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Evolution of Natural Frequencies of Longitudinal Vibrations of a Bar as Its Cross-Section Defect Increases

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Abstract—We study the evolution of characteristics of natural longitudinal vibrations of a circular bar in the case of increasing defect in its cross-section. It is shown that, in the limit case where the defect separates the bar into two independent fragments, the natural frequencies of the initially defect-free bar pass into the natural frequencies of its separate parts. The respective evolution of the natural modes of vibrations is observed. The evolution predicted by the theoretical analysis can be observed experimentally by using the resonance method and constantly increasing the defect till the final separation of the bar into two parts.

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1. STATEMENT OF THE PROBLEM

The studies of natural frequencies of inhomogeneous systems with distributed parameters and defects of various nature are of great interest in the science-cognitive, general-theoretical, and applied aspects [1–8]. Their results are especially important and actual for the purposes of defect diagnostics based on the data of resonance vibration measurements in the nondestructive testing. But there are practically no meaningful publications describing the qualitative evolution of natural frequencies and natural modes of vibrations in the case of increasing defect in the cross-section [3]. In what follows, we analyze the properties of vibrations in the case where the cross-section decreases to the values at which the bar practically splits into two independent fragments.

So there is a bar with a defect in the cross-section. In the linear approximation, we study the natural vibrations of the bar of length l with a defect in the cross-section (Fig. 1). It is required to reveal how the frequency and mode shapes of longitudinal vibrations vary as the defect of a certain shape increases until the bar splits into two parts.

Assume that longitudinal vibrations are excited in the bar. When studying the influence of the defects on the natural vibrations, it is expedient to assume that the bar ends are free, because the vibrating system has the largest quality factor in this case. This property allows one to use experimental methods [3] to discover the influence of small and significant defects on the natural frequencies and mode shapes of vibrations; moreover, the boundary conditions of the second kind are the simplest to be realized experimentally.

In the dimensionless variables, we have the eigenvalue (eigenfrequency) problem, i.e., the problem of determining the eigenvalues and eigenfunctions (mode shapes) [2, 3, 9–11],

$$[p(x)u']' + \lambda r(x)u = 0, \quad p(x) \geq p_0 > 0, \quad r(x) \geq r_0 > 0, \tag{1.1}$$

$$u'(0) = u'(1) = 0, \quad 0 \leq x \leq 1. \tag{1.2}$$

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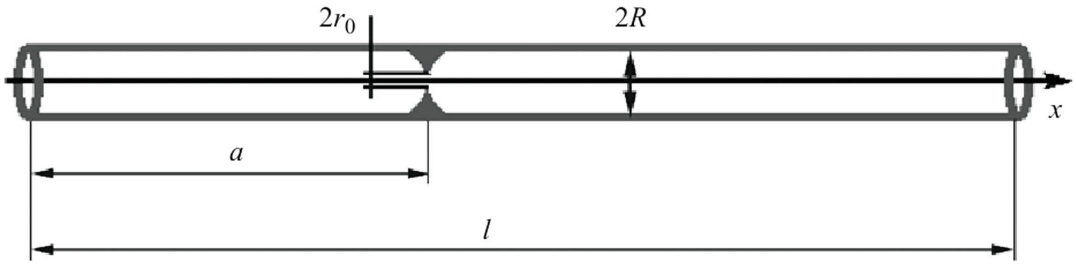

Fig. 1.

In the dimensionless variables, the desired quantities are the parameter λ, i.e., the eigenvalue ($\lambda = \omega^2$ and $\omega = \Omega l/c$ are the dimensionless natural frequency of vibrations, c is the speed of sound), and $u(x, \lambda)$, i.e., the eigenfunction (shape). The boundary conditions (1.2) correspond to the case of free ends of the bar.

For $p(x) = p_0 = 1$ and $r(x) = r_0 = 1$, we have (conditionally) ideal bar without defects; the desired quantities are

$$\omega_n = \sqrt{\lambda_n} = \pi n, \quad u_n = c_n \cos(\pi n x), \quad c_n = \text{const}, \quad c_0 = 1, \quad c_1 = \sqrt{2}, \quad n \geq 1. \tag{1.3}$$

The defect in the circular cross-section is modeled by the functions contained in (1.1), which are described by the relations

$$p(x) = r(x) = S(x), \quad S(x) = \pi R^2(x). \tag{1.4}$$

2. SOLUTION OF THE SPECTRUM PROBLEM FOR A BAR WITH A DEFECT IN THE CROSS-SECTION

The radius of the cross-section in (1.4) is given as

$$R(x) = R_0[1 \pm h(x)], \quad h(x) = \frac{1-\xi}{1+(x-a)^2/\alpha^2} \quad (R_0 = 1). \tag{2.1}$$

Here R_0 is the bar radius in the defect-free case ($\xi = 1$), the parameter ξ characterized the defect depth (minimal radius of the bar), α characterizes the defect width, and a characterizes the defect position. The maximal value of the defect is attained at the point $x = a$. The choice of the negative (positive) sign in expression (2.1) means the narrowing (extension) of the cross-section in a sufficiently narrow (for $\alpha \ll 1$) neighborhood of the point $x = a$. In the further calculations, to be definite, we consider an example where the defect in at an arbitrary point $x = a$ ($0 \leq a \leq 1$) and assume that the defect "depth" parameter ξ is positive and ranges from 1 to $\xi = 5 \times 10^{-4}$, while the value of the defect "width" parameter α is small (for definiteness, we set $\alpha = 0.001 \ll 1$).

For all admissible values of the defect "depth" parameter ξ, using the accelerated convergence method (described in detail in [3] and tested in [2, 3]), we calculated the first six eigenvalues, correspondingly, the first six frequencies $\omega_n = \sqrt{\lambda_n}$ with relative error $O(10^{-6})$. The results of calculations are given in the table. We note that the problem has the solution $\omega_0 = \lambda_0 = 0$ and $\omega_0 = 2\pi$ (see [3]).

A numerical analysis (up to $O(10^{-7})$) of the behavior of natural frequencies depending on the defect position a ($0 \leq a \leq 1$) is shown in Fig. 2 for the first four natural frequencies for $\alpha = 0.001$. The arrows show the limit values to which the natural frequencies tend as the defect disappears ($\xi \to 1$) and when the bar splits into parts ($\xi \to 0$, $\alpha \to 0$). Using A. Fock's model [12, 13], one can analytically determine the defect positions corresponding to the limit values of the frequencies from the condition of solvability of the boundary-value problem which can be written as the transcendental equation relating the vibration frequency, defect position, and minimal radius of the bar as follows:

$$\Delta(\omega) = \sin\omega + \frac{\pi}{4}\frac{\psi(\xi)}{\xi}\omega \sin(a\omega)\sin[(a-1)\omega] = \sin\omega + \frac{\pi}{8}\frac{\psi(\xi)}{\xi}\omega\{\cos\omega - \cos[(2a-1)\omega]\} = 0, \tag{2.2}$$

$$\psi(\xi) \approx \psi^*(\xi) = 1 - 1.41\xi + 0.34\xi^3 + 0.07\xi^5, \quad \psi^*(0) = 1, \quad \psi^*(1) = 0.$$

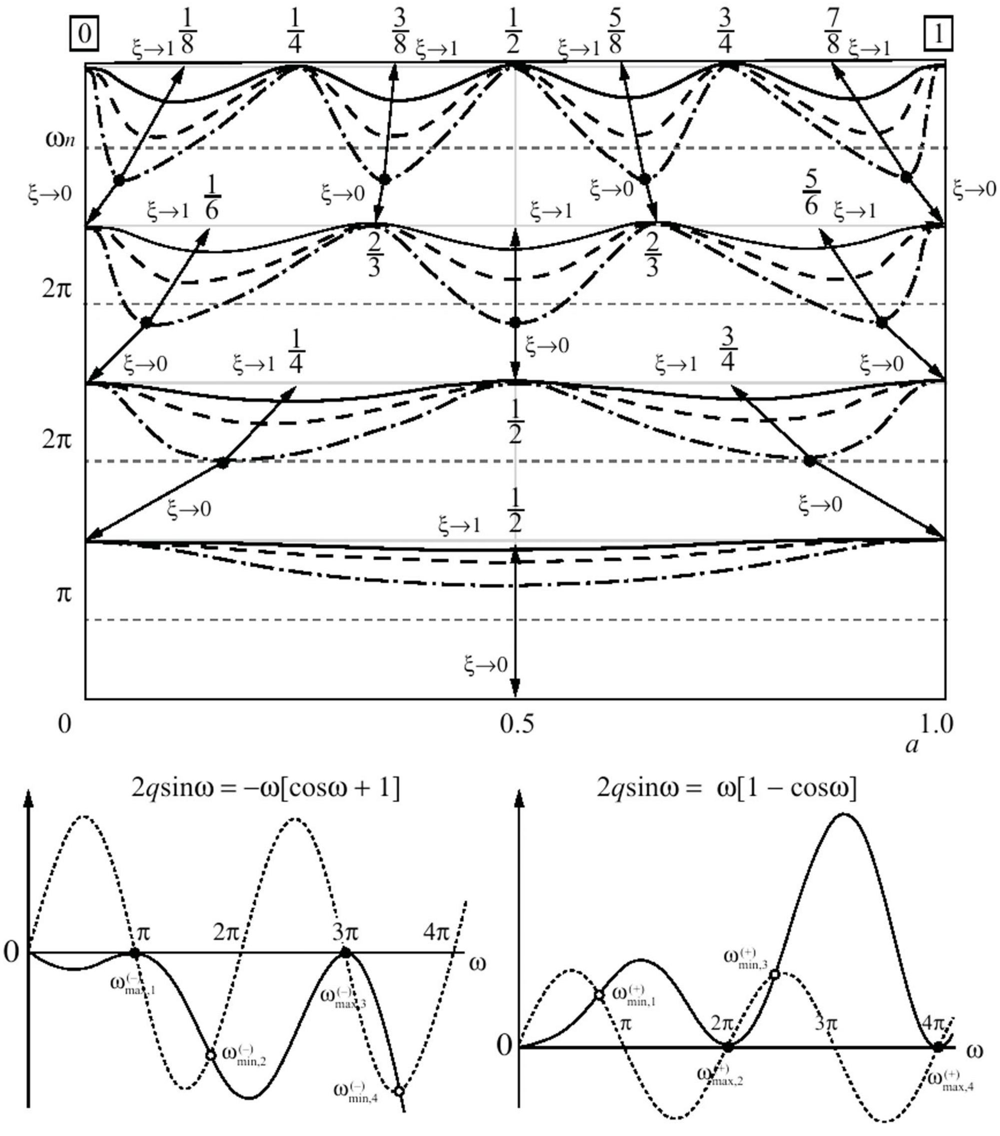

Fig. 2.

Then the extrema of the dependence $\omega_n(a)$ are determined by the system of equations

$$\begin{gathered} \sin[(2a-1)\omega] = 0, \\ 2q\sin\omega = -\omega(\cos\omega \pm 1), \quad q = \frac{4}{\pi}\frac{\xi}{\psi(\xi)}. \end{gathered} \tag{2.3}$$

From the first equation in system (2.3), we have $(2a-1)\omega = \pi n$, and the plus (minus) sign in the right-hand side of the second equation corresponds to even (odd) values of n. The solution of the second equation in system (2.2) is graphically illustrated in Fig. 2 a, b. Since the vibration frequencies are positive definite, it follows from the implicit dependence of the frequency on the defect position (2.2) that the graph $\omega(a)$ is symmetric with respect to the bar center $a = 1/2$.

The values of the frequency maxima are independent of the defect depth, and their positions are fixed:

$$\begin{gathered} \omega = \pi, \quad a = 0, 1; \quad \omega = 2\pi, \quad a = 0, \frac{1}{2}, 1; \\ \omega = 3\pi, \quad a = 0, \frac{1}{3}, \frac{2}{3}, 1; \quad \omega = 4\pi, \quad a = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1. \end{gathered} \tag{2.4}$$

The frequency minima change their position. Their limit values (their positions are shown in Fig. 2)

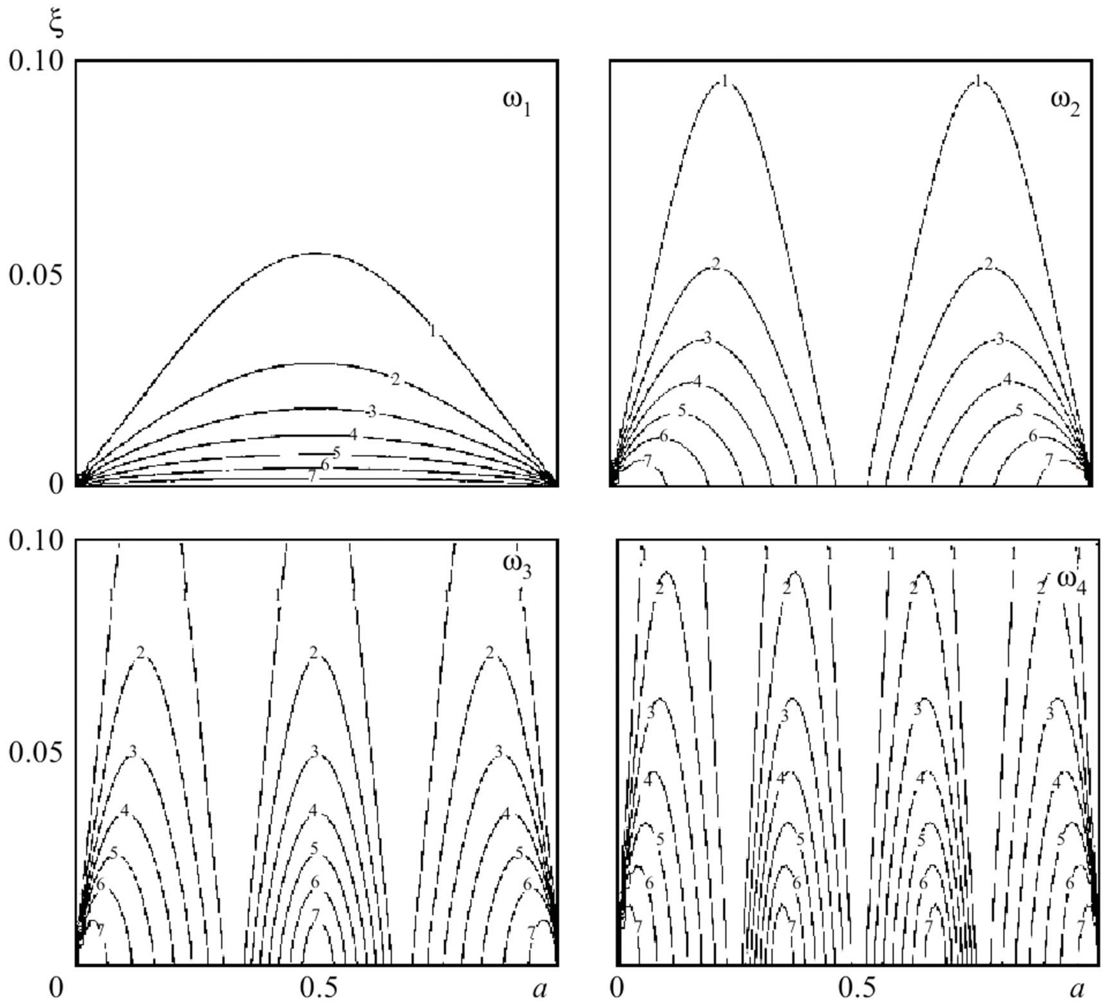

Fig. 3.

can be determined taking into account the fact that the parameter q vanishes as $\xi \to 0$ and unboundedly increases as $\xi \to 1$ (the defect-free case). They are equal to

$$\begin{aligned}
&\xi \to 0\colon \quad a = \frac{1}{2}, \quad \omega \to 0_+, 2\pi_+,\\
&\omega \to \pi_+, \quad a \to 0, 1; \quad \omega \to 2\pi_+, \quad a \to 0, 1; \quad \omega \to 3\pi_+, \quad a \to 0, \frac{1}{3}, \frac{2}{3}, 1,\\
&\xi \to 1: \quad a = \frac{1}{2}, \quad \omega \to \pi_-, 3\pi_-,\\
&\omega \to 2\pi_-, \quad a \to \frac{1}{4}, \frac{3}{4}; \quad \omega \to 3\pi_-, \quad a \to \frac{1}{6}, \frac{5}{6}; \quad \omega \to 4\pi_-, \quad a \to \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}.
\end{aligned} \tag{2.5}$$

The dependence of the vibration natural frequencies on the parameters ξ and a (their charts are shown in Fig. 3, where the digits from 1 to 7 indicate the level lines for the following frequency values: in Fig. 3 *a*: 2.775, 2.390, 2.005, 1.620, 1.235, 0.850, 0.465; in Fig. 3 *b*: 5.928, 5.535, 5.143, 4.750, 4.357, 3.965, 3.573; in Fig. 3 *c*: 9.081, 8.693, 8.304, 7.915, 7.526, 7.138, 6.749; in Fig. 3 *d*: 12.244, 11.858, 11, 471, 11.085, 10.699, 10, 313, 9.926) numerically calculated by the accelerated convergence method implies that, for the values of the hole relative radius ξ greater than 0.2, the vibration natural frequencies ω_n remain practically unchanged (the defect influence is small and independent of its position).

Let us consider the evolution of the bar frequencies and mode shapes for the defect position $a = 0.4$ in more detail. We note that, for $\xi = 5.07 \times 10^{-4}$, the ratio of the area of the cross-section of the partition between the left and right parts of the bar to the area of the cross-section of the bar itself is $S(\xi)/S(0) \approx 2.57 \times 10^{-7}$. From the standpoint of computational practice, we can assume that these

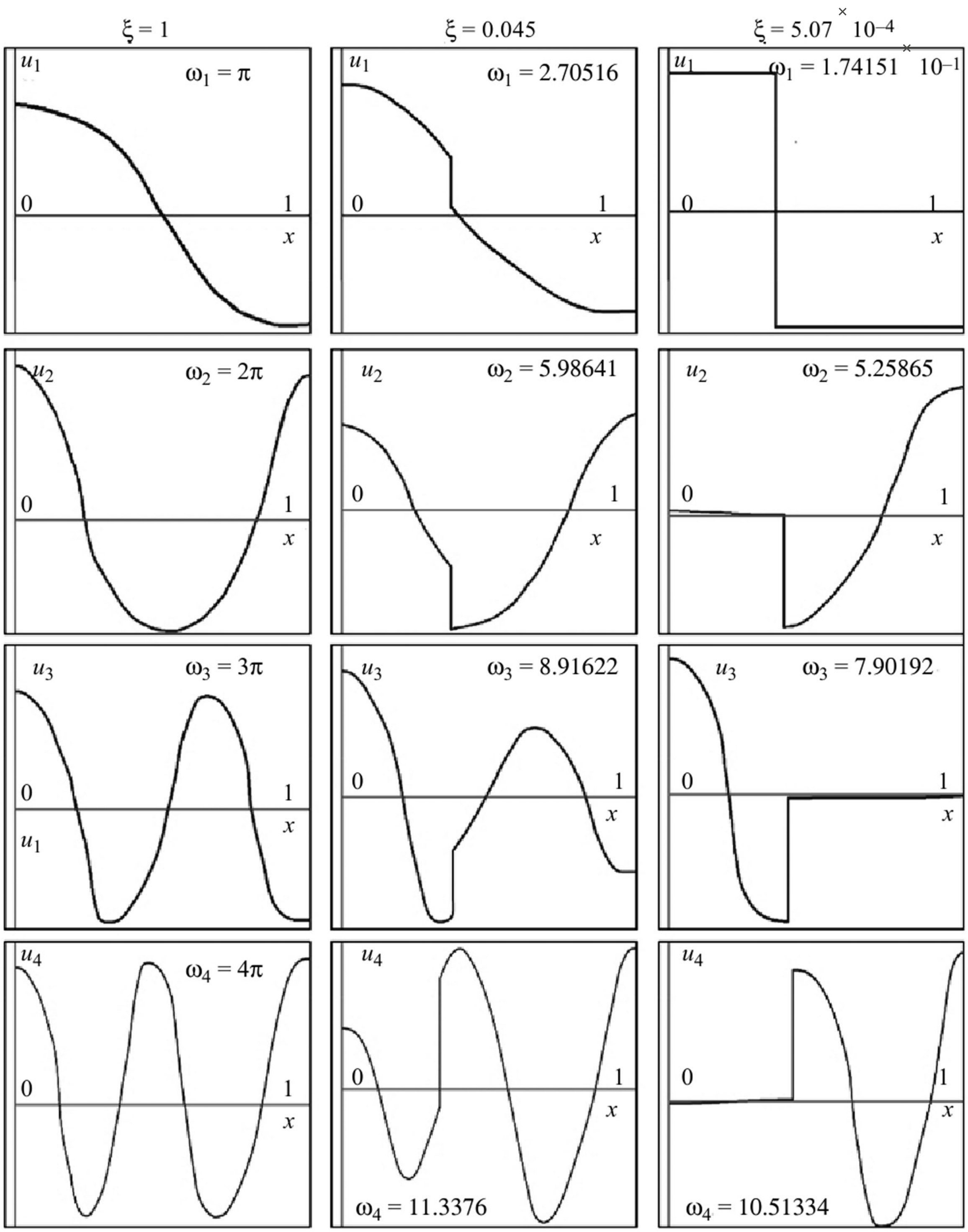

Fig. 4.

two parts are independent of each other, but passing to the limit is still singular. The natural mode shapes observed in the process of calculations are qualitatively shown in the figure for $\xi = 1$ and $\xi = 0.045$.

The eigenvalues (natural frequencies) obtained in calculations permit, with the required accuracy $O(10^{-7})$, constructing the eigenfunctions (natural mode shapes) which satisfy the boundary conditions (see Fig. 4).

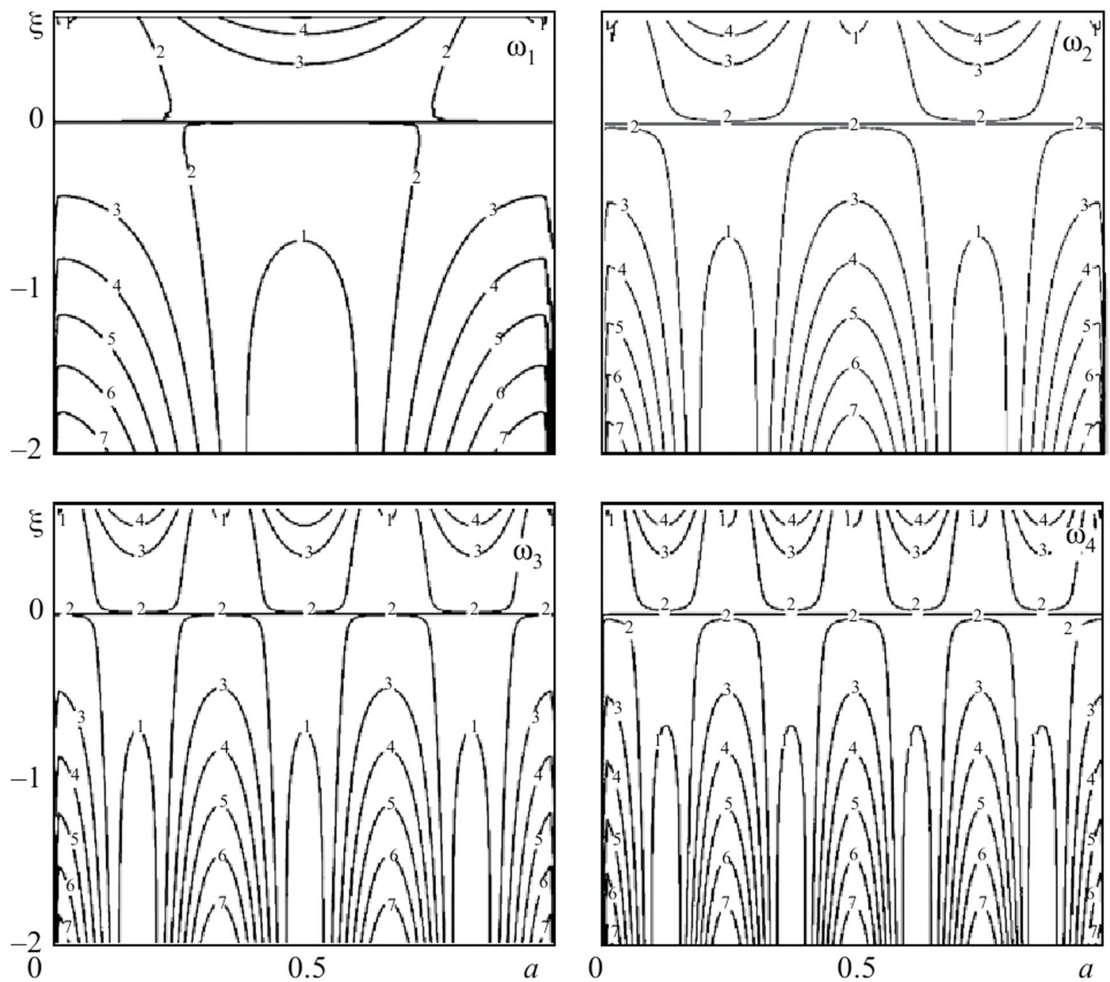

Fig. 5.

3. EVOLUTION OF NATURAL FREQUENCIES OF VIBRATIONS

The results of calculations of the first six natural frequencies depending on the defect depth are shown in the table. Let us analyze the relations between the natural frequencies and mode shapes of the entire bar with defect and of its separate parts into which the bar practically splits when the partition diameter becomes extremely small. So for example, for $\xi = 5.07 \times 10^{-4}$, the ratio of the area of the cross-section of the partition joining the two parts of the bar to the area of the cross-section of the bar itself is $S(\xi)/S(0) = 2.57 \times 10^{-7}$, i.e., the bar parts are practically separated from each other.

The second mode of the entire bar evolves into the first mode of the right part of the bar whose natural frequency is equal to $\omega_2 = 2\pi \times 0.5/0.6 = 5.23578$.

The third mode of the entire bar evolves into the first mode of the left part of the bar whose natural frequency is equal to $\omega_3 = 2\pi \times 0.5/0.4 = 7.85398$.

The fourth mode of the entire bar evolves into the second mode of the right part of the bar whose natural frequency is equal to $\omega_4 = 2\pi \times 1.0/0.6 = 10.46779$.

The fifth mode of the entire bar evolves into the third mode of the right part of the bar whose natural frequency is equal to $\omega_5 = 2\pi \times 1.5/0.6 = 15.70796$.

The sixth mode of the entire bar evolves into the second mode of the right part of the bar whose natural frequency is equal to $\omega_2 = 15.70796$. But the further decrease in the parameter ξ to the value $\xi = 5.07 \times 10^{-4}$ results in the instability of calculations. The amplitude of the second mode of the left part of the bar sharply decreases for $\xi = 8.25 \times 10^{-4}$ and the picture takes the form described below.

The further decrease in the parameter ξ leads to $\xi = 1.89 \times 10^{-4}$, $\lambda_5 = 248.6231884$, and $\omega_5 = 15.76778$.

We arrive at the final result of evolution when the right part of bar has natural frequency of the third mode $\bar{\omega}_3 = 15.70796$.

An "attempt" to evolve into the second mode of the left part of the bar failed, because both parts have the same natural frequencies for $a = 0.4$.

The studies showed that, in the case of positive sign of the defect (the choice of the sign "+" in formula (2.1)), i.e., in the case of extension of the bar cross-section, the natural frequencies vary insignificantly (1–2%) with respect to the unperturbed value up to the defect heights equal to the doubled radius of the bar. The calculated charts of the dependence of the vibration natural frequencies ω_n on the defect position a and the defect height ξ (in Fig. 5, the digits from 1 to 7 indicate the level lines for the following frequency values: in Fig. 5 *a*: 3.151, 3.142, 3.132, 3.123, 3.113, 3.104, 3.094; in Fig. 5 *b*: 6.301, 6.282, 5.263, 6.244, 6.225. 6.206, 6.187; in Fig. 5 *c*: 9.453, 9.424, 9.396, 9.368, 9.339, 9.311, 9.282; in Fig. 5 *d*: 12.602, 12.564, 12.527, 12.490, 12.452, 12.415, 12.377) showed that the position of the maxima (minima) of the frequency dependence $\omega_n(a)$ for positive defects coincides with the position of the minima (maxima) for the negative defects.

4. CONCLUSIONS

The natural vibrations of an elastic system with distributed parameters and defects were investigated. By using numerical-analytical methods, the frequencies and mode shapes of an elastic bar with free ends and variable cross-section area were analyzed with high accuracy. The influence of the values of the defect parameters varying in a wide range (nearly admissible in the limit) was studied. The theoretical and experimental results obtained by the resonance method of nondestructive testing developed by the authors were compared.

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