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On the Constructive Algorithm for Stability Investigation of an Equilibrium Point of a Periodic Hamiltonian System with Two Degrees of Freedom in First-Order Resonance Case

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Abstract—We consider a non-autonomous Hamiltonian system with two degrees of freedom, whose Hamiltonian function is a 2π-periodic function of time and is analytic in the neighborhood of an equilibrium point. It is assumed that the system exhibits a first-order resonance, i.e., the linearized system in the neighborhood of the equilibrium point has a unit multiplier of multiplicity two. The case of the general position is considered when the monodromy matrix is not reduced to the diagonal form, and the equilibrium point is linearly unstable. In this case, a nonlinear analysis is required to draw conclusions on the stability (or instability) of the equilibrium point in the complete system. In this paper, a constructive algorithm for the rigorous-stability analysis of the equilibrium point of the above-mentioned system is presented. This algorithm has been developed on the basis of a method proposed by Markeev. The sufficient conditions for the instability of the equilibrium position, as well as the conditions for its formal stability and stability in the third approximation, are expressed in terms of the coefficients of the normalized map. Explicit formulas are obtained that allow one to calculate the coefficients of the normal form of the Hamiltonian in terms of the coefficients of the generating function of the symplectic map. The developed algorithm is used to solve the problem of the stability of the resonant rotation of a symmetric satellite.

Keywords: Hamiltonian system, stability, resonance of essential type, symplectic map, normalization, resonant rotation of a symmetric satellite

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The main idea of the methodology proposed by Markeev [1] and used below in the development of the stability analysis algorithm, consists in constructing and normalizing the symplectic map generated by the phase flow of the system under consideration.

1. INTRODUCTION

Consider a mechanical system with two degrees of freedom whose motion is described by a non-autonomous canonical system of differential equations

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}, \qquad j = 1, 2. \tag{1.1}$$

We assume that the Hamiltonian H is a 2π-periodic function of time t, analytic in the neighborhood of the equilibrium point of the system, which coincides with the origin of coordinates $q_1 = q_2 = p_1 = p_2 = 0$. This means that in a small neighborhood of the origin, the Hamiltonian can be represented as a convergent series

$$H = H_2 + H_3 + H_4 + \dots, \tag{1.2}$$

where H_m is the form of degree m with coefficients 2π-periodic in t.

If, among the roots of the characteristic equation of the linear canonical system with the Hamiltonian H_2, there is a root with an absolute value that is not equal to unity, then the equilibrium point is Lyapunov unstable [2, 3]. If the critical case takes place when all these roots are equal to unity in absolute value, a nonlinear analysis taking into account the terms of degree higher than two in the expansion of the Hamiltonian (1.2) is required to draw conclusions on the stability of the equilibrium point.

The most general approach to the study of the stability of the equilibrium point of Hamiltonian systems in critical cases is constructing a canonical change of variables that reduces the system (1.1) to a form convenient for stability analysis, for which the expansion of the Hamiltonian (1.2) to terms of some finite degree has the simplest form, the normal form (NF). The conclusions regarding the stability or instability of the equilibrium point can be drawn on the basis of the sufficient conditions [4], which are written in the form of inequalities containing the NF coefficients of the Hamiltonian (1.2). In the general case, these coefficients can only be calculated numerically. Both the classical Birkhoff method [5] and the Deprit–Hori method [6] lead to calculations that are rather cumbersome from an algorithmic point of view.

The procedure for finding the NF of the Hamiltonian is essentially simplified if we normalize not the Hamiltonian system itself, but the symplectic map generated by it. Knowing the generating function of the normalized map, we can obtain the Hamilton function of the corresponding canonical system of differential equations. The Hamiltonian function thus constructed will be the desired NF of the original Hamiltonian. An algorithm for constructing the NF of the Hamiltonian (1.2) was proposed, based on the symplectic mapping method [1]. It was developed for the case when the system parameters lie inside the linear stability region of the equilibrium point, i.e., when all the roots of the characteristic equation of a linear canonical system with the Hamiltonian H_2 are simple and their absolute values are equal to unity. A similar algorithm [7] was developed for the case when the characteristic equation has two simple roots with modules equal to unity and a double root equal to -1 (the case of a second-order resonance). This algorithm was used in the problem of the orbital stability of plane oscillations of a satellite-plate in a circular orbit [8].

In this paper, it is assumed that the system (1.1) has a first-order resonance, i.e., its characteristic equation has a double root equal to unity. This means that the system parameters lie on the boundary of the stability region of the equilibrium point. In what follows, we will consider only the case of general position, when the monodromy matrix of a linear system has nonsimple elementary divisors. In this case, the linear system with the Hamiltonian H_2 is unstable; however, this does not imply the instability of the equilibrium point in the complete nonlinear system. Rigorous conclusions on the stability of the equilibrium point of the system (1.1) may be drawn from analysis of the NF coefficients of the Hamiltonian (1.2) [9].

The goal of this paper is to develop an efficient algorithm for constructing the NF of the Hamiltonian (1.2) in the presence of a first-order resonance in the system.

2. LINEAR NORMALIZATION

In the case of first-order resonance, the characteristic equation of the linear system with a Hamiltonian H_2 can be written as

$$(\rho-1)^2(\rho^2-2a\rho+1)=0; \quad a=\frac{1}{2}\sum_{s=1}^{4}x_{ss}(2\pi)-1, \quad |a|<1, \tag{2.1}$$

where $x_{ss}(t)$ are the diagonal elements of the fundamental matrix $\mathbf{X}(t)$ of a linear system. This matrix satisfies the differential equation

$$\frac{d\mathbf{X}}{dt}=\mathbf{IH}_2\mathbf{X}; \quad \mathbf{I}=\begin{Vmatrix}0 & \mathbf{E}_2\\ -\mathbf{E}_2 & 0\end{Vmatrix} \tag{2.2}$$

with the initial condition

$$\mathbf{X}(0)=\mathbf{E}_4.$$

The Hess matrix of the Hamiltonian H_2 of the linear system is denoted by $\mathbf{H}_2$, $\mathbf{E}_2$, and $\mathbf{E}_4$ are the identity matrices of order two and four, respectively.

In addition to the multiple root $\rho = 1$ the characteristic equation (2.1) has two simple complex conjugate roots ρ_* and $\bar{\rho}_*$, where $\rho_* = e^{i2\pi\lambda}$, and the quantity λ is found from the relation $\cos 2\pi\lambda = a$.

Consider the following symplectic map generated by a linear system with the Hamiltonian H_2:

$$\begin{Vmatrix} q_1^{(1)} \\ q_2^{(1)} \\ p_1^{(1)} \\ p_2^{(1)} \end{Vmatrix} = \mathbf{X}(2\pi) \begin{Vmatrix} q_1^{(0)} \\ q_2^{(0)} \\ p_1^{(0)} \\ p_2^{(0)} \end{Vmatrix}, \tag{2.3}$$

where $q_j^{(0)}, p_j^{(0)}$ denotes the initial values of the variables q_j, p_j, and $q_j^{(1)}, p_j^{(1)}$ denotes their values at $t = 2\pi$ $(j = 1, 2)$.

By making a linear canonical change of variables,

$$\begin{Vmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{Vmatrix} = \mathbf{N} \begin{Vmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{Vmatrix} \tag{2.4}$$

we can reduce the map (2.3) to a simpler form:

$$\begin{Vmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ P_1^{(1)} \\ P_2^{(1)} \end{Vmatrix} = \mathbf{G} \begin{Vmatrix} Q_1^{(0)} \\ Q_2^{(0)} \\ P_1^{(0)} \\ P_2^{(0)} \end{Vmatrix}; \quad \mathbf{G} = \begin{Vmatrix} 1 & 0 & \delta_1 & 0 \\ 0 & \cos 2\pi\sigma & 0 & \sin 2\pi\sigma \\ 0 & 0 & 1 & 0 \\ 0 & -\sin 2\pi\sigma & 0 & \cos 2\pi\sigma \end{Vmatrix} \tag{2.5}$$

$$\sigma = \delta_2\lambda, \quad \delta_1 = \operatorname{sgn}(\mathbf{u}^T\mathbf{I}\mathbf{v}), \quad \delta_2 = \operatorname{sgn}(\mathbf{r}^T\mathbf{I}\mathbf{s}),$$

where **r** and **s** denote, respectively, the real and the imaginary part of the eigenvector, which corresponds to the simple complex root $\rho = e^{i2\pi\lambda}$ of the Eq. (2.1), and **u** and **v** denote the eigenvector and adjoined vector, which correspond to the multiple root $\rho = 1$.

The matrix (2.4) of the linear change has the form

$$\mathbf{N} = (\delta_1 c_1 \mathbf{u}, \delta_2 c_2 \mathbf{r}, c_1 \mathbf{v}, c_2 \mathbf{s}); \quad c_1 = \frac{1}{\sqrt{|\mathbf{u}^T\mathbf{I}\mathbf{v}|}}, \quad c_2 = \frac{1}{\sqrt{|\mathbf{r}^T\mathbf{I}\mathbf{s}|}}. \tag{2.6}$$

This can be shown by an immediate verification. The coefficients c_1 and c_2 are obtained from the condition that the change (2.4) be univalent.

It is straightforward to show that in a linear system with the following Hamiltonian

$$\Gamma_2 = \frac{\delta_1}{4\pi} P_1^2 + \frac{\sigma}{2}(Q_2^2 + P_2^2), \tag{2.7}$$

one can choose as a linear canonical system whose phase flow generates a map (2.5), which indicates the NF of the quadratic part H_2 of the initial Hamiltonian (1.2).

3. NONLINEAR NORMALIZATION

We show how, in the case of a first-order resonance, using a non-linear symplectic map generated by the phase flow of the initial canonical system, obtain the NF of Hamiltonian (1.2), up to terms of degree four inclusive.

Having performed the linear change of variables (2.4), we arrive at the canonical system with the Hamiltonian H^*, which is obtained by substituting the linear change formulas into the Hamilton function (1.2). Following the method of [1], the symplectic map generated by a canonical system with the Hamiltonian H^*, in variables Q_j, P_j $(j = 1, 2)$ can be obtained in the form

$$\begin{Vmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ P_1^{(1)} \\ P_2^{(1)} \end{Vmatrix} = \mathbf{G} \begin{Vmatrix} \tilde{Q}_1^{(0)} \\ \tilde{Q}_2^{(0)} \\ \tilde{P}_1^{(0)} \\ \tilde{P}_2^{(0)} \end{Vmatrix}; \quad \tilde{Q}_j^{(0)} = Q_j^{(0)} - \varkappa_+ \left(\frac{\partial F_3}{\partial P_j^{(0)}} - \sum_{s=1,2} \frac{\partial^2 F_3}{\partial P_j^{(0)} \partial Q_s^{(0)}} \frac{\partial F_3}{\partial P_s^{(0)}} + \frac{\partial F_4}{\partial P_j^{(0)}} \right) + O_4, \tag{3.1}$$

$$(Q_j, +) \leftrightarrow (P_j, -), \quad \varkappa_\pm = \pm 1.$$

Here and what follows, O_k denotes convergent power series of canonical variables that contain no terms of degree lower than k.

The forms F_k $(k = 3, 4)$ are

$$F_k(Q_1^{(0)}, Q_2^{(0)}, P_1^{(0)}, P_2^{(0)}) = \sum_{i_1+i_2+j_1+j_2=k} f_{i_1 i_2 j_1 j_2} Q_1^{(0)\,i_1} Q_2^{(0)\,i_2} P_1^{(0)\,j_1} P_2^{(0)\,j_2}; \quad f_{i_1 i_2 j_1 j_2} = \phi_{i_1 i_2 j_1 j_2}(2\pi).$$

The functions $\phi_{i_1 i_2 j_1 j_2}(t)$ are defined by solving equations

$$\frac{d\phi_{i_1 i_2 j_1 j_2}}{dt} = g_{i_1 i_2 j_1 j_2}, \quad i_1 + i_2 + j_1 + j_2 = 3, 4 \tag{3.3}$$

with the initial conditions

$$\phi_{i_1 i_2 j_1 j_2}(0) = 0.$$

The quantities $g_{i_1 i_2 j_1 j_2}$ on the right-hand sides of Eqs. (3.3) are coefficients of the forms

$$G_k(U_1, U_2, V_1, V_2, t) = \sum_{i_1+i_2+j_1+j_2=k} g_{i_1 i_2 j_1 j_2} U_1^{\,i_1} U_2^{\,i_2} V_1^{\,j_1} V_2^{\,j_2}, \quad k = 3, 4. \tag{3.4}$$

Explicit expressions for coefficients $g_{i_1 i_2 j_1 j_2}$ of the forms G_k are obtained from the relations:

$$G_3 = -\Gamma_3, \quad G_4 = -\Gamma_4 - \sum_{s=1}^{2} \frac{\partial \Gamma_3}{\partial V_s} \frac{\partial \Phi_3}{\partial U_s}, \tag{3.5}$$

where

$$\Phi_3(U_1, U_2, V_1, V_2, t) = \sum_{i_1+i_2+j_1+j_2=3} \varphi_{i_1 i_2 j_1 j_2} U_1^{\,i_1} U_2^{\,i_2} V_1^{\,j_1} V_2^{\,j_2}. \tag{3.6}$$

The forms $\Gamma_k\,(U_1, U_2, V_1, V_2, t)$ have been obtained by substituting

$$\begin{Vmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{Vmatrix} = \mathbf{Y}(t) \begin{Vmatrix} U_1 \\ U_2 \\ V_1 \\ V_2 \end{Vmatrix} \tag{3.7}$$

into the forms $H_k^*(Q_1, Q_2, P_1, P_2)$ $(k = 3, 4)$. Elements of the matrix $\mathbf{Y}(t)$ satisfy the differential equations

$$\frac{dy_{s,l}}{dt} = \frac{\partial H_2^{(l)}}{\partial y_{s+2,l}}, \quad \frac{dy_{s+2,l}}{dt} = -\frac{\partial H_2^{(l)}}{\partial y_{s,l}}; \quad H_2^{(l)} = H_2^*(y_{1,l}, y_{2,l}, y_{3,l}, y_{4,l}, t); \tag{3.8}$$

$$s = 1, 2; \quad l = 1, 2, 3, 4,$$

with the initial condition

$$\mathbf{Y}(0) = \mathbf{E}_4.$$

Thus, the coefficients of the forms F_3 and F_4 are obtained as a result of numerical integration on the interval $[0, 2\pi]$ of a system of 71 equations (55 equations (3.3) for $\phi_{i_1 i_2 j_1 j_2}$ and 16 equations (3.8) for $y_{s,l}$).

We now make a linear univalent change of variables using the formulas

$$Q_1 = x_1, \quad Q_2 = \frac{i+1}{2}(x_2 + y_2), \quad P_1 = y_1, \quad P_2 = \frac{i-1}{2}(x_2 - y_2), \tag{3.9}$$

where i is the imaginary unit.

In the new variables x_j, y_j, $(j = 1,2)$, the map (3.1) takes the following form:

$$x_1^{(1)} = \tilde{x}_1^{(0)} + \delta_1 \tilde{y}_1^{(0)}, \quad x_2^{(1)} = \rho^* \tilde{x}_2^{(0)}; \quad y_1^{(1)} = \tilde{y}_1^{(0)}, \quad y_2^{(1)} = \frac{1}{\rho^*}\tilde{y}_2^{(0)}, \tag{3.10}$$

$$\tilde{x}_j^{(0)} = x_j^{(0)} - \varkappa_+\left(\frac{\partial Z_3}{\partial y_j^{(0)}} - \sum_{s=1,2}\frac{\partial^2 Z_3}{\partial y_j^{(0)}\partial x_s^{(0)}}\frac{\partial Z_3}{\partial y_s^{(0)}} + \frac{\partial Z_4}{\partial y_j^{(0)}}\right) + O_4, (x_j, \varkappa_+) \leftrightarrow (y_j, \varkappa_-),$$

where

$$Z_3 = F_3^*, \quad Z_4 = F_4^* + \frac{1}{4}\left(\frac{\partial F_3^*}{\partial x_2^{(0)}}\right)^2 - \frac{1}{4}\left(\frac{\partial F_3^*}{\partial y_2^{(0)}}\right)^2 + \frac{1}{2}\frac{\partial F_3^*}{\partial x_2^{(0)}}\frac{\partial F_3^*}{\partial y_2^{(0)}}. \tag{3.11}$$

Here, F_k^* denote the forms F_k in which Q_j, P_j are expressed in terms of x_j, y_j, through the formula (3.9)

Let us construct a canonical change of variables $x_j, y_j \to \xi_j, \eta_j$, which allows a simplification of the form of the symplectic map (3.10). We will search for the generating function of this change in the form

$$R = x_1\eta_1 + x_2\eta_2 + R_3 + R_4, \tag{3.12}$$

$$R_n = \sum_{i_1+i_2+j_1+j_2=n} r_{i_1 i_2 j_1 j_2} x_1^{i_1} x_2^{i_2} \eta_1^{j_1} \eta_2^{j_2}. \tag{3.13}$$

The old and new variables are related in the following way:

$$y_j = \frac{\partial R}{\partial x_j}, \quad \xi_j = \frac{\partial R}{\partial \eta_j}, \quad j = 1,2 \tag{3.14}$$

Taking into account the structure of the generating function R, we have from Eqs. (3.14) the following explicit formulas that express the old variables in terms of new ones:

$$x_j = \xi_j - \varkappa_+\left(\frac{\partial R_3}{\partial \eta_j} - \sum_{s=1,2}\frac{\partial^2 R_3}{\partial \eta_j \partial \xi_s}\frac{\partial R_3}{\partial \eta_s} + \frac{\partial R_4}{\partial \eta_j}\right) + O_4, \quad (x_j, \xi_j, \varkappa_+) \leftrightarrow (y_j, \eta_j, \varkappa_-). \tag{3.15}$$

Substituting these expressions into the map (3.10), we get

$$\xi_1^{(1)} = \tilde{\xi}_1^{(0)} + \delta_1 \tilde{\eta}_1^{(0)}, \quad \xi_2^{(1)} = \rho_* \tilde{\xi}_2^{(0)}; \quad \eta_1^{(1)} = \tilde{\eta}_1^{(0)}, \quad \eta_2^{(1)} = \frac{1}{\rho_*}\tilde{\eta}_2^{(0)}, \tag{3.16}$$

$$\tilde{\xi}_j^{(0)} = \xi_j^{(0)} - \varkappa_+\left(\frac{\partial W_3}{\partial \eta_j^{(0)}} - \sum_{s=1,2}\frac{\partial^2 W_3}{\partial \eta_j^{(0)}\partial \xi_s^{(0)}}\frac{\partial W_3}{\partial \eta_s^{(0)}} + \frac{\partial W_4}{\partial \eta_j^{(0)}}\right) + O_4, \quad (\xi_j, \varkappa_+) \leftrightarrow (\eta_j, \varkappa_-).$$

Here,

$$W_3 = R_3 + Z_3 - R_3^*,$$
$$W_4 = R_4 + Z_4 - R_4^* + \sum_{s=1,2}\left(\frac{\partial R_3}{\partial \xi_s^{(0)}}\frac{\partial Z_3}{\partial \eta_s^{(0)}} - \frac{\partial W_3}{\partial \xi_s^{(0)}}\frac{\partial R_3^*}{\partial \eta_s^{(0)}}\right) - \frac{1}{2}\delta_1\left(\frac{\partial R_3^*}{\partial \xi_1^{(0)}}\right)^2, \tag{3.17}$$
$$R_k^* = R_k\left(\delta_1\eta_1^{(0)} + \xi_1^{(0)}, \rho^*\xi_2^{(0)}, \eta_1^{(0)}, \frac{\eta_2^{(0)}}{\rho^*}\right), \quad k = 3,4.$$

The still undetermined coefficients $r_{i_1 i_2 j_1 j_2}$ of the forms R_3 and R_4 are defined from Eqs. (3.17) in such a way that the maximal number of terms in the forms W_3 and W_4 vanishes. Calculations show that for the specified choice of the coefficients $r_{i_1 i_2 j_1 j_2}$ of the forms W_3 and W_4, they will have the form

$$W_3 = w_{3000}{\xi_1^{(0)}}^3 + w_{1101}\xi_1^{(0)}\xi_2^{(0)}\eta_2^{(0)},$$
$$W_4 = w_{4000}{\xi_1^{(0)}}^4 + w_{2101}{\xi_1^{(0)}}^2\xi_2^{(0)}\eta_2^{(0)} + w_{0202}{\xi_2^{(0)}}^2{\eta_2^{(0)}}^2, \tag{3.18}$$

where

$$w_{3000} = f_{3000}, \quad w_{1101} = i(f_{1002} + f_{1200}),$$
$$w_{4000} = \frac{1}{4}(f_{2001}^2 + f_{2100}^2)\cot \pi\sigma + f_{4000} - \frac{3}{2}f_{3000}(f_{1020}\delta_1 + f_{2010}) - \frac{1}{2}f_{2001}f_{2100} + \frac{1}{2}f_{2010}^2\delta_1 \quad (3.19)$$

Since the formulas for the coefficients w_{0202} and w_{2101} are rather cumbersome, they are presented separately in Section 5.

Formulas (3.16) and (3.18) define the explicit form of the normalized map (3.1) up to third-degree terms inclusive. On the other hand, the initial map (3.1) is generated by the phase flow of the canonical system with the Hamiltonian H^*. Therefore, by an appropriate choice of canonical variables, the Hamiltonian H^* can be reduced to a form in which it will correspond to the normalized map (3.16). Note that that the above-mentioned choice of variables (and hence the type of the Hamiltonian) is not uniquely defined [4]. In particular, the Hamilton function corresponding to the map (3.16) can be searched for in the form

$$K = \frac{\delta_1}{4\pi}\eta_1^2 + i\sigma\xi_2\eta_2 + K_3 + K_4 + K^{(5)}, \quad (3.20)$$

where

$$K_m = \sum_{i_1+i_2+j_1+j_2=m} k_{i_1i_2j_1j_2}\xi_1^{i_1}\xi_2^{i_2}\eta_1^{j_1}\eta_2^{j_2}, \quad m = 3, 4 \quad (3.21)$$

and the still undefined coefficients $k_{i_1i_2j_1j_2}$ are taken to be constant. Here, $K^{(5)}$ denotes a convergent power series that begins with terms of degree at least five in the canonical variables ξ_j, η_j $(j = 1, 2)$ whose coefficients 2π-periodically depend on t.

The symplectic map generated by the phase flow of the canonical system with Hamiltonian (3.20) has the form (3.1). Since the quadratic part of the Hamiltonian (3.20) does not explicitly depend on t, one can solve analytically Eqs. (3.3) and (3.8) and obtain expressions for the coefficients $f_{i_1i_2j_1j_2}$ of the forms F_3 and F_4 in terms of the coefficients $k_{i_1i_2j_1j_2}$ of the Hamiltonian (3.20), which should obviously be chosen in such a way that the equalities

$$F_3 = W_3, \quad F_4 = W_4 \quad (3.22)$$

are satisfied identically.

Equating the coefficients with equal monomials in the left-hand and right-hand sides of Eqs. (3.22), one can obtain a system of algebraic equations from which the coefficients $k_{i_1i_2j_1j_2}$ of the Hamiltonian (3.20) are uniquely defined in terms of the coefficients $w_{i_1i_2j_1j_2}$ of the forms W_3 and W_4. The explicit form of the coefficients $k_{i_1i_2j_1j_2}$ is presented in Section 5.

The canonical univalent near-identity change of variables $\xi_j, \eta_j \to u_j, v_j$ $(j = 1, 2)$ which is given by the generating function

$$S = \xi_1 v_1 + \xi_2 v_2 + S_3 + S_4. \quad (3.23)$$

The Hamiltonian (3.20) is reduced to the NF

$$\begin{aligned} H = {} & \frac{1}{4\pi}\delta_1 Y_1^2 + \frac{1}{2}\sigma(X_2^2 + Y_2^2) - \frac{1}{2\pi}w_{3000}X_1^3 - \frac{1}{4\pi}b_{1101}X_1(X_2^2 + Y_2^2) \\ & - \frac{1}{16\pi}(8w_{4000} - 3\delta_1 w_{3000}^2)X_1^4 - \frac{1}{4\pi}b_{2101}X_1^2(X_2^2 + Y_2^2) \\ & - \frac{1}{192\pi}(\delta_1 b_{1101}^2 + 24w_{0202})(X_2^2 + Y_2^2)^2 + O_5, \end{aligned} \quad (3.24)$$

where $b_{1101} = f_{1002} + f_{1200}$, and the expression for b_{2101} is given in section 5 (see Eqs. (5.2)). Real variables X_j, Y_j are introduced by the formulas

$$u_1 = X_1, \quad u_2 = -\frac{i+1}{2}(iX_2 + Y_2), \quad v_1 = Y_1, \quad v_2 = -\frac{i+1}{2}(iX_2 - Y_2) \quad (3.25)$$

The expressions for the coefficients $s_{i_1i_2j_1j_2}$ of the forms S_3 and S_4 are given in Section 5.

The problem of the stability of the equilibrium point of a system with the Hamiltonian (3.24) was investigated in detail in [9]. Using the results obtained in [9] and the notations therein, we formulate sufficient conditions for stability and instability as follows. If $w_{3000} \neq 0$, then the equilibrium point is unstable. If $w_{3000} = 0$, however, when $\delta_1 w_{4000} > 0$ the equilibrium point is unstable, when $\delta_1 w_{4000} < 0$ there is stability when the terms of degree higher than four in the Hamiltonian (3.24) are taken into account, and when the inequalities $\delta_1 w_{4000} < 0$ and $\delta_1 \sigma > 0$ are satisfied simultaneously, the equilibrium point is formally stable. In the case of $w_{3000} = w_{4000} = 0$ an additional analysis is required to solve the stability problem. It can be performed by taking into account terms of degree higher than four in the expansion of the Hamiltonian (3.24).

Conclusions on the stability of a system with a normalized Hamilton function (3.24) also hold for a system with an initial Hamilton function (1.2). Thus, the problem of investigating the stability of the trivial equilibrium point of system (1.1) amounts to the following. It is necessary to construct a symplectic map (3.1) generated by the phase flow of system (1.1) and, using formulas (3.19), to calculate the coefficient w_{3000} and if it turns out to be zero, then the coefficient w_{4000}. Then, based on the above-mentioned sufficient conditions, to draw conclusions on the stability of the equilibrium point.

4. ON THE STABILITY OF THE RESONANT ROTATION OF A DYNAMICALLY SYMMETRIC SATELLITE

Consider a satellite moving in a central Newtonian gravitational force field. The satellite is modeled by a dynamically symmetric rigid body. To describe the motion of the satellite with respect to the center of mass, we introduce an orbital coordinate system *OXYZ* and a satellite-fixed coordinate system *Oxyz*. The axes *X*, *Y*, and *Z* of the orbital coordinate system are directed along the radius vector of the center of mass relative to the attracting center, along the transversal and along the normal to the orbit, respectively.

The axes of the coordinate system rigidly connected with the satellite *Oxyz* are directed along its principal central axes of inertia; the axis z is directed along the axis of symmetry. The orientation of the satellite-fixed coordinate system relative to the orbital coordinate system is given by the Euler angles ψ, θ, φ.

Introducing the generalized momenta p_ψ, p_θ, p_φ, which canonically conjugate angles ψ, θ, φ, we can write the equations of motion in Hamiltonian form. By virtue of the dynamical symmetry of the satellite the angle of proper rotation, φ, is a cyclic coordinate and, therefore, the corresponding momentum p_φ keeps a constant value $p_\varphi = \text{const}$ on the satellite's motions.

In what follows, we assume $p_\varphi = 0$, that is, we will consider the limited problem of the motion of a symmetric satellite, assuming that the projection of the absolute angular velocity onto the axis of its dynamic symmetry is zero. Under this assumption the Hamiltonian of the problem has the following form [10, 11]:

$$H = \frac{1}{2}\frac{p_\psi^2}{\zeta^2 \sin^2\theta} + \frac{1}{2}\frac{p_\theta^2}{\zeta^2} - p_\psi + \frac{1}{2}\zeta\alpha \sin^2\psi \sin^2\theta, \qquad \zeta = (1 + e\cos\nu), \tag{4.1}$$

where $\alpha = 3(C - A)/A$, e is the eccentricity of the orbit, ν is the true anomaly, and A and C are, respectively, the equatorial and the polar moment of inertia.

If the parameters α and e satisfy the ratio

$$\alpha - 2e(0 < e < 1),$$

then the system of equations with Hamiltonian (4.1) has an exact solution [12]

$$\psi^* = -\frac{1}{2}\nu, \quad p_\psi^* = \frac{1}{2}\zeta^2, \quad \theta^* = \frac{1}{2}\pi, \quad p_\theta^* = 0 \tag{4.2}$$

and corresponds to the planar motion of the satellite, in which one of its main central axes of inertia is perpendicular to the plane of the orbit, and the satellite completes one rotation in absolute space during two orbital revolutions of its center of mass.

The stability of the resonant rotations (4.2) in various formulations was investigated earlier in [13–17]. It was established that as the eccentricity approaches unity, intervals of linear stability and intervals of instability alternate.

The stability of the resonant rotation (4.2) in the case of a dynamically symmetric satellite was examined in [17], and five intervals of linear stability were found inside which rigorous conclusions on stability for most initial conditions, formal stability, stability in the third approximation, or instability were obtained based on a nonlinear analysis. The values of eccentricity regarding the boundaries of the above-mentioned intervals, which correspond to first-order and second-order resonances, remained unexplored. The first-order resonance takes place at the following boundary points of stability intervals:

$$\begin{aligned} e_1 &= 0.917910, \quad e_2 = 0.990545, \\ e_3 &= 0.999304, \quad e_4 = 0.999919. \end{aligned} \tag{4.3}$$

The problem of the stability of the resonant rotation for these values of eccentricity can be solved using the results of Sections 2 and 3. As before [17], the stability study will be performed with respect to perturbations that retain the zero value of the projection of the satellite's absolute angular velocity onto its axis of dynamic symmetry, i.e., in the framework of the limited problem considered here.

To describe the motion in the vicinity of the resonant rotation (4.2), we introduce the perturbations q_j, p_j $(j = 1, 2)$ with the following formulas:

$$\begin{aligned} \psi &= \psi^* + \frac{q_1}{\zeta}, \quad p_\psi = p_\psi^* + \zeta p_1 + e \sin\nu q_1, \\ \theta &= \theta^* + \frac{q_2}{\zeta}, \quad p_\theta = p_\theta^* + \zeta p_2 + e \sin\nu q_2. \end{aligned} \tag{4.4}$$

The canonical equations of perturbed motion have the following form

$$\frac{dq_j}{d\nu} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{d\nu} = -\frac{\partial H}{\partial q_j}, \quad j = 1, 2. \tag{4.5}$$

We present the necessary terms of series expansion of the Hamilton function H in the neighborhood of $q_j = p_j = 0$ $(j = 1, 2)$ [17]

$$\begin{aligned} H_2 &= \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 - \frac{1}{2}\frac{e\cos\nu q_1^2}{\zeta} + \frac{1}{8}\frac{(\zeta + 4e)q_2^2}{\zeta}, \\ H_3 &= \frac{1}{2}\frac{p_1 q_2^2}{\zeta} - \frac{1}{2}\frac{e\sin\nu q_1 q_2^2}{\zeta^2} - \frac{2}{3}\frac{e\sin\nu q_1^3}{\zeta^2}, \\ H_4 &= \frac{1}{2}\frac{q_2^2 p_1^2}{\zeta^2} + \frac{1}{12}\frac{(3\zeta - 2e - 2)q_2^4}{\zeta^3} + \frac{e\sin\nu q_1 q_2^2 p_1}{\zeta^3}, \\ &+ \frac{1}{2}\frac{(\zeta^2 + e^2 - 1)q_1^2 q_2^2}{\zeta^4} + \frac{1}{3}\frac{e\cos\nu q_1^4}{\zeta^3}. \end{aligned} \tag{4.6}$$

For $e = e_1$, the numerically found monodromy matrix of the linear system has the form

$$\mathbf{X}(2\pi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.81307 & 0 & 0.70665 \\ 1.31451 & 0 & 1 & 0 \\ 0 & -0.47960 & 0 & -0.81307 \end{bmatrix} \tag{4.7}$$

As a result of a linear change of variables (2.4) with a matrix

$$\mathbf{N} = \begin{bmatrix} 0 & 0 & 0.87220 & 0 \\ 0 & -1.10175 & 0 & 0 \\ -1.14652 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.90765 \end{bmatrix} \tag{4.8}$$

and using the method of Section 3, a symplectic map (3.1) was constructed, for which

$$\begin{aligned} F_3 &= -202.39Q_1^3 + 303.59Q_1^2 P_1 - 153.28Q_1 P_1^2 - 11.183Q_1 Q_2^2 \\ &+ 2.7761Q_1 Q_2 P_2 - 7.3063Q_1 P_2^2 + 26.042P_1^3 + 5.5613P_1 Q_2^2 \end{aligned}$$

$$-1.4730P_1Q_2P_2 + 3.6835P_1P_2^2. \tag{4.9}$$

Because the coefficient $w_{3000} = f_{3000} = -202.39$ is non-zero, in this case the instability of the resonant rotation (4.2) takes place.

Similarly, stability was studied for eccentricity values e_2, e_3, and e_4. Calculations showed that for all these values the resonant rotation is unstable.

5. CALCULATED FORMULAS

We present formulas for calculating the coefficients of the normalized map (3.16):

$$\begin{aligned} w_{0202} &= -\frac{1}{2}(3f_{0004} + f_{0202} + 3f_{0400}) - \frac{\delta_1}{2}(f_{0012} + f_{0210})^2 \\ &+\frac{9}{4}(f_{0003}f_{0102} + f_{0003}f_{0300} + f_{0201}f_{0300}) + \frac{5}{4}(f_{0012}f_{1002} + f_{0102}f_{0201} + f_{0210}f_{1200}) \\ &+\frac{1}{4}(3f_{0012}f_{1200} + f_{0111}f_{1101} + 3f_{0210}f_{1002}) + \frac{\delta_1}{16}\frac{(f_{1002} - f_{1200})^2 + f_{1101}^2}{\sin^2 2\pi\sigma} \\ &- \frac{1}{16\sin 2\pi\sigma(2\cos 2\pi\sigma + 1)}[(72(f_{0003}^2 + f_{0300}^2) + 24(f_{0102}^2 + f_{0201}^2) \\ &- 8f_{1101}(f_{0012} - f_{0210}) + 8f_{0111}(f_{1002} - f_{1200}))\cos^2 2\pi\sigma \\ &+ (90(f_{0003}^2 + f_{0300}^2) + 18(f_{0102}^2 + f_{0201}^2) + 36(f_{0003}f_{0201} + f_{0102}f_{0300}) \\ &- 4f_{1101}(f_{0012} - f_{0210}) + 4f_{0111}(f_{1002} - f_{1200}))\cos 2\pi\sigma + 18(f_{0003}^2 + f_{0300}^2) \\ &+ 36(f_{0003}f_{0201} + f_{0102}f_{0300}) - 6(f_{0102}^2 + f_{0201}^2)], \end{aligned} \tag{5.1}$$

$$w_{2101} = a_{2101} + ib_{2101}, \tag{5.2}$$

$$a_{2101} = -\frac{1}{2}(f_{1002} + f_{1200})^2,$$

$$\begin{aligned} b_{2101} &= f_{2002} + f_{2200} + \delta_1 f_{2010}(f_{0012} + f_{0210}) - \frac{1}{2}\delta_1 f_{1020}(f_{1002} + f_{1200}) \\ &-\frac{3}{2}f_{0003}f_{2100} - 3f_{3000}(f_{0012} + f_{0210}) - \frac{3}{2}f_{0300}f_{2001} \\ &-f_{1101}(f_{1200} + f_{1002}) - \frac{1}{2}f_{0102}f_{2001} - \frac{1}{2}f_{0201}f_{2100} - \frac{1}{2}f_{2010}(f_{1002} + f_{1200}) \\ &- f_{2001}f_{1011} - f_{2100}f_{1110} + \frac{\delta_1(f_{2001}^2 + f_{2100}^2)}{\cos 2\pi\sigma - 1} \\ &+ \frac{1}{2\sin 2\pi\sigma}[(3f_{0003}f_{2001} + f_{0102}f_{2100} + f_{0201}f_{2001} + 3f_{0300}f_{2100} \\ &+ (f_{1002} - f_{1200})^2 - 2f_{1011}f_{2100} + f_{1101}^2 + 2f_{1110}f_{2001})\cos 2\pi\sigma + 3f_{0003}f_{2001} \\ &+f_{0102}f_{2100} + f_{0201}f_{2001} + 3f_{0300}f_{2100} - 2f_{1011}f_{2100} + 2f_{1110}f_{2001}] \end{aligned}$$

to calculate the coefficients of the Hamiltonian (3.20):

$$k_{3000} = -\frac{1}{2\pi}w_{3000}, \quad k_{2010} = \frac{3}{4\pi}\delta_1 w_{3000}, \quad k_{1101} = -\frac{1}{2\pi}w_{1101},$$

$$k_{1020} = -\frac{1}{4\pi}w_{3000}, \quad k_{0111} = \frac{1}{4\pi}\delta_1 w_{1101},$$

$$k_{4000} = -\frac{\delta_1}{2}k_{3010}, \quad k_{3010} = \frac{1}{4\pi}(4\delta_1 w_{4000} - 3w_{3000}^2),$$

$$k_{2101}=\frac{1}{4\pi}(w_{1101}^2-2w_{2101}+\delta_1 w_{1101}w_{3000}),\qquad k_{2020}=-\frac{\delta_1}{2}k_{3010}.$$

$$k_{1111}=-\frac{1}{4\pi}(\delta_1 w_{1101}^2-2\delta_1 w_{2101}+w_{1101}w_{3000}), \tag{5.3}$$

$$k_{0202}=\frac{1}{24\pi}(\delta_1 w_{1101}^2-12w_{0202}),$$

$$k_{0121}=\frac{1}{120\pi}(5w_{1101}^2-10w_{2101}+3\delta_1 w_{1101}w_{3000}),$$

$$k_{0040}=\frac{1}{840\pi}(14w_{4000}-9\delta_1 w_{3000}^2)$$

and to calculate the coefficients of the generating function (3.23):

$$s_{3000}=-\frac{1}{2}w_{3000},\qquad s_{2010}=\frac{1}{4}\delta_1 w_{3000},\qquad s_{1101}=-\frac{1}{2}w_{1101},$$

$$s_{4000}=-\frac{1}{2}w_{4000},\qquad s_{3010}\frac{1}{24}(8\delta_1 w_{4000}-3w_{3000}^2),$$

$$s_{2101}=\frac{1}{8}(3w_{1101}^2-4w_{2101}), \tag{5.4}$$

$$s_{1111}=\frac{1}{60}(10\delta_1 w_{2101}-5\delta_1 w_{1101}^2-3w_{1101}w_{3000}),$$

$$s_{1030}=\frac{1}{420}(9w_{3000}^2-14\delta_1 w_{4000}).$$

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