ISSN 0025-6544, Mechanics of Solids, 2018, Vol. 53, No. 4, pp. 440–453. © Allerton Press, Inc., 2018.
Original Russian Text © S.A. Lurie, D.B. Volkov–Bogorodskii, 2018, published in Izvestiya Akademii Nauk, Mekhanika Tverdogo Tela, 2018, No. 4, pp. 100–114.

Green Tensor and Solution of the Boussinesq Problem in the Generalized Theory of Elasticity

S. A. Lurie[1,2*] and D. B. Volkov–Bogorodskii[2]

[1] *Ishlinsky Institute for Problems in Mechanics RAS, pr. Vernadskogo 101, str. 1, Moscow, 119526 Russia*
[2] *Institute of Applied Mechanics of Russian Academy of Sciences, Leningradskii pr. 7, Moscow, 125040 Russia*

Received March 2, 2018

Abstract—The fundamental spatial problems of the theory of elasticity such as the problem of constructing Green tensor and the Boussinesq problem of the action of a concentrated force on a half-space are considered. According to the classical theory of elasticity, these problems are singular. It is shown that an analytical solution of such problems can be constructed by the Papkovich–Neuber representation without invoking symmetry conditions. This makes it possible to present the solution of the problems under consideration in a single form and allows us to write an explicit solution of half-space loaded by a concentrated vector-force having non-zero projections onto the normal to the plane bounding the half-space and onto the plane itself.
This paper deals with the generalized regular solutions of the considered fundamental problems of the elasticity. The solutions are limited at a singular point and damp at infinity.

DOI: 10.3103/S0025654418040106

Keywords: *generalized theory of elasticity, Papkovich–Neuber representation, regular solution, Green tensor, Boussinesq problem.*

1. INTRODUCTION

The solutions of the fundamental problems of the theory of elasticity such as Boussinesq and Flamant problems, the problems of constructing Green functions are the basis for obtaining many important theoretical and applied results in the study of the stress-strain state of elastic solids and also in estimating the strength and fracture of materials. However, the classical solutions of these problems are singular for displacement fields, deformations, and stresses. As noted in [1, 2], the singularity of the solution in the problems of the theory of elasticity is not physically determined and appears due to a mismatch between the mathematical and physical models of a continuum. In recent works [3–8], it has been shown that the use of the generalized theory of elasticity makes it possible to obtain regular solutions for problems that are singular in the classical theory. The relations of the generalized theory of elasticity are constructed on the basis of consideration of the equilibrium of a finite medium fragment, but not of an infinitely small medium element as it is done in the conventional elasticity. Generalized fields of displacements, deformations, and stresses are introduced by averaging over a representative fragment and they take into account high gradients. It is shown that the constructed generalized theory makes it possible to implement the regularization of a number of singular solutions and obtain regular ones for the rigid die problem [7], problems in the theory of cracks [6], etc. Note that the gradient theories of elasticity were also previously used to regularize classical singular solutions for problems of continuous dislocation theory [9]. In [10], the simplest one-parameter gradient theory was used to obtain a regular solution of the Flamant problem, and in [11] the regular solutions for the Green function in gradient theories of elasticity were constructed by Fourier transforms. We note that to obtain regular solutions for point dislocations [9] and the Green function [11], it was not necessary to use boundary conditions, but only general representations of fundamental solutions were used. Such problems do not require a clear definition of the physical meaning of generalized solutions, as well as the constitutive relations, which,

*e-mail: salurie@mail.ru

as is well known, are often contradictory in gradient theories. In the Flamant problem [10], there is a question about the fulfillment of boundary conditions; it has not been solved correctly. In [7], an analytic regular generalized solution of the Boussinesq problem was first obtained for loading a half-space with a force normal to the half-space boundary under the assumption that the solution was constructed in cylindrical coordinates and had the properties of the generalized solution in the direction of the radial coordinate in the boundary plane.

In this article, we give complete generalized solutions for the Green function and for the Boussinesq problem. They are constructed by the Papkovich–Neuber representation. It is shown that the generalized theory of elasticity makes it possible to find regular, every where limited solutions for the displacement and stress fields that damp at infinity. The explicit expressions are given for the components of displacements and stresses for the cases of local loading of the surface of a half-space by the force applied along the normal to the surface of the boundary and by the force lying in the plane.

2. BASIC EQUATIONS OF THE GENERALIZED THEORY OF ELASTICITY

In [3, 4, 8] a version of the theory of elasticity was developed on the basis of the analysis of a medium element having a form of a parallelepiped, which has small but finite dimensions. The definitions of generalized displacement fields, deformations, and stresses are proposed to introduce as a special case of the determination of a generalized tensor field as described below. Suppose that there is some tensor field. For its components in Cartesian coordinates, it is suggested to introduce the corresponding averaging over the considered medium fragment with a certain weight. Further, for the tensor components in the expression under the integral sign, Taylor series expansions in local coordinates are used in the neighborhood of the origin of the fragment coordinates and it is proposed to remain in these expansions the terms containing derivatives up to the third order (inclusive). The result of integrating the expressions obtained over a representative fragment is the definition of a generalized non-local tensor field. In the general case, the generalized tensor field is formally defined in terms of the tensor field and its derivatives in the considered fragment with an accuracy up to three structural parameters having the dimension of the order of a squared length. For an isotropic medium, it can be assumed that these parameters are equal. Thus, the generalized tensor field takes into account the variability of the tensor field over a representative fragment and is determined with an accuracy of a structural dimensional parameter.

When averaging, the appropriate choice of the weight function allows us to introduce the following definitions of generalized non-local fields of displacements U_i, deformations E_{ij}, and stresses T_{ij}, respectively, in terms of the fields of true displacements $R_i(x, y, z)$, deformations $\varepsilon_{ij}(x, y, z)$, and stresses $\sigma_{ij}(x, y, z)$:

$$U_i = R_i - s^2\Delta R_i, \quad E_{ij} = \varepsilon_{ij} - s^2\Delta\varepsilon_{ij}, \quad T_{ij} = \sigma_{ij} - s^2\Delta\sigma_{ij}, \tag{2.1}$$
$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} = f_{,kk}.$$

It was shown [8] that nonlocal partial derivatives of functions $R_i(x, y, z)$ coincide with traditional derivatives of a generalized function $U_i(x, y, z)$. Therefore, for generalized stresses and strains, the Cauchy relations remain valid:

$$E_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i}), \quad U_{i,j} = \frac{\partial U_i}{\partial x_j}. \tag{2.2}$$

The direct derivation of the equilibrium equations in terms of generalized stresses is presented in [3, 4] as a result of the analysis of the equilibrium of a representative fragment. These equations have a classical form and can be written in the divergence form if we take into account the definitions of generalized derivatives. In the absence of bulk forces, the equilibrium equations for generalized stresses have the classical form

$$T_{ij,j} = 0. \tag{2.3}$$

The generalized stresses T_{ij} are associated with generalized deformations E_{mn} by Hooke law, that is,

$$T_{ii} = C_{ijmn}E_{mn}, \quad C_{ijmn} = \lambda\delta_{ij}\delta_{mn} + \mu(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}), \tag{2.4}$$

where λ and μ are Lame coefficients.

Taking (2.4) into account, we can write the equilibrium equations for a vector in terms of generalized displacements $\mathbf{U}$ in a vector form. For an elastic isotropic solid, these equations completely coincide with the equilibrium equations for classical theory of elasticity.

$$\mathbf{L}(\mathbf{U}) = \mathbf{0}, \quad \mathbf{L}(\mathbf{U}) = \mu\Delta\mathbf{U} + (\mu+\lambda)\boldsymbol{\nabla}\,\mathrm{div}\,\mathbf{U}. \tag{2.5}$$

If we use the virtual displacements principle and write the expression for the work of the resulting forces applied on variations of generalized displacements (mean displacements for this element), then we get

$$\iiint \Big[T_{ij,j}\delta U_i\Big] dx_1 dx_2 dx_3 = 0.$$

This equality allows us to write down the natural boundary conditions, which at the point of surface intersection with the normal n_j for generalized stresses and displacements have the traditional form

$$T_{ij}n_j\delta U_i = 0. \tag{2.6}$$

As a result, a generalized theory of elasticity has been constructed. In the framework of this theory, the generalized fields of displacements and stresses are determined via the true fields of displacements and stresses by relations (2.1). For the generalized theory the Cauchy relations (2.2) are valid and the tensor of generalized stresses satisfies the equilibrium equations (2.5) written in the classical form. A feature of this theory is that the physical relations are formulated for generalized stresses and strains (2.4), and static boundary conditions are written by the traditional way for generalized stresses (2.6).

These features of the theory allow us to formulate a consistent algorithm for constructing solutions of the problems under static boundary conditions. This solution has two stages. At the first stage, the traditional boundary value problem for the equilibrium equation in terms of displacements with traditional boundary conditions is considered and the field of classical displacements and stresses is determined. Thus, as a solution at the first stage, it is proposed to use the traditional problem solution of mathematical physics. It can be regular or singular. At the second stage, the first equation from (2.1) is considered and the traditional solution obtained above is substituted into its left-hand side as a function $U_i(x_1,\ x_2,\ x_3)$. The complete solution of the desired problem includes a particular solution defined by a function f and a general solution of the corresponding homogeneous equation, i.e. Helmholtz equations. Hereafter, the solutions of the Helmholtz equations found by the classical traditional solutions for displacements $U_i(x_1,\ x_2,\ x_3)$ (or stresses $T_{ij}(x_1,\ x_2,\ x_3)$) will be called generalized solutions. In [5–8], it has been shown that such a structure of generalized solutions allows us to eliminate the singularity, if the traditional solution (the left side of the Helmholtz equation (2.1)) is singular.

Then we construct regular solutions of two problems: the problem of determining the Green function and the Boussinesq problem for a half-space. Both fundamental problems are singular in the classical theory of elasticity. To solve these problems, the representation of the general solution of the theory of elasticity in the form of the Papkovich–Neuber representation [12, 13] in terms of scalar and vector potentials that satisfy the Laplace equation is used.

3. GREEN TENSOR IN THE SPATIAL PROBLEM OF THE CLASSICAL THEORY OF ELASTICITY

The system of Lame equilibrium equations in terms of displacements for an elastic isotropic body in the classical theory of elasticity is analyzed.

$$\mathbf{L}(\mathbf{U}) = \mathbf{h}, \quad \mathbf{L}(\mathbf{U}) = \mu\Delta\mathbf{U} + (\mu+\lambda)\ \boldsymbol{\nabla}\,\mathrm{div}\,\mathbf{U}. \tag{3.1}$$

The solutions of the Lame equation (3.1), as shown in [12, 13], can be represented as a differential expression in terms of auxiliary potentials that satisfy the Poisson equation

$$\mathbf{U}(P) = \frac{\mathbf{f}(P)}{\mu} + \frac{\nabla(\phi - \mathbf{r}\ \mathbf{f})}{4\mu(1-\nu)}, \quad \Delta\mathbf{f} = \mathbf{h}(P), \quad \Delta\phi = \mathbf{r}\ \mathbf{h}(P). \tag{3.2}$$

Here $\mathbf{h}(P)$ is the density vector of the bulk forces acting on an elastic body at a point P; $\mathbf{r}$ is the radius vector from the origin to the point $P = (x_1, x_2, x_3)$.

The representation (3.2) allows one to write the fundamental solution of the Lame equations (3.1) using the fundamental solution of the Laplace equation via a harmonic vector potential $\mathbf{f}$ acting in a certain direction. As shown in [13, 14], the Green tensor for the Lame equation is three column vectors

$\mathbf{G}_i = G_{ik}$, which are the response of the displacement vector with components U_k to the action of a concentrated force in the direction of one of the coordinate axes x_i. The vector of a concentrated force acting in the direction of one of the coordinate axes x_i corresponds to the vector potential $\mathbf{f}_i$ in the Papkovich–Neuber representation (3.2).

Since the fundamental solution of the 3D Laplace equation is a function $(4\pi r)^{-1}$, then the components of the Green tensor $\mathbf{G}_i = \{G_{ik}\}$, being the response to the concentrated force acting in the direction of the axis x_i, correspond to the potential $\mathbf{f}_i = \{\delta_{ik}/4\pi r\}$. Substituting this expression into (3.2), we reduce the Green tensor (see [14]) to the differential superposition of the fundamental solutions of the biharmonic r and harmonic r^{-1} equations

$$G_{ik} = \frac{1}{4\pi\mu}\Big(\frac{\delta_{ik}}{r} - \frac{1}{4(1-\nu)}\frac{\partial^2 r}{\partial x_i \partial x_k}\Big). \tag{3.3}$$

4. THE GENERALIZED GREEN TENSOR FOR THE SPATIAL PROBLEM OF THE GENERALIZED THEORY OF ELASTICITY

We denote the generalized solutions for the Green function as $\tilde{G}_{ik}$. According to the relations (2.1)–(2.3) and the algorithm for obtaining the generalized solutions described above, the generalized Green tensor $\tilde{G}_{ik}$ is determined using the classical Green tensor G_{ik} by the gradient smoothing formula with the scale parameter s

$$\tilde{G}_{ik} - s^2\Delta\tilde{G}_{ik} = G_{ik}. \tag{4.1}$$

Considering the representation of the Green tensor in the differential form (3.3), we construct the generalized Green tensor $\tilde{G}_{ik}$ in the differential form. We find the representation for the generalized Green tensor $\tilde{G}_{ik}$ in (3.1) in terms of the fundamental solutions $\phi(r)$ and $\psi(r)$ of the Helmholtz equations

$$\tilde{G}_{ik} = \frac{1}{4\pi\mu}\Big(\delta_{ik}\phi - \frac{1}{4(1-\nu)}\frac{\partial^2\psi}{\partial x_i \partial x_k}\Big), \quad \phi - s^2\Delta\phi = r^{-1}, \quad \psi - s^2\Delta\psi = r. \tag{4.2}$$

In order to construct a generalized Green tensor $\tilde{G}_{ik}$, we first find particular solutions of the Helmholtz equation $\phi^{(0)}$and $\psi^{(0)}$ that compensate the right-hand side r^{-1} and r in (4.2). Then, as a general solution of the homogeneous Helmholtz equation, we add the fundamental solution of the Helmholtz equation $h_0 = e^{-r/s}/r$ damping at infinity by choosing the coefficient at it so as to compensate the singularities of the functions $\phi^{(0)}$ and $\psi^{(0)}$.

Since the function on the right-hand side of the Helmholtz equation for a function $\psi(r)$ is a function that satisfies the biharmonic equation, the particular solution of the Helmholtz equation for the function $\psi(r)$ is represented as the sum of two terms

$$\psi^{(0)} = r + s^2\Delta r = r + 2s^2 r^{-1}.$$

Obviously, the right-hand side itself $\phi^{(0)} = r^{-1}$ is a particular solution of the Helmholtz equation for the function $\phi(r)$, since it is a harmonic function. The general solutions for the function ϕ and ψ with smoothed singularities determined by the function r^{-1} are built by adding the corresponding fundamental solutions $-e^{-r/s}$ and $-2s^2e^{-r/s}/r$:

$$\phi(r) = \frac{1}{r} - h_0(r), \quad \psi(r) = r + 2s^2\Big(\frac{1}{r} - h_0(r)\Big), \quad h_0(r) = \frac{e^{-r/s}}{r}. \tag{4.3}$$

As a result, the generalized Green tensor $\tilde{G}_{ik}$ is written as a differential combination of the difference between the fundamental solutions of the Laplace and Helmholtz equations.

$$\tilde{G}_{ik} = \frac{1}{4\pi\mu}\Big\{\delta_{ik}\Big(\frac{1}{r} - h_0\Big) - \frac{1}{4(1-\nu)}\frac{\partial^2}{\partial x_i \partial x_k}\Big[r + 2s^2\Big(\frac{1}{r} - h_0\Big)\Big]\Big\}, \quad h_0 = \frac{e^{-r/s}}{r}. \tag{4.4}$$

The asymptotic behavior of functions (4.3) with smoothed singularities and their derivatives in a neighborhood of the origin follows from the expansions of the fundamental solutions (4.3) into series that absolutely converge everywhere in powers of the parameter r/s:

$$\phi(r)=\frac{1}{s}\sum_{k=0}^{\infty}\frac{(-1)^k(rs^{-1})^k}{(k+1)!},\quad \psi(r)=2s\Big(1+\sum_{k=0}^{\infty}\frac{(-1)^k(rs^{-1})^{k+2}}{(k+3)!}\Big). \tag{4.5}$$

Taking into account (4.4), (4.5) we find the convergent expanding for the Green tensor

$$\tilde{G}_{ik}=\frac{s^{-1}}{8\pi(1-\nu)\mu}\sum_{k=0}^{\infty}\frac{(-1)^k(rs^{-1})^k}{k!}\Big[\frac{2(1-\nu)\delta_{ik}}{k+3}+\frac{s^{-1}}{(k+2)(k+4)}\frac{x_ix_k}{r}\Big]. \tag{4.6}$$

The generalized Green tensor $\tilde{G}_{ik}$ has no singularities at the origin, since all coefficients in the expansion (4.6) (starting from s^{-2}) are equal to zero and take a finite value $\tilde{G}_{ik}(0)=s^{-1}[\delta_{ik}/(12\pi\mu)]$ that grows inversely proportional to the scale parameter.

5. THE GENERALIZED BOUSSINESQ PROBLEM FOR A HALF SPACE

Let us consider the classical problem of loading a half-space with a concentrated force in the direction normal to the half-space boundary under condition of a free half-space surface, i.e. the Boussinesq problem [13]. We construct a generalized solution of the Boussinesq problem. This solution unlike the classical case does not have a singularity at the point of applying a load.

To construct the solution, we use the Papkovich–Neuber representation (3.2) with the potential $\mathbf{f}(P)=\{f_x,\ f_y,\ f_z\}$ corresponding to the fundamental solution of the Laplace equation $\Delta\mathbf{f}=\mathbf{h}$ with a vector delta function $\mathbf{h}(P)$ directed along the normal to the half-space surface, i.e. in the direction of the axis z. This potential determines the local force acting in the direction of the axis z.

$$f_z(P)=\frac{A}{r},\quad f_x=f_y=0. \tag{5.1}$$

A static boundary condition for stresses should be satisfied on the surface of a half-space $z=0$

$$\sigma_{zz}=\delta(0),\quad \sigma_{xz}=\sigma_{yz}=0\quad (\sigma_{13}\equiv\sigma_{xz},\quad \sigma_{23}\equiv\sigma_{yz},\quad \sigma_{33}\equiv\sigma_{zz}), \tag{5.2}$$

where $\delta(x)$ is the Dirac delta function.

The stress tensor expressed in the Papkovich–Neuber representation (3.2) in terms of auxiliary potentials $\mathbf{f}$ and ϕ is written as follows:

$$\sigma_{ij}(\mathbf{U})=f_{i,j}+f_{j,i}+\frac{(\phi-\mathbf{r}\mathbf{f})_{,ij}+2\nu\delta_{ij}\,\mathrm{div}\,\mathbf{f}}{2(1-\nu)}. \tag{5.3}$$

The relation (5.3) can be obtained by substituting (3.2) into the expression for isotropic body stresses expressed via the displacements

$$\sigma_{ij}(\mathbf{U})=2\mu\varepsilon_{ij}+\lambda\,\delta_{ij}\,\mathrm{div}\,\mathbf{U},\quad \frac{\lambda}{\mu}=\frac{2\nu}{1-2\nu},\quad \mathrm{div}\,\mathbf{U}=\frac{\mathrm{div}\,\mathbf{f}}{2\mu+\lambda}.$$

The solution of the Boussinesq problem (5.2) for the Lame equation that has been constructed using the Green tensor based on potentials (5.1) does not satisfy the boundary conditions (5.2) on the free surface. Nevertheless, we will show further that one can use an arbitrary (for now) harmonic function ϕ so that the static conditions (5.2) are satisfied on the border of the half-space. These conditions determine the equalities to zero of normal and tangential forces everywhere except the point of applying the local load.

Prove the following statement.

Theorem 1. For potentials determining the components of the Green tensor in the Papkovich–Neuber $\mathbf{f}(P)$ representation one can always find a harmonic function $\phi\,(P)$ such that solution (5.3) satisfies the static boundary conditions of the free surface on a plane $z=0$ every where except at the point of applying the load $x=y=z=0$. The harmonic function that ensures the fulfillment of conditions (5.2) for potentials (5.1) has the form

$$\phi(P)=-A\,(1-2\nu)\ln(r+z). \tag{5.4}$$

Proof. Convert the components σ_{xz}, σ_{yz}, σ_{zz}. Of the stress tensor (5.3) on the boundary of the half-space so as to clearly distinguish nonzero components of the surface forces at $z = 0$. To compensate for the residual errors from these components in the boundary conditions at $z = 0$ we use additional potential $\phi\,(P)$.

We rewrite equation (5.3) for $j = 3$ in the form

$$\sigma_{iz} = f_{z,i} + f_{i,z} + \frac{(\phi - xf_x - yf_y - zf_z)_{,iz} + 2\nu\delta_{i3}\,\mathrm{div}\,\mathbf{f}}{2(1-\nu)}.$$

In the third term of the written equation, we open the derivatives $(z\,f_z)_{,iz}$, select the term $f_{z,i}$, and group the expressions as follows:

$$\sigma_{iz} = f_{i,z} - \frac{zf_{z,iz}}{2(1-\nu)} + \frac{\delta_{i3}(2\nu\,\mathrm{div}\,\mathbf{f} - f_{z,z})}{2(1-\nu)} + \frac{(1-2\nu)f_{z,i} + (\phi - xf_x - yf_y)_{,iz}}{2(1-\nu)}.$$

The first two terms in the written expression vanish on the surface $z = 0$ outside the point of applying the load. It is enough to take into account that the derivative of the potential $\mathbf{f}$ with respect to a variable z gives a factor z outside the regular function, since $\mathbf{f}$ is a function of the radial coordinate. We now select the harmonic combination in the last term.

$$\sigma_{iz} = f_{i,z} + \frac{\delta_{i3}(2\nu\,\mathrm{div}\,\mathbf{f} - f_{z,z}) - zf_{z,zi}}{2(1-\nu)} + \frac{1}{2(1-\nu)}\frac{\partial}{\partial x_i}\Big[\phi_{,z} + z(f_{x,x} + f_{y,y})$$
$$- (xf_{x,z} + yf_{y,z}) - z(f_{x,x} + f_{y,y}) + (1-2\nu)f_z\Big].$$

The main goal of the transformations is to compensate for the residual error in static boundary conditions. The residual error is associated with the term in square brackets. It can be verified that the combination of functions $z(f_{x,x} + f_{y,y}) - (xf_{x,z} + yf_{y,z})$ satisfies the Laplace equation, that is, it is a harmonic function, and the term $[z(f_{x,x} + f_{y,y})]_{,i}$ in the last equality is transformed as follows:

$$\frac{\partial}{\partial x_i}\Big[z(f_{x,x} + f_{y,y})\Big] = \delta_{i3}(f_{x,x} + f_{y,y}) + z(f_{x,x} + f_{y,y})_{,i}.$$

As a result, we obtain the expression

$$\sigma_{iz} = f_{i,z} + \frac{\delta_{i3}(2\nu\,\mathrm{div}\,\mathbf{f} - f_{x,x} - f_{y,y} - f_{z,z}) - z(f_{x,x} + f_{y,y} + f_{z,z})_{,i}}{2(1-\nu)}$$
$$+ \frac{1}{2(1-\nu)}\frac{\partial}{\partial x_i}\Big[\phi_{,z} + z(f_{x,x} + f_{y,y}) - (xf_{x,z} + yf_{y,z}) + (1-2\nu)f_z\Big].$$

The last equation can be rewritten in the following form:

$$\sigma_{iz} = f_{i,z} - \frac{\delta_{i3}(1-2\nu)\,\mathrm{div}\,\mathbf{f} + z\,\mathrm{div}\,\mathbf{f}_{,i}}{2(1-\nu)} + \frac{1}{2(1-\nu)}\frac{\partial}{\partial x_i}\Big[(\phi - xf_x - yf_y)_{,z} + z(f_{x,x} + f_{y,y})$$
$$+ (1-2\nu)f_z\Big]. \tag{5.5}$$

We insert the second term in square brackets in formula (5.5) into the sign of the derivative:

$$(\phi - xf_x - yf_y)_{,z} + z(f_{x,x} + f_{y,y}) = (\phi - xf_x - yf_y + z(F_{x,x} + F_{y,y}))_{,z} - (F_{x,x} + F_{y,y})$$
$$= (\phi + z\,\mathrm{div}\,\mathbf{F} - \mathbf{r}\,\mathbf{f})_{,z} - \mathrm{div}\,\mathbf{F} + f_z.$$

Here $\mathbf{F}$ is the primitive function of potential $\mathbf{f}$ for the integration with respect to z, that is $\mathbf{f} = \mathbf{F}_{,z}$. It is possible to verify that for such a transformation, the combination $z\,\mathrm{div}\,\mathbf{F} - \mathbf{rf}$ is a harmonic function, and one can consider a harmonic function $\psi = \phi + z\,\mathrm{div}\,\mathbf{F} - \mathbf{rf}$ instead of a function ϕ.

In the equality (5.5), the first two terms satisfy the static boundary conditions, since, according to (5.1), $(1-2\nu)\,\mathrm{div}\,\mathbf{f} = 2(1-\nu)f_{z,z}$ and $f_{x,z} = f_{y,z} = 0$ on the boundary of the half-space. Then we rewrite the expression for surface stresses (5.5) in the following form

$$\sigma_{iz} = f_{i,z} - \frac{\delta_{i3}(1-2\nu)\,\mathrm{div}\,\mathbf{f} + z\,\mathrm{div}\,\mathbf{f}_{,i}}{2(1-\nu)} + \frac{\partial}{\partial x_i}\Big[\frac{\psi_{,z} - \mathrm{div}\,\mathbf{F} + 2(1-\nu)f_z}{2(1-\nu)}\Big]. \tag{5.6}$$

In order to satisfy boundary conditions for stresses σ_{iz} for $z = 0$, it is necessary to require the square bracket in formula (5.6) to be equal to zero. Thus, the harmonic potential $\phi(P)$ is determined by the equality

$$\frac{\partial}{\partial z}\Big[\phi + z\,\mathrm{div}\,\mathbf{F} - \mathbf{r}\,\mathbf{f}\Big] - \mathrm{div}\,\mathbf{F} + 2(1-\nu)f_z = 0. \tag{5.7}$$

From here we find that for the fundamental solution (5.1) the harmonic potential $\phi(P)$ is determined explicitly by integrating the fundamental solution f_z:

$$\mathrm{div}\,\mathbf{F} = f_z = \frac{A}{r}, \quad z\,\mathrm{div}\,\mathbf{F} - \mathbf{r}\mathbf{f} = 0, \quad \int \frac{dz}{r} = \ln(r+z).$$

As a result we obtain

$$\phi(P) = -A(1-2\nu)\ln(r+z)$$

and the theorem is proved. □

Note that the considered theorem is valid for potentials of a more general form than those dictated by conditions (5.1). It is enough to require for them the harmonicity in the half-space, the fulfillment of equality (5.7) and the following conditions on the border of the half-space:

$$f_{z,z} - (1-2\nu)(f_{x,x} + f_{y,y}) = 0, \quad f_{x,z} = f_{y,z} = 0, \quad z = 0. \tag{5.8}$$

This generalization turns out to be essential if we consider the more general conditions for loading a half-space.

Let us return to the potentials (5.1). When (5.7) is satisfied, the stresses are found by explicit formulas and contain derivatives with respect to the argument z of the fundamental solution

$$\sigma_{iz} = -\frac{zf_{z,zi}}{2(1-\nu)}, \quad i \neq 3, \quad \text{and} \quad \sigma_{zz} = \frac{f_{z,z} - zf_{z,zi}}{2(1-\nu)}, \quad f_{z,z} = -\frac{A\,z}{r^3}.$$

It can be seen that for the recorded stresses at the boundary of the half-space, static boundary conditions (5.2) are satisfied at all points except the origin.

By opening all derivatives in these representations, we obtain

$$\sigma_{xz} = -\frac{3A}{2(1-\nu)}\frac{xz^2}{r^5}, \quad \sigma_{yz} = -\frac{3A}{2(1-\nu)}\frac{yz^2}{r^5}, \quad \sigma_{zz} = -\frac{3A}{2(1-\nu)}\frac{z^3}{r^5}.$$

The choice of the constant A is determined by the properties of the delta function of the component of the surface stress σ_{zz} and the static conditions (5.2). The integral action of the force must be equal to -1, since the outward normal to the surface is directed in the negative direction. Performing integration σ_{zz} over a parallel surface $\{z = d\}$, we find

$$-\frac{3A}{2(1-\nu)}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{z^3dxdy}{r^5} = -\frac{3A}{2(1-\nu)}\int\limits_{0}^{2\pi}\int\limits_{0}^{\infty} d^3(t^2+d^2)^{-5/2}t\,dt\,d\varphi$$

$$= -\frac{3A\pi}{2\,(1-\nu)}\int\limits_{0}^{\infty} d^3(t^2+d^2)^{-5/2}d(t^2+d^2) = \frac{A\pi}{(1-\nu)}d^3(t^2+d^2)^{-3/2}\Big|_0^{\infty} = -\frac{A\,\pi}{(1-\nu)} = -1.$$

Consequently, the constant in the definition of potentials (5.1) is $A = (1-\nu)/\pi$.

Thus, the solution of the Boussinesq problem on the action of a unit concentrated force on a free half-space is given by the Papkovich–Neuber representation (3.2) with the following potentials

$$f_x = f_y = 0, \quad f_z = \frac{1-\nu}{\pi r}, \quad \phi = -\frac{(1-2\nu)(1-\nu)}{\pi}\ln(r+z), \tag{5.9}$$

and the surface forces $\boldsymbol{\sigma}_z = \{\sigma_{iz}\}$ acting on a small area oriented perpendicular to the axis z, are determined by the formula $\boldsymbol{\sigma}_z = -3\mathbf{r}z^2/2\pi r^5$ ($\mathbf{r}$ is the radius vector of the current point) and satisfy the static boundary conditions (5.2) at $z = 0$.

6. DETERMINATION OF DISPLACEMENTS IN THE BOUSSINESQ PROBLEM

The components of the displacement vector are calculated directly using the Papkovich–Neuber representation (3.2) and considering the specific form (5.9) for the potentials. As a result, the displacement field for the classical Boussinesq problem on a half-space $\{z \geq 0\}$ loaded by a concentrated force applied at a point $x = y = z = 0$ is determined by the relations

$$U_x = -\frac{1}{4\pi\mu}\Big[\frac{(1-2\nu)x}{r\,(r+z)} - \frac{xz}{r^3}\Big], \quad U_y = -\frac{1}{4\pi\mu}\Big[\frac{(1-2\nu)y}{r\,(r+z)} - \frac{yz}{r^3}\Big], \tag{6.1}$$

$$U_z = \frac{1}{4\pi\mu}\Big[\frac{2(1-\nu)}{r} + \frac{z^2}{r^3}\Big]. \tag{6.2}$$

Note that in the solution structure of the Boussinesq problem, there are harmonic functions $x/(r+z)$ and $y/(r+z)$ depending only on the angular coordinates $\varphi = \arctan(y/x)$ and $\theta = \arccos(z/r)$:

$$\frac{x}{r+z} = \frac{\cos\varphi\sin\theta}{1+\cos\theta}, \quad \frac{y}{r+z} = \frac{\sin\varphi\sin\theta}{1+\cos\theta}, \quad \Delta\Big(\frac{x}{r+z}\Big) = \Delta\Big(\frac{y}{r+z}\Big) = 0. \tag{6.3}$$

These functions are also the fundamental solution for the Laplace operator. However, they are bounded across the space, in contrast to the singular function r^{-1}.

The solution of the Boussinesq problem (6.1), (6.2) in terms of displacements can be written by analogy with formula (3.3) for the Green tensor in differential form by highlighting the fundamental solution of the Laplace operator and derivatives of the fundamental solution of the biharmonic equation. There is a following statement.

Theorem 2. The displacements in the problem of a concentrated force action on a free half-space can be represented as a superposition of angular harmonic functions (6.3), a fundamental solution of a harmonic equation, and derivatives of a fundamental solution of a biharmonic equation:

$$U_x = -\frac{1}{4\pi\mu}\Big(\frac{x}{r+z}\frac{1-2\nu}{r} + \frac{\partial^2 r}{\partial x\,\partial z}\Big), \quad U_y = -\frac{1}{4\pi\mu}\Big(\frac{y}{r+z}\frac{1-2\nu}{r} + \frac{\partial^2 r}{\partial y\,\partial z}\Big), \tag{6.4}$$

$$U_z = \frac{1}{4\pi\mu}\Big(\frac{3-2\nu}{r} - \frac{\partial^2 r}{\partial z^2}\Big). \tag{6.5}$$

The proof of the last assertion can be carried out by direct verification of relations (6.4), (6.5).

Note also that the stresses (5.8) in the classical Boussinesq problem are also rewritten in differential form via two fundamental solutions of the harmonic and biharmonic equation (the angular functions (6.3) are not involved in the formula):

$$\sigma_{xz} = \frac{1}{2\pi}\frac{\partial}{\partial x}\Big(\frac{1}{r} - \frac{\partial^2 r}{\partial z^2}\Big), \quad \sigma_{yz} = \frac{1}{2\pi}\frac{\partial}{\partial y}\Big(\frac{1}{r} - \frac{\partial^2 r}{\partial z^2}\Big), \quad \sigma_{zz} = \frac{1}{2\pi}\frac{\partial}{\partial z}\Big(\frac{3}{r} - \frac{\partial^2 r}{\partial z^2}\Big). \tag{6.6}$$

7. THE DETERMINATION OF DISPLACEMENTS AND STRESSES IN THE GENERALIZED BOUSSINESQ PROBLEM

The generalized theory of elasticity deals with displacements **R** that are associated with classical displacements **U** by the Helmholtz equation with a parameter s^2 that is

$$\mathbf{R} - s^2\Delta\mathbf{R} = \mathbf{U}. \tag{7.1}$$

We construct a generalized solution of equation (7.1) that has no singularities at the origin. To do this, we use the representation of displacements in differential form (6.4), (6.5). Taking into account the structure of the solution, it can be seen that a part of the necessary functions $\phi(r)$ and $\psi(r)$ from (4.2) for the components of generalized displacements have already been obtained in (4.3). Thus, it suffices to determine the solutions of the equations for the terms containing the angular functions (6.3):

$$\phi_x - s^2\Delta\phi_x = \frac{x}{r+z}r^{-1}, \quad \phi_y - s^2\Delta\phi_y = \frac{y}{r+z}r^{-1}. \tag{7.2}$$

Using the properties of harmonic functions (6.3) (dependent only on the angular coordinates), the particular solution of equations (7.2) have the forms $\phi_x^{(0)} = r^{-1}x/(r+z)$ and $\phi_y^{(0)} = r^{-1}y/(r+z)$, and the compensating singularity general solution of the homogeneous Helmholtz equation has a form of product of the angular function and the fundamental solution $h_0(r)$ of the Helmholtz equation. As a result, the solution of equations (7.2) has the form of the product of the angular function and the solution already constructed $\phi(r)$. The asymptotic behavior of ϕ_x, ϕ_y and their regularity follows from the expanding (4.5) for the function $\phi(r)$.

As a result, the generalized solutions for displacements in the Boussinesq problem that are regular at the point of applying the load are represented as follows:

$$R_x = -\frac{1}{4\pi\mu}\Big\{(1-2\nu)\Big(\frac{1}{r}-h_0\Big)\frac{x}{r+z}+\frac{\partial^2}{\partial x\,\partial z}\Big[r+2s^2\Big(\frac{1}{r}-h_0\Big)\Big]\Big\}, \quad h_0 = \frac{e^{-r/s}}{r}, \tag{7.3}$$

$$R_y = -\frac{1}{4\pi\mu}\Big\{(1-2\nu)\Big(\frac{1}{r}-h_0\Big)\frac{y}{r+z}+\frac{\partial^2}{\partial y\partial z}\Big[r+2s^2\Big(\frac{1}{r}-h_0\Big)\Big]\Big\}, \tag{7.4}$$

$$R_z = \frac{1}{4\pi\mu}\Big\{(3-2\nu)\Big(\frac{1}{r}-h_0\Big)-\frac{\partial^2}{\partial z^2}\Big[(r+2s^2\Big(\frac{1}{r}-h_0\Big)\Big]\Big\}. \tag{7.5}$$

The regularity of functions (7.3)–(7.5) follows from the expansion of displacements over the whole space into series obtained on the basis of expansions (4.5):

$$R_x = xd, \quad R_y = yd, \quad d = -\frac{s^{-1}}{4\pi\mu}\sum_{k=0}^{\infty}\frac{(-1)^k(r/s)^k}{k!}\Big[\frac{1-2\nu}{(k+1)(r+z)}-\frac{2s^{-1}}{(k+2)(k+4)}\frac{z}{r}\Big], \tag{7.6}$$

$$R_z = \frac{s^{-1}}{4\pi\mu}\sum_{k=0}^{\infty}\frac{(-1)^k(r/s)^k}{k!}\Big[\frac{1}{k+1}\Big(3-2\nu-\frac{2}{k+3}\Big)+\frac{2s^{-1}}{(k+2)(k+4)}\frac{z^2}{r}\Big]. \tag{7.7}$$

From these expansions, it follows that at the origin the displacement component takes a finite value $R_z = s^{-1}[(7-6\nu)/12\pi\mu]$ inversely proportional to the scale parameter, that is, the displacements in the generalized theory of elasticity have no singularities at the origin. Similarly, using (6.6) and (4.3), the generalized stresses are written in the Boussinesq problem

$$\sigma_{xz} = \frac{1}{2\pi}\frac{\partial}{\partial x}\Big\{\Big(\frac{1}{r}-h_0\Big)-\frac{\partial^2}{\partial z^2}\Big[r+2s^2\Big(\frac{1}{r}-h_0\Big)\Big]\Big\}, \quad h_0 = \frac{e^{-r/s}}{r}, \tag{7.8}$$

$$\sigma_{yz} = \frac{1}{2\pi}\frac{\partial}{\partial y}\Big\{\Big(\frac{1}{r}-h_0\Big)-\frac{\partial^2}{\partial z^2}\Big[r+2s^2\Big(\frac{1}{r}-h_0\Big)\Big]\Big\}, \tag{7.9}$$

$$\sigma_{zz} = \frac{1}{2\pi}\frac{\partial}{\partial z}\Big\{3\Big(\frac{1}{r}-h_0\Big)-\frac{\partial^2}{\partial z^2}\Big[r+2s^2\Big(\frac{1}{r}-h_0\Big)\Big]\Big\}. \tag{7.10}$$

The expansions of generalized stresses into series converging over the whole space have the form

$$\sigma_{xz} = -\frac{xd}{r}, \quad \sigma_{yz} = -\frac{yd}{r},$$

$$d = \frac{s^{-2}}{2\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(r/s)^k}{k!}\Big[\frac{1}{k+1}\Big(\frac{2(z^2/r^2-1)}{k+4}+1\Big)+\frac{2s^{-1}}{(k+3)(k+5)}\frac{z^2}{r}\Big], \tag{7.11}$$

$$\sigma_{zz} = -\frac{z}{r}\frac{s^{-2}}{2\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(r/s)^k}{k!}\Big[\frac{1}{k+1}\Big(\frac{2(z^2/r^2-3)}{k+4}+3\Big)+\frac{2s^{-1}}{(k+3)(k+5)}\frac{z^2}{r}\Big]. \tag{7.12}$$

The generalized stresses have a finite value at the origin. This value is inversely proportional to the square of the scale parameter.

Writing in (7.3)–(7.5), (7.8)–(7.10) the derivatives in an explicit form, we can present the solution of the generalized Boussinesq problem in algebraic form separating the classical part of the generalized solution, that is,

$$R_x = -\frac{x}{4\pi\mu}\Big[\frac{1-2\nu}{r(r+z)}-\frac{z}{r^3}-\frac{2z(r+z)+(1-2\nu)r^2}{r^2(r+z)}h_0+\frac{6s^2z}{r^2}\Big(\frac{1}{r^3}+h_1\Big)\Big],$$

$$R_y = -\frac{y}{4\pi\mu}\Big[\frac{1-2\nu}{r(r+z)}-\frac{z}{r^3}-\frac{2z(r+z)+(1-2\nu)r^2}{r^2(r+z)}h_0+\frac{6s^2z}{r^2}\Big(\frac{1}{r^3}+h_1\Big)\Big],$$

$$R_z = \frac{1}{4\pi\mu}\Big[\frac{2(1-\nu)}{r} + \frac{z^2}{r^3} + \frac{2z^2-(3-2\nu)r^2}{r^2}h_0 - \frac{2s^2(3z^2-r^2)}{r^2}\Big(\frac{1}{r^3}+h_1\Big)\Big],$$

$$\sigma_{xz} = \frac{x}{2\pi}\Big[-\frac{3z^2}{r^5} + \frac{2z^2-r^2}{r^2}h_1 + \frac{2s^2(5z^2-r^2)}{r^2}\Big(\frac{3}{r^5}-h_2\Big)\Big],$$

$$\sigma_{yz} = \frac{y}{2\pi}\Big[-\frac{3z^2}{r^5} + \frac{2z^2-r^2}{r^2}h_1 + \frac{2s^2(5z^2-r^2)}{r^2}\Big(\frac{3}{r^5}-h_2\Big)\Big],$$

$$\sigma_{zz} = \frac{z}{2\pi}\Big[-\frac{3z^2}{r^5} + \frac{2z^2-3r^2}{r^2}h_1 + \frac{2s^2(5z^2-3r^2)}{r^2}\Big(\frac{3}{r^5}-h_2\Big)\Big],$$

$$h_0 = \frac{e^{-r/s}}{r}, \quad h_1 = \frac{h_0'}{r} = -\frac{(1+r/s)e^{-r/s}}{r^3}, \quad h_2 = \frac{h_1'}{r} = \frac{\Big[3(1+r/s)+(r/s)^2\Big]e^{-r/s}}{r^5}.$$

8. COMPARISON WITH THE SOLUTION OBTAINED BY SMOOTHING OVER THE BOUNDARY OF THE HALF-SPACE

In [7], a generalized solution of the Boussinesq problem on the surface of applying a concentrated force was obtained by using the smoothing operator (7.1) to the projection of the deflection function (6.2) onto the plane $\{z = 0\}$. In other words, a model in which the representative fragment is a plane square and is determined by one scale parameter s has been used. It can be seen that this solution is reduced to the construction of a fundamental solution $\phi - s^2\Delta\,\phi = 1/r$ on the plane by using the plane Laplace operator $\Delta\phi = \phi'' + \phi'/r$. The solution is expressed in terms of the Macdonald and Bessel functions of zero-order [7]:

$$\phi(r) = \Big[\frac{\pi}{2} - \int_0^{\bar r} K_0(\bar r)d\bar r\Big]I_0(\bar r) + K_0(\bar r)\int_0^{\bar r} I_0(\bar r)d\bar r, \quad \bar r = r/s, \quad r = \sqrt{x^2+y^2}.$$

The corresponding generalized displacement can be expressed via this function in the form $w_1(r) = (1-\nu^2)\phi'(r)/(\pi E)$ and expanded by analogy with (37) in a series of powers r/s

$$w_1(r) = \frac{1-\nu^2}{2\pi E\, s}\sum_{k=0}^{\infty}\Big(\frac{r}{2s}\Big)^{2k}\Big[\frac{\pi}{(k!)^2} + \frac{r}{2s}\sum_{l=0}^{k}\frac{1}{(l!\,(k-l)!)^2(l+1/2)}\Big(\frac{(k-2j)h_j}{k-j-1/2} - \frac{1}{2j+1}\Big)\Big], \quad (8.1)$$

where $h_j = \sum_{l=1}^{j} 1/l$, E is the Young modulus, ν is Poisson ratio.

The comparison of solution (8.1) with expansion (7.7) obtained by smoothing the fundamental solution by the spatial Laplace operator indicates a discrepancy in the asymptotic behavior of the displacements at the origin. The discrepancy corresponds to the multiplier $3\pi(1-\nu)/(7-6\nu)$ and indicates the need to fit the scale parameter s in the 2D and 3D case. The difference disappears if in the 2D case the scale parameter is increased by a factor of $3\pi(1-\nu)/(7-6\nu)$. In this case, the main term of the asymptotics for generalized displacements is the same in both cases. This problem of matching is similar to that when, for example, the effective properties of a discretely reinforced composite are compared in 2D and 3D cases. In this case, the scale parameter is found by matching the relative concentrations of inclusions having different dimensions in the 2D and 3D cases.

Fig. 1 demonstrates a comparison of the graphs for the generalized displacement $w_1(r)$ from [7] obtained with the help of a plane operator (thin line) and the function $R_z(r)$ (7.7) obtained with the help of a spatial operator (ordinary line) on a plane $\{z = 0\}$ at matching the scale parameters and at $E = 1.2$ GPa, $\nu = 0.3$. The results of the comparison show a good agreement of functions in the neighborhood of the singular point.

9. GENERALIZATION: THE BOUSSINESQ PROBLEM FOR A CONCENTRATED FORCE ACTING TANGENTIALLY TO THE HALF-SPACE BOUNDARY

The solution constructed in Section 7 is easily generalized using Theorem 1 to a problem in which the concentrated force acts tangentially to the half-space boundary, i.e. in the direction of the axis x or y. The vector delta function $\mathbf{h}(P)$ from the Papkovich—Neuber representation (3.2) in this case has a

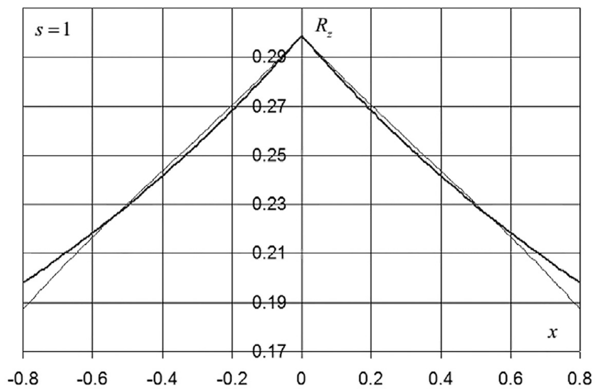

Fig. 1.

component directed along the x or y axis. Thus, we believe that the Papkovich–Neuber potential $\mathbf{f}(P)$ is a fundamental solution of the Laplace equation with a non-zero component $f_x(P) = A/r$.

To provide the necessary static boundary conditions in a problem with a force acting tangentially to the half-space boundary

$$\sigma_{xz} = \delta(0), \quad \sigma_{yz} = \sigma_{zz} = 0, \quad (\sigma_{13} \equiv \sigma_{xz}, \quad \sigma_{23} \equiv \sigma_{yz}, \quad \sigma_{33} \equiv \sigma_{zz}) \tag{9.1}$$

in accordance with Theorem 1, it is necessary to introduce an additional component f_z from the condition that follows from (5.8)

$$\frac{\partial f_z}{\partial z} - (1-2\nu)\frac{\partial f_x}{\partial x} = 0. \tag{9.2}$$

For the fundamental solution that determines the action of a concentrated force in the direction of the axis x, we obtain from (5.7) and (9.2)

$$\phi = (1-2\nu)^2\frac{Ax}{r+z}, \quad f_x = \frac{A}{r}, \quad f_z = (1-2\nu)\frac{Ax}{r(r+z)}. \tag{9.3}$$

Performing the derivation of first and last relations (9.3), the formulas have been used to integrate the fundamental solution

$$\int\frac{dz}{r^3} = -\frac{1}{r(r+z)}, \quad \int\frac{dz}{r(r+z)} = -\frac{1}{r+z}, \quad \int\frac{dz}{r} = \ln(r+z).$$

As a result, when conditions (5.7) and (9.2) are fulfilled, the components of the stress tensor are equal to zero on the half-space surface and are determined by the formulas

$$\sigma_{iz} = f_{i,z} - \frac{z\,\mathrm{div}\,\mathbf{f}_{,i}}{2(1-\nu)}, \quad \sigma_{zz} = -\frac{z\,\mathrm{div}\,\mathbf{f}_{,i}}{2(1-\nu)}. \tag{9.4}$$

The normalization of the solution (the choice of a constant A) is determined by the properties of the component σ_{xz} of surface stresses (9.4) as δ–function and is found when integrating over a parallel surface $\{z = d\}$. Calculating the derivatives of potentials in (9.3), we obtain $\sigma_{xz} = -3Azx^2/r^5$. Carrying out integration over a parallel surface, we find

$$-3A\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{zx^2dxdy}{r^5} = -3A\int\limits_{0}^{2\pi}\int\limits_{0}^{\infty} d(t^2+d^2)^{-5/2}t^3\cos^2\varphi\,dtd\varphi$$

$$= -\frac{3A\pi}{2}\int\limits_0^{\infty} d(t^2+d^2)^{-5/2}t^2d(t^2+d^2) = -\frac{3A\pi}{2}\int\limits_0^{\infty} d(t^2+d^2)^{-3/2}d(t^2+d^2)$$

$$+\frac{3A\pi}{2}\int\limits_0^{\infty} d^3(t^2+d^2)^{-5/2}d(t^2+d^2) = A\pi\Big(3d(t^2+d^2)^{-1/2} - d^3(t^2+d^2)^{-3/2}\Big)\Big|_0^{\infty} = 2A\pi.$$

Taking into account that the integral action of the tangential force σ_{xz} must be equal to 1, we obtain a constant $A = 1/(2\pi)$.

Thus, the solution of the Boussinesq problem for a concentrated force acting tangentially to the half-space surface is obtained. This solution determines the potentials in the representation (3.2) in the form (9.4) with constant $A = 1/(2\pi)$. By opening the derivatives in (9.4), we obtain the formula $\boldsymbol{\sigma}_z = -3\mathbf{r}xz/2\pi r^5$ for the surface forces $\boldsymbol{\sigma}_z = \{\sigma_{iz}\}$ acting on the platform orthogonal to z.

Based on (3.2) and (9.4), we calculate the components of the displacement vector in differential form via the biharmonic function$r$, the angular functions (of a new type), and the fundamental solution of the Laplace equation r^{-1}

$$U_x = \frac{1}{4\pi\mu}\Big[\frac{2}{r} + (1-2\nu)\Big(\frac{r}{r+z} - \frac{x^2}{(r+z)^2}\Big)\frac{1}{r} - \frac{\partial^2 r}{\partial x^2}\Big], \tag{9.5}$$

$$U_y = -\frac{1}{4\pi\mu}\Big[\frac{(1-2\nu)xy}{(r+z)^2}\frac{1}{r} + \frac{\partial^2 r}{\partial x\partial y}\Big], \quad U_z = -\frac{1}{4\pi\mu}\Big[\frac{(1-2\nu)x}{r+z}\frac{1}{r} + \frac{\partial^2 r}{\partial x\,\partial z}\Big]. \tag{9.6}$$

We construct a smoothed solution of the generalized Boussinesq problem. Here the structure of the solution is determined not only by the functions r, r^{-1}, but also by angular harmonic functions $x/(r+z)$ and $xy/(r+z)^2$, and $r/(r+z) - x^2/(r+z)^2$ depending only on the angular variables in the spherical coordinate system:

$$\frac{r}{r+z} - \frac{x^2}{(r+z)^2} = \frac{1}{1+\cos\theta} - \Big(\frac{\cos\varphi\sin\theta}{1+\cos\theta}\Big)^2, \quad \nabla^2\Big[\frac{r}{r+z} - \frac{x^2}{(r+z)^2}\Big] = 0,$$
$$\frac{xy}{(r+z)^2} = \sin\varphi\cos\varphi\Big(\frac{\sin\theta}{1+\cos\theta}\Big)^2, \quad \nabla^2\Big[\frac{xy}{(r+z)^2}\Big] = 0.$$

The product of these functions by the fundamental solution r^{-1} is a singular harmonic function with a first order singularity depending on the angular coordinates and the generalized solution is constructed, as before, using the product of these angular functions by the smoothed fundamental solution $\phi(r)$.

The differential stresses have the form

$$\sigma_{xz} = \frac{1}{2\pi}\frac{\partial}{\partial z}\Big(\frac{1}{r} - \frac{\partial^2 r}{\partial x^2}\Big), \quad \sigma_{yz} = -\frac{1}{2\pi}\frac{\partial^3 r}{\partial x\partial y\partial z}, \quad \sigma_{zz} = \frac{1}{2\pi}\frac{\partial}{\partial x}\Big(\frac{1}{r} - \frac{\partial^2 r}{\partial z^2}\Big). \tag{9.7}$$

The representation for displacements in algebraic form is obtained by expanding the derivatives in (9.5), (9.6):

$$U_x = \frac{1}{4\pi\mu}\Big[\frac{1}{r} + (1-2\nu)\Big(\frac{1}{r+z} - \frac{x^2}{r(r+z)^2}\Big) + \frac{x^2}{r^3}\Big],$$
$$U_y = -\frac{1}{4\pi\mu}\Big[\frac{(1-2\nu)xy}{r(r+z)^2} - \frac{xy}{r^3}\Big], \quad U_z = -\frac{1}{4\pi\mu}\Big[\frac{(1-2\nu)x}{r(r+z)} - \frac{xz}{r^3}\Big].$$

In accordance with the structure of solution (9.5)–(9.7) and the properties of the angular functions, the solution of the Boussinesq problem in the generalized theory of elasticity is written in differential form and is determined by the difference between the fundamental solutions of the Laplace and Helmholtz equations and the angular functions. (The solution has no singularities at the origin)

$$R_x = \frac{1}{4\pi\mu}\Big\{\Big[2 + (1-2\nu)\Big(\frac{r}{r+z} - \frac{x^2}{(r+z)^2}\Big)\Big]\Big(\frac{1}{r} - h_0\Big) - \frac{\partial^2}{\partial x^2}\Big[r + 2s^2\Big(\frac{1}{r} - h_0\Big)\Big]\Big\}, \tag{9.8}$$

$$R_y = -\frac{1}{4\pi\mu}\Big\{\frac{(1-2\nu)xy}{(r+z)^2}\Big(\frac{1}{r} - h_0\Big) + \frac{\partial^2}{\partial x\partial y}\Big[+2s^2\Big(\frac{1}{r} - h_0\Big)\Big]\Big\}, \quad h_0 = \frac{e^{-r/s}}{r}, \tag{9.9}$$

$$R_z = -\frac{1}{4\pi\mu}\Big\{\frac{(1-2\nu)x}{r+z}\Big(\frac{1}{r} - h_0\Big) + \frac{\partial^2}{\partial x\partial z}\Big[+2s^2\Big(\frac{1}{r} - h_0\Big)\Big]\Big\}, \tag{9.10}$$

$$\sigma_{xz} = \frac{1}{2\pi}\frac{\partial}{\partial z}\Big\{\Big(\frac{1}{r} - h_0\Big) - \frac{\partial^2}{\partial x^2}\Big[r + 2s^2\Big(\frac{1}{r} - h_0\Big)\Big]\Big\}, \tag{9.11}$$

$$\sigma_{yz} = -\frac{1}{2\pi}\frac{\partial^3}{\partial x\partial y\partial z}\Big[r + 2s^2\Big(\frac{1}{r} - h_0\Big)\Big], \tag{9.12}$$

$$\sigma_{zz} = \frac{1}{2\pi}\frac{\partial}{\partial x}\Big\{\Big(\frac{1}{r} - h_0\Big) - \frac{\partial^2}{\partial z^2}\Big[r + 2s^2\Big(\frac{1}{r} - h_0\Big)\Big]\Big\}. \tag{9.13}$$

The regularity of the functions (9.8)–(9.13) follows from the asymptotics (4.5) for smoothed fundamental solutions $\phi(r)$ and $\psi(r)$ used in the construction of the solution

$$R_x = \frac{1}{4\pi\mu}\Big[\frac{4}{3} + (1-2\nu)\Big(\frac{r}{r+z} - \frac{x^2}{(r+z)^2}\Big)\Big]s^{-1} + O(s^{-2}), \quad R_y = -\frac{1-2\nu}{4\pi\mu}\frac{xy}{(r+z)^2}s^{-1} + O(s^{-2}),$$

$$R_z = -\frac{1-2\nu}{4\pi\mu}\frac{x}{r+z}s^{-1} + O(s^{-2}), \quad \sigma_{xz} = -\frac{1}{8\pi}\frac{z(r^2+x^2)}{r^3}s^{-2} + O(s^{-3}),$$

$$\sigma_{yz} = -\frac{1}{8\pi}\frac{xyz}{r^3}s^{-2} + O(s^{-3}), \quad \sigma_{zz} = -\frac{1}{8\pi}\frac{x(r^2+z^2)}{r^3}s^{-2} + O(s^{-3}).$$

According to these asymptotic formulas, displacements and stresses in the generalized theory of elasticity do not have singularities at the origin, but take a finite value inversely proportional to the scale parameter or its square (for stresses).

10. EXAMPLES

Figure 2 shows the graphs of the function R_z on the boundary of the half-space along the axis x for different values of the scale parameter s in the Boussinesq problem with the force applied along the normal to the surface of the half-space. Characteristics of the material are: $E = 1.2$ GPa, $\nu = 0.3$. Figure 3 shows the graphs of the function σ_{zz}, (formula (7.10)) at different distances d from the boundary of the half-space along a straight line $y = 0$ for a fixed value of the scale parameter $s = 1$ in the Boussinesq problem with a force applied normal to the surface of the half-space.

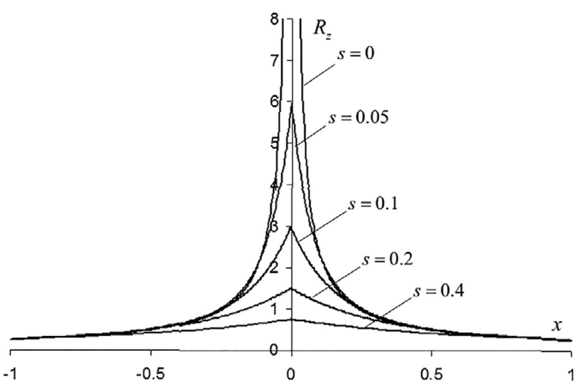

Fig. 2.

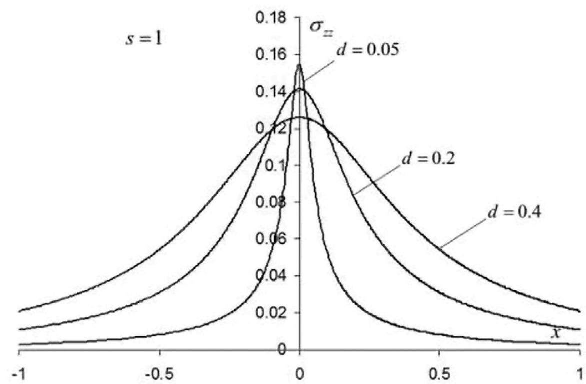

Fig. 3.

11. CONCLUSION

In the class of solutions for the generalized theory of elasticity, a regular representation for the Green tensor is found and a solution of the Boussinesq problem is constructed. For the Boussinesq problem, we obtain explicit solutions that are regular everywhere in the half-space and damp at infinity. Finite relations for displacements and stresses are written and explicit expressions that determine the asymptotic behavior of a regular solution in a neighborhood of a singular point are presented.

ACKNOWLEDGMENTS

This research was supported by the Russian Foundation for Basic Research (grant No. 16-01-00623) for building a generalized solution, the part of constructing a common solution for classical problems carried out within the frameworks of the state task 0050-2016-0001.

REFERENCES

1. V. V. Vasiliev, “Stress Tensor Symmetry and Singular Solutions in the Theory of Elasticity,” Izv. Akad. Nauk. Mekh. Tverd. Tela, No. 2, 62–72 (2010) [Mech. Sol. (Engl. Transl.) **45** (2), 205–213 (2010)].
2. V. V. Vasiliev and S. A. Lurie, “On the Solution Singularity in the Plane Elasticity Problem for a Cantilever Strip,” Izv. Akad. Nauk. Mekh. Tverd. Tela, No. 4, 40–49 (2013) [Mech. Sol. (Engl. Transl.) **48** (4), 388–396 (2013)].
3. V. V. Vasiliev and S. A. Lurie, “Continuous Model with Microstructure,” Comp. Nanostruct. **7** (1), 2–10 (2015).
4. V. V. Vasiliev and S. A. Lurie, “Generalized Theory of Elasticity,” Izv. Akad. Nauk. Mekh. Tverd. Tela, No. 4, 16–27 (2015) [Mech. Sol. (Engl. Transl.) **50** (4), pp. 379–388 (2015)].
5. V. V. Vasiliev and S. A. Lurie, “Generalized Solution of the Problem on a Circular Membrane Loaded by a Lumped Force,” Izv. Akad. Nauk. Mekh. Tverd. Tela, No. 3, 115–119 (2061) [Mech. Sol. (Engl. Transl.) **51** (3), 334–338 (2011)].
6. V. V. Vasiliev and S. A. Lurie, “New Solution of the Plane Problem for an Equilibrium Crack,” Izv. Akad. Nauk. Mekh. Tverd. Tela, No. 5, 61–67 (2016) [Mech. Sol. (Engl. Transl.) **51** (5), 557–561 (2016)].
7. V. V. Vasiliev and S. A. Lurie, “New Solution of Axisymmetric Contact Problem of Elasticity,” Izv. Akad. Nauk. Mekh. Tverd. Tela, No. 5, 12–21 (2017) [Mech. Sol. (Engl. Transl.) **52** (5), 479–487 (2017)].
8. V. V. Vasiliev and S. A. Lurie, “Non-local Solutions of Singular Problems of Mathematical Physics and Mechanics,” Prikl. Mat. Mech. **82** (4), 459–471 (2018).
9. M. Yu. Gutkin, “Nanoscopics of Dislocations and Disclinations in Gradient Elasticity,” Rev. Adv. Mater. Sci. No. 1, 27–60 (2000).
10. M. Lazar and G. A. Maugin, “A Note on Line Forces in Gradient Elasticity,” Mech. Res. Commun. **33**, 674–680 (2006).
11. M. Lazar, “The Fundamentals of Non-singular Dislocations in the Theory of Gradient Elasticity: Dislocation Loops and Straight Dislocations,” Int. J. Sol. Struct. **50**, 352–362 (2013).
12. P. F. Papkovich, *Theory of Elasticity* (Oborongiz, Moscow–Leningrad, 1939) [in Russian].
13. V. Novatskii, *Theory of elasticity* (Nauka, Moscow, 1975) [in Russian].
14. L. D. Landau and E. M. Lifschitz, *Fluid Mechanics* (Nauka, Moscow, 1986) [in Russian].