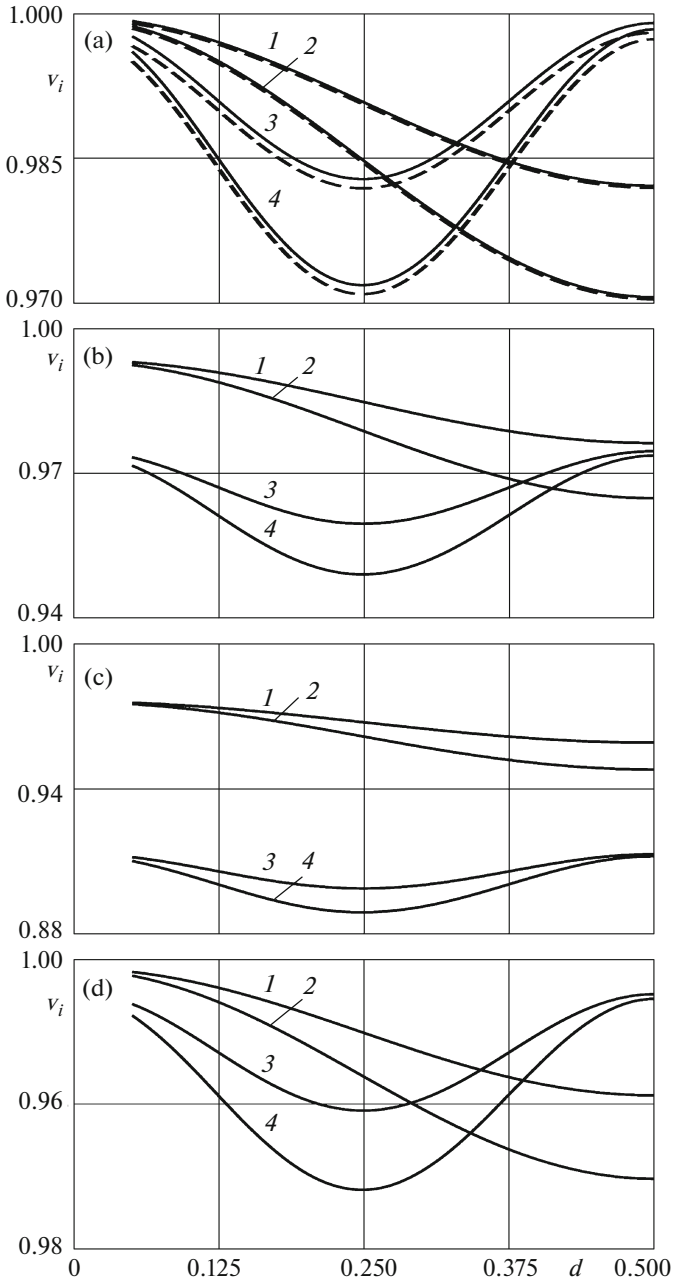

Fig. 2.

using the dashed lines we indicate the EV curves for oscillations with the Rayleigh correction, whereas in Fig. 2d, we use dashed lines in the Timoshenko model. The following scaling is utilized:

$$\nu_1 = 4\lambda_1 100^2/\pi^4, \quad \nu_2 = 4\lambda_2 100^2/(2^4\pi^4).$$

Note that the EVs in the Euler model for the short, $\lambda^s_{1,2}$, and long, $\lambda^l_{1,2}$, rods are here linked with the relationship $\lambda^l_{1,2} = 25\lambda^s_{1,2}$.

The behavior of EVs in our case is similar to the case of longitudinal oscillations of the free rod [21]; the variation of them is moderate (several percent), and the maximum shear appears near the node points (the ends and the middle of the segment). As for the homogeneous rod [40], an increase in the rod length leads to the approaching of EVs found with different models.

The results allow for the suggestion that the method can be applied to investigate a wide range of boundary value problems for EVs and to calculate them with a high degree of accuracy.

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