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Quaternion Regularization of the Equations of the Perturbed Spatial Restricted Three-Body Problem: I

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Abstract—We develop a quaternion method for regularizing the differential equations of the perturbed spatial restricted three-body problem by using the Kustaanheimo–Stiefel variables, which is methodologically closely related to the quaternion method for regularizing the differential equations of perturbed spatial two-body problem, which was proposed by the author of the present paper.
A survey of papers related to the regularization of the differential equations of the two- and three-body problems is given. The original Newtonian equations of perturbed spatial restricted three-body problem are considered, and the problem of their regularization is posed; the energy relations and the differential equations describing the variations in the energies of the system in the perturbed spatial restricted three-body problem are given, as well as the first integrals of the differential equations of the unperturbed spatial restricted circular three-body problem (Jacobi integrals); the equations of perturbed spatial restricted three-body problem written in terms of rotating coordinate systems whose angular motion is described by the rotation quaternions (Euler (Rodrigues–Hamilton) parameters) are considered; and the differential equations for angular momenta in the restricted three-body problem are given.
Local regular quaternion differential equations of perturbed spatial restricted three-body problem in the Kustaanheimo–Stiefel variables, i.e., equations regular in a neighborhood of the first and second body of finite mass, are obtained. The equations are systems of nonlinear nonstationary eleventh-order differential equations. These equations employ, as additional dependent variables, the energy characteristics of motion of the body under study (a body of a negligibly small mass) and the time whose derivative with respect to a new independent variable is equal to the distance from the body of negligibly small mass to the first or second body of finite mass.
The equations obtained in the paper permit developing regular methods for determining solutions, in analytical or numerical form, of problems difficult for classical methods, such as the motion of a body of negligibly small mass in a neighborhood of the other two bodies of finite masses.

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1. REGULARIZATION PROBLEMS FOR DIFFERENTIAL EQUATIONS OF PERTURBED SPATIAL TWO-BODY PROBLEM AND PERTURBED BOUNDED THREE-BODY PROBLEM

1.1. Kustaanheimo–Stiefel Regularization of Differential Equations of Perturbed Spatial Two-Body Problem

The celestial mechanics and astrodynamics are based on the vector Newtonian differential equation of perturbed spatial two-body problem

$$\frac{d^2\mathbf{r}}{dt^2} + f(m+M)r^{-3}\mathbf{r} = \mathbf{p}\left(t, \mathbf{r}, \frac{d\mathbf{r}}{dt}\right), \tag{1.1}$$

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where $\mathbf{r}$ is the radius vector of the center of mass of the second body (under study) which is drawn from the center of mass of the first (central) body, $r = |\mathbf{r}|$, m and M are the masses of the second and first bodies, f is the gravitation constant, $\mathbf{p}$ is the vector of perturbing acceleration of the center of mass of the second body, and t is the time.

This equation degenerates in collision of the second body with the central body (when the distance r between the bodies is zero), and hence it is inconvenient to use this equation to study the motion of the second body in a small neighborhood of the central body or its motion in strongly elongated orbits. The singularity at the origin creates not only theoretical but also practical (computational) difficulties.

The problem of removing this singularity, which is known in the celestial mechanics and astrodynamics as the problem of regularization of differential equations of perturbed two-body problem, dates back to L. Euler [1] and T. Levi-Civita [2–4], who solved the one- and two-dimensional problems of collision of two bodies (in the cases of rectilinear and plane motions). The most efficient regularization of equations of perturbed spatial two-body problem, the so-called spinor or KS-regularization, was proposed by P. Kustaanheimo and E. Stiefel [5, 6]. This regularization is a generalization of the Levi-Civita regularization of equations of plane motion, and its most complete presentation is given in the widely known monograph by E. Stiefel and G. Scheifele [7].

The KS-regularization is based on a nonlinear non-unique transformation of Cartesian coordinates of the body under study, the so-called KS-transformation, which generalizes the Levi-Civita transformation and has the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_0 \\ u_2 & u_1 & -u_0 & -u_3 \\ u_3 & u_0 & u_1 & u_2 \\ u_0 & -u_3 & u_2 & -u_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_0 \end{pmatrix} = L(\mathbf{u}_{KS}) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_0 \end{pmatrix}, \tag{1.2}$$

where x_k $(k = 1, 2, 3)$ are the coordinates of the center of mass of the body considered in inertial coordinates with origin at the center of mass of the central body and coordinate axes directed to remote stars, u_j $(j = 0, 1, 2, 3)$ are new variables (KS-variables), and $L(\mathbf{u}_{KS})$ is the generalized Levi-Civita matrix, called the KS-matrix, which contains the two-dimensional square Levi-Civita matrix in its left upper corner.

In scalar form, transformation (1.2) becomes

$$x = u_0^2 + u_1^2 - u_2^2 - u_3^2, \quad x_2 = 2(u_1 u_2 - u_0 u_3), \quad x_3 = 2(u_1 u_3 + u_0 u_2), \tag{1.3}$$

and up to permutation of indices, it coincides with the Hopf mapping [8].

The regular Kustaanheimo–Stiefel differential equations of perturbed spatial two-body problem in scalar form become [7]

$$\frac{d^2 u_j}{d\tau^2} - \frac{h}{2} u_j = \frac{r}{2} q_j \quad (j = 0, 1, 2, 3), \tag{1.4}$$

$$\frac{dh}{d\tau} = 2\left(q_0 \frac{du_0}{d\tau} + q_1 \frac{du_1}{d\tau} + q_2 \frac{du_2}{d\tau} + q_3 \frac{du_3}{d\tau} \right), \tag{1.5}$$

$$\frac{dt}{d\tau} = r, \quad r = |\mathbf{r}| = u_0^2 + u_1^2 + u_2^2 + u_3^2, \tag{1.6}$$

$$q_0 = u_0 p_1 - u_3 p_2 + u_2 p_3, \quad q_1 = u_1 p_1 + u_2 p_2 + u_3 p_3,$$
$$q_2 = -u_2 p_1 + u_1 p_2 + u_0 p_3, \quad q_3 = -u_3 p_1 - u_0 p_2 + u_1 p_3.$$

Here τ is a new independent variable called the fictive time which is related to the time t by differential equation (1.6), h is an additional variable which has the meaning of Kepler energy and is determined by the relation $h = \frac{1}{2}v^2 - f(m + MN)r^{-1}$ ($v = |\mathbf{v}|$, $\mathbf{v} = d\mathbf{r}/dt$), and p_k $(k = 1, 2, 3)$ is the projection of perturbed acceleration $\mathbf{p}$ of the center of mass of the second body on the axes of the inertial coordinate system.

These equations form a system of ten ordinary nonlinear nonstationary differential equations with respect to the Kustaanheimo–Stiefel variables u_j, the Kepler energy h, and the time t. Equations (1.4) are equivalent to the matrix equation

$$\frac{d^2\mathbf{u}_{KS}}{d\tau^2} - \frac{h}{2}\mathbf{u}_{KS} = \frac{r}{2}L^T(\mathbf{u}_{KS})\mathbf{P}_{KS}, \tag{1.7}$$

where $\mathbf{u}_{KS}$ is the four-dimensional column vector of KS-variables $\mathbf{u}_{KS} = (u_1, u_2, u_3, u_0)$, and $\mathbf{P}_{KS}$ is the four-dimensional column vector associated with the three-dimensional vector of perturbed acceleration $\mathbf{p}$, $\mathbf{P}_{KS} = (p_1, p_2, p_3, 0)$; here the superscript T is the transposition symbol.

Let us note the following advantages of Kustaanheimo–Stiefel equations [7, 9–15]:

— they, in contrast to Newtonian equations, are regular at the center of attraction;

— they are liner for unperturbed Kepler motions and, in this case, have the form

$$\frac{d^2u_j}{d\tau^2} - \frac{h}{2}u_j = 0, \quad h = \text{const} \quad (j = 0, 1, 2, 3)$$

(for the elliptic Kepler motion with the Kepler energy $h < 0$, these equations are equivalent to the equations of motion of four-dimensional one-frequency harmonic oscillator whose squared frequency is equal to half the Kepler energy with minus sign);

— they permit developing a unified approach to studying all three types of Kepler motion;

— they are closely related to linear equations for perturbed Kepler motions;

— they permit representing the right-hand sides of differential equations of motion of celestial and cosmic bodies in polynomial form which is convenient for solving them by computers.

These facts allowed one to develop efficient methods for determining solutions in analytic or numerical form of problems difficult for the classical methods such as the study of motion near attracting masses or motion in large eccentricity orbits. So in [7, 10, 11], it was shown that the use of regular equations in KS-variables permits increasing the accuracy of numerical solution of several problems in celestial mechanics and astrodynamics (for example, the problem of motion of artificial satellite of the Earth (AES) in large eccentricity orbits) from three to five orders of magnitude as compared to the solutions obtained using the classical (Newtonian) equations in rectangular coordinates.

As was already noted, the KS-regularization is based on a nonlinear non-unique transformation of Cartesian coordinates (1.3), and this transformation consists in the transition from the three-dimensional space of Cartesian coordinates x_k to a four-dimensional space of new coordinates u_k. Therefore, according to E. Stiefel and G. Scheifele, it is impossible to derive regular equations directly in the three-dimensional (i.e., spatial) case. In the book [7], they postulate a matrix regular equation of spatial two-body problem (1.7), which they wrote by analogy with the Levi-Civita matrix regular equation of plane motion, and use several theorems to prove that the old vector Newtonian equation (1.1) is satisfied in this case. Such an approach to the construction of regular equations of spatial two-body problem is artificial in many aspects and hardly visual.

1.2. Quaternion Regularization of Differential Equations of Perturbed Spatial Two-Body Problem Given by the Author of the Paper, and Its Generalization

Soon after the KS-regularization was discovered, it was proposed to use quaternions (four-dimensional hypercomplex numbers) and four-dimensional quaternion matrices to regularize the equations spatial two-body problems. But in their book, E. Stiefel and G. Scheifele completely rejected this idea. They wrote ([7, p. 288]): "Any attempt to replace the theory of KS-matrices by a more popular theory of quaternion matrices fails or, in any case, leads to a very cumbersome formalism." This assertion was first disproved by the author of this paper who, in the end of the 1970s and at the beginning of the 1980s, showed in [16–19] that, in fact, the quaternion approach to the regularization permits obtaining a direct and visual derivation of regular equations in KS-variables, makes the basic postulates underlying the KS-regularization more natural and visual, and permits constructing a theory generalizing the KS-regularization.

The author of this paper showed that the regularizing KS-transformation of coordinates (1.2) or (1.3) means the transition from Cartesian coordinates of the center of mass of the second body in inertial coordinates to new variables which are components (normalized in certain way) of the conjugate rotation quaternion characterizing the orientation of a rotating coordinate system η in the inertial coordinate system. The axis η_1 of this rotating coordinate system is directed along the radius vector $\mathbf{r}$ of the center of mass of the second body. The normalizing coefficient is equal to the square root of the distance r

from the center of mass of the second body to the attraction center. The bilinear Kustaanheimo–Stiefel relation

$$u_1\frac{du_0}{d\tau} - u_0\frac{du_1}{d\tau} + u_3\frac{du_2}{d\tau} - u_2\frac{du_3}{d\tau} = 0, \tag{1.8}$$

relating the KS-variables and their first derivatives and, according to E. Stiefel and G. Scheifele, playing the key role in their construction of regular celestial mechanics [7, p. 29], imposes an additional (nonholonomic) condition the motion of the trihedron η, which means that the projection ω_1 of the vector of absolute angular velocity of the trihedron η on the direction of the radius vector $\mathbf{r}$ (axis η_1) is zero.

Thus, the transition in the equations of spatial two-body problem from Cartesian coordinates of the center of mass of the second body to the KS-variables actually means that these equations are written in the rotating coordinate system η, where the Euler (Rodrigues–Hamilton) parameters, which are components of the quaternion of rotation of this coordinate system, are taken as the parameters of orientation of this rotating coordinate system. The further transformations of these equations are related to the normalization of the Euler parameters (rotation quaternion) by the above-described method with introduction of additional dependent variables of the Kepler energy and time and with transition to a new independent variable (fiction time).

The quaternion regular equations of perturbed spatial two-body problem in KS-variables has the form [16, 17] (also see [14, 15])

$$\begin{aligned}
&\frac{d^2\mathbf{u}}{d\tau^2} - \frac{h}{2}\mathbf{u} = \frac{r}{2}\mathbf{q},\\
&\frac{dh}{d\tau} = 2\mathrm{scal}\left(\frac{d\bar{\mathbf{u}}}{d\tau}\circ\mathbf{q}\right), \quad \frac{dt}{d\tau} = r,
\end{aligned} \tag{1.9}$$

$$\begin{aligned}
&r = \|\mathbf{u}\|^2 = \mathbf{u}\circ\bar{\mathbf{u}} = \bar{\mathbf{u}}\circ\mathbf{u} = u_0^2 + u_1^2 + u_2^2 + u_3^2,\\
&\mathbf{q} = -\mathbf{i}\circ\mathbf{u}\circ\mathbf{P}, \quad \mathbf{P} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k},\\
&\mathbf{R} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} = \bar{\mathbf{u}}\circ\mathbf{i}\circ\mathbf{u},
\end{aligned} \tag{1.10}$$

$$\mathbf{V} = \frac{d\mathbf{R}}{dt} = 2\bar{\mathbf{u}}\circ\mathbf{i}\circ\frac{d\mathbf{u}}{dt} = \frac{2}{r}\bar{\mathbf{u}}\circ\mathbf{i}\circ\frac{d\mathbf{u}}{d\tau}. \tag{1.11}$$

Here and below, the symbol $\circ$ denote the quaternion multiplication; $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the Hamiltonian vector imaginary units; the upper bar is the symbol of quaternion conjugation; scal( ) is the scalar part of quaternion placed in parentheses; $\mathbf{u}$ is the quaternion regular variable defined by the relations

$$\begin{aligned}
&\mathbf{u} = u_0 + u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} = r^{1/2}\bar{\lambda}, \quad \lambda = r^{-1/2}\bar{\mathbf{u}},\\
&u_0 = r^{1/2}\lambda_0, \quad u_k = -r^{1/2}\lambda_k \quad (k = 1, 2, 3),\\
&\|\mathbf{u}\|^2 = \mathbf{u}\circ\bar{\mathbf{u}} = \bar{\mathbf{u}}\circ\mathbf{u} = u_0^2 + u_1^2 + u_2^2 + u_3^2 = r,
\end{aligned}$$

where u_j are still regular Kustaanheimo–Stiefel variables, λ_j are Rodrigues–Hamilton (Euler) parameters characterizing the orientation of the coordinate system η in inertial coordinates. Quaternion relations (1.10) and (1.11) are used to determine the Cartesian coordinates of the second (studied) body in inertial coordinates and the projections of its velocity vector on the axes of this coordinate system.

In quaternion equations (1.9), the role of variables is played by the quaternion $\mathbf{u}$ whose components are regular KS-variables u_j, the Kepler energy h, and the time t. In scalar form, equations (1.9) coincide with regular equations (1.4)–(1.6) derived by P. Kustaanheimo and E. Stiefel. Therefore, quaternion equations (1.9) have all previously mentioned advantages of regular Kustaanheimo–Stiefel equations. At the same time, they permit using the convenient and well-developed formalism of quaternion algebra in analytic and numerical studies.

The author of this paper also obtained [16, 17] (also see [14, 15]) more general quaternion regular equations of perturbed spatial two-body problem in KS-variables under the assumption that the bilinear Kustaanheimo–Stiefel relations (1.8) do not hold. These equations contain additional terms with the projections ω_1 and ε_1 of the vectors of angular velocity and angular acceleration of the accompanying trihedron η on the direction of the radius vector $\mathbf{r}$ of the center of mass of the second body (one of these projections is an arbitrarily given parameter) and are more complicated.

We note that these more general quaternion regular equations of perturbed spatial two-body problem were obtained by the author of this paper in [17], while in the previous paper [16], he obtained their matrix analog in quaternion matrices. It should be also noted that another more general interpretation of the regularizing KS-transformation was obtained in [15, 16] by considering the helical motion of the above-introduced trihedron η and using the quaternion differential equations of motion of the two-body problem in parameters of the helical motion of this trihedron. This approach allowed one to reveal the relationship between the regularizing KS-transformation (1.2) and the E. Study formula relating the rectangular coordinates of the origin of the introduced moving coordinate system η to the components of the biquaternion of its helical finite displacement [20, p. 146]; this transformation is a particular case of the E. Study formula.

Thus, the quaternion approach to regularization of equations of perturbed spatial two-body problem proposed by the author of this paper, in contrast to the approach based on the KS-matrix apparatus, allows one to obtain clear geometric and kinematic interpretations of the regularizing KS-transformation, to reveal the geometric meaning of its ambiguity, and to give a direct visual derivation of regular equations of spatial two-body problem one of whose particular cases is the regular Kustaanheimo–Stiefel equations.

The author of the paper also proposed regular equations of perturbed spatial two-body problem in quaternion osculating elements (i.e., in quaternion slowly varying variables), which were obtained from the quaternion regular equations in KS-variables by the method of variation of arbitrary quaternion constants and were published in [15]. These equations are a quaternion analog of equations of spatial two-body problem in regular elements obtained in [7, p. 93]. We point out the following advantages of these equations: first, they are regular (have no singularity at the origin), and second, their right-hand sides are uniformly and slowly varying functions in the case of perturbed elliptic motion, and in the case of unperturbed Kepler motion, the equations can be integrated without methodological errors. A drawback of these equations is the fact that the region of their applicability is bounded by motions of elliptic type (for the Kepler energy $h < 0$).

Later [18, 19, 21–24] (also see [14, 15]), the ideas of quaternion regularization of equations of the two-body problem were used by the author of this paper to develop the theory of quaternion regularization of the vector differential equation of perturbed central motion of material point:

$$\begin{gathered}\frac{d^{\mathbf{r}}t}{d =^{\mathbf{r}}} - \frac{1}{m}\left(\frac{d\Pi}{dr}\frac{\mathbf{r}}{r} + \frac{\partial\Pi^*}{\partial\mathbf{r}}\right) + \mathbf{p},\\ r = |\mathbf{r}|, \quad \Pi = \Pi(r), \quad \Pi^* = \Pi^*(t,\mathbf{r}), \quad \mathbf{p} = \mathbf{p}\left(t, \mathbf{r}, \frac{d\mathbf{r}}{dt}\right).\end{gathered} \tag{1.12}$$

This equation describes the motion of material point with mass m in a central force field with potential Π, which is an arbitrary differentiable function of the distance r from the point to the force field center, under the action of a perturbing force equal to the geometric sum of the force with potential Π^* and the force $m\mathbf{p}$. Here $\mathbf{r}$ is the radius vector of material point drawn from the attraction center O, and $\mathbf{p}$ is the perturbing acceleration due to the force $m\mathbf{p}$. The equation of unperturbed central motion of material point is obtained from equation (1.12) it we set $\Pi^* = 0$ and $\mathbf{p} = 0$ in it.

In [18, 19, 21–24] (also see [14, 15]), the general quaternion differential equations of perturbed central motion of material point with regularizing functions were obtained; necessary and sufficient conditions for their reducibility to the oscillatory form convenient for analytical and numerical studies (i.e., to the form of equations of motion of a four-dimensional perturbed oscillator which harmonically oscillates with the same frequency in the case of unperturbed central motion) were established; different (including new regular) systems of quaternion differential equations of perturbed central motion of material point in normal and oscillatory form, which differ in their structure, dimension, and the employed dependent and independent variables, were obtained; the obtained systems were compared, and their properties and regions of applicability were determined.

So, regular differential equations of perturbed central motion of material point of oscillatory form, which are regular for the potential

$$\Pi(r) = -a_1 r^{-1} - a_2 r^{-2} - a_3 r^{-3} - a_4 r^{-4}, \quad a_i = \text{const},$$

i.e., a polynomial of negative fourth degree of the distance r to the attraction center, were obtained in [18, 21, 24]. (Recall that the Kustaanheimo–Stiefel equations are regular only for a polynomial of the first negative degree of the distance r.)

In [18, 22], new equations of satellite motion in the terrestrial gravitational field (neglecting the tesseral and sectorial harmonics) were obtained in new variables, which have all advantages of the known equations in KS-variables [7] but have a simple and symmetric structure and their order can be decreases by two units.

E. Stiefel and G. Scheifele [7] wrote that Levi-Civita made a very good effort to find a generalization of his method to regularize the differential equations of plane motion in the two-body problem to the general spatial two-body problem, but without success. In [25], it is noted that because of fundamental difficulties which were first explained by Hopf [26] and Hurwitz [27], it is impossible to generalize the Levi-Civita transformation to the case of three-dimensional space. Nevertheless, the author of this paper showed [28] that the Levi-Civita regularization can successfully be used to construct regular equations of perturbed spatial two-body problem. This can be done [28] by using ideal rectangular Hasen coordinates, regular Levi-Civita variables U_0 and U_3, the Kepler energy h as an additional variable, and a new independent variable τ (new time) and using the the quaternion variable $\boldsymbol{\Lambda}$ describing the orientation of an ideal (in the sense of A. Deprit [29]) coordinate system in which the differential equations of perturbed spatial two-body problem must be written (or by using the Euler (Rodrigues—Hamilton) parameters Λ_j describing the orientation of this coordinate system).

The new, proposed by the author of this paper, regular equations of perturbed spatial two-body problem in scalar form become [28]

$$\frac{d^2U_0}{d\tau^2} - \frac{h}{2}U_0 = \frac{r}{2}Q_0, \quad \frac{d^2U_3}{d\tau^2} - \frac{h}{2}U_3 = \frac{r}{2}Q_3, \quad \frac{dh}{d\tau} = 2\left(Q_0\frac{dU_0}{d\tau} + Q_3\frac{dU_3}{d\tau}\right),$$
$$2\frac{d\Lambda_0}{d\tau} = -r(\Omega_1\Lambda_1 + \Omega_2\Lambda_2), \quad 2\frac{d\Lambda_1}{d\tau} = r(\Omega_1\Lambda_0 - \Omega_2\Lambda_3),$$
$$2\frac{d\Lambda_2}{d\tau} = r(\Omega_2\Lambda_0 + \Omega_1\Lambda_3), \quad 2\frac{d\Lambda_3}{d\tau} = r(\Omega_2\Lambda_1 - \Omega_1\Lambda_2),$$
$$\frac{dt}{d\tau} = r, \quad r = |\mathbf{r}| = U_0^2 + U_3^2,$$
$$\Omega_1 = c^{-1}(U_0^2 - U_3^2)p_{\xi 3}, \quad \Omega_2 = -2c^{-1}U_0U_3p_{\xi 3}, \quad c = 2\left(U_3\frac{dU_0}{d\tau} - U_0\frac{dU_3}{d\tau}\right),$$
$$Q_0 = U_0p_{\xi 1} - U_3p_{\xi 2}, \quad Q_3 = -U_3p_{\xi 1} - U_0p_{\xi 2},$$

where $o_{\xi k}$ $(k = 1, 2, 3)$ are projections of the vector $\mathbf{p}$ of the perturbing acceleration of the center of mass of the second body on the axes of the ideal coordinate system ξ.

These equations form a system of nonlinear nonstationary tenth-order differential equations in the variables U_0, U_3, h, Λ_j $(j = 0, 1, 2, 3)$ (all regular KS-equations have the same dimension) and have all advantages of the Kustaanheimo—Stiefel equations. In contrast to the Newtonian equations, they are regular at the center of attraction and linear in the case of unperturbed Kepler motions; permit developing a unified approach to studying all three types of the Kepler motion; are close to linear equations for perturbed Kepler motions; permit representing the right-hand sides of differential equations of motion of celestial and cosmic bodies in polynomial form convenient for solving them by computers.

In contrast to the Kustaanheimo—Stiefel equations, the regular equations of elliptic Kepler motion, proposed by the author of this paper, are equivalent to the equations of motion of not a four-dimensional but a two-dimensional one-frequency harmonic oscillator whose square frequency is equal to half the Kepler energy h with minus sign. The quaternion $\boldsymbol{\Lambda}$ of orientation of the ideal coordinate system (constant for the Kepler motion), which is used in the proposed regular equations, is a slow (slightly varying) variable in the perturbed two-body problem which makes the proposed regular equations convenient for the application of methods of nonlinear mechanics.

We note that, in [9, pp. 63—66], V. A. Brumberg described how the Euler parameters can be used to derive the equations of perturbed motion of spatial two-body problem in Hansen coordinations and pointed out that further transformation of the obtained equations of perturbed motion is possible by using the parabolic Levi-Civita coordinates.

1.3. Works of Other Authors in the Field of KS and Quaternion Regularization of Differential Equations of Two-Body Problem

The perturbed Kepler motion was studied by E. Stiefel and G. Scheifele [7] not only by using the regular equations in oscillatory form and methods of the oscillation theory but also by using the regular equations in canonical form, for which they developed the theory of canonical KS-transformation. Such a canonical approach to the regularization problem, based on the use of KS-transformation, was developed by M. D. Lidov [30–32]. The application of a generalized KS-matrix and related transformations in the theory of regularization of canonical equations of the two-body problem was considered later in [33].

We also note the paper [34], where the differential equations of motion of an artificial satellite of the Earth were obtained in the orbital coordinate system. The rotation quaternion, normalized by a factor equal to the square root of the absolute value of the vector of satellite velocity moment, was used to describe the motion of this coordinate system in the inertial space. These equations are linear for the unperturbed motion of the satellite.

As for the works of foreign authors, it is worth to note the papers [35–37], where the applicability of quaternions to regularization of equations of celestial mechanics was demonstrated later but apparently independently of the author of the present paper. In the last years, Waldvogel's papers [38, 39] were written about the quaternion regularization of differential equations of perturbed spatial two-body problem. (We note that Waldvogel has earlier works in the problem of regularization written together with one of the authors of the KS-regularization, E. Stiefel [40].) In [39], it is said that "it is a true way to use quaternions to regularize the celestial mechanics" and the quaternions "are an ideal tool for describing and developing the theory of spatial regularization in celestial mechanics."

In our opinion, the quaternion method for regularizing the differential equations of perturbed spatial two-body problem described in [39] has no advantage over the quaternion regularization method proposed by the author of this paper much earlier. Moreover, the forme is worth that the latter in geometric and kinematic visualization and in the possibility of further generalization.

We note that the priority of the author of this paper in the field of quaternion regularization was acknowledged in [39], where it was said that "This assertion[1)] was first disproved by Chelnokov (1981) who used geometric representations in a rotating coordinate system and quaternion matrices to represent the theory of regularization of spatial Kepler problem. An a series of papers (for example, 1992 and 1999), the same author extended the theory of quaternion regularization and gave practical applications."

Now we point out the basic specific features of the quaternion method of the Waldvogel regularization [39]. For regularization, he proposed to use the "star conjugate" quaternion

$$\mathbf{u}^* = -k\bar{\mathbf{u}}k = u_0 + iu_1 + ju_2 + ku_3 \tag{1.13}$$

(quaternion "star conjugate" to the quaternion $\mathbf{u} = u_0 + iu_1 + ju_2 + ku_3$) and the mapping

$$\mathbf{u} \in U \to \mathbf{x} = \mathbf{u}\mathbf{u}^*, \tag{1.14}$$

which is based on the nontraditional representation of three-dimensional vector $\mathbf{x}$ by the quaternion $\mathbf{x} = x_0 + ix_1 + jx_2$ whose kth component is zero (we note that Waldvogel does not use the special symbol $\circ$ of quaternion produce). Such a quaternion $\mathbf{x}$ is a formal generalization (accretion) of the complex variable $\mathbf{x} = x_0 + ix_1$ used by Levi-Civita in the theory of regularization of equations of plane motion.

The mapping (1.14) with regard to (1.13) becomes

$$\mathbf{x} = \mathbf{u}\mathbf{u}^* = -\mathbf{h}k\bar{\mathbf{u}}k. \tag{1.15}$$

In scalar form, (1.15) implies

$$x_0 = u_0^2 - u_1^2 - u_2^2 + u_3^2, \quad x_1 = 2(u_0u_1 - u_2u_3), \quad x_2 = 2(u_0u_2 + u_1u_3), \tag{1.16}$$

"which is precisely a KS-transformation in its classical form or – before the permutation of indices – a Hopf transformation" (these are words of the author of [39]).

In the classical theory of quaternions, a three-dimensional vector $\mathbf{x}$ is associated with the quaternion $\mathbf{x} = ix_1 + jx_2 + kx_3$ with zero scalar part. In the works of the author of this paper, the regularization

[1)]The author means the above-cited assertion by E. Stiefel and G. Scheifele that the use of quaternion matrices in the theory of regularization has no perspectives.

is based on the use of the quaternion variable $\mathbf{u} = u_0 + iu_1 + ju_2 + ku_3$, which does not coincide (in the meaning) with the Waldvogel quaternion variable, and the quaternion with zero scalar part. In these works (and in the present paper), the mapping

$$\mathbf{x} = \bar{\mathbf{u}}i\mathbf{u}$$

and the mapping

$$\mathbf{x} = \bar{\mathbf{u}}k\mathbf{u}$$

are used.

In scalar form, the first of these mapping is precisely the Kustaanheimo–Stiefel transformation (1.3) which differs from (1.16) in form.

We note that no actually (not formally) new regular equations of the two-body problem were obtained in [39]. Nevertheless, because of great importance of the regularization problem in celestial mechanics and astrodynamics, the quaternion method of Waldvogel regularization of equations of perturbed two-body problem is undoubtedly of great interest. The elegant quaternion representation of the Birkhoff spatial mapping obtained by Waldvogel in [39], which is used in the theory of regularization of equations of bounded three-body problem, is also undoubtedly of great interest. This representation is given by Waldvogel in addition to his earlier results in the theory of regularization 40–42].

To conclude this section, we note the book [14] of the author of this paper published in 2011, where, in particular, the quaternion method for regularizing the differential equations of perturbed spatial two-body problem and perturbed central motion of material point is presented and the quaternion regular models of celestial mechanics and astrodynamics and their applications to solving the problems of optimal control of the spacecraft trajectory motion are given. We also note the surveys [15, 18] of the author of this paper in the regularization of equations of celestial mechanics and astrodynamics.

1.4. Regularization of Differential Equations of the Three-Body Problem

In [25], Aarseth and Zare noted that the history of regularization of the three-body problem begins in the famous works by Poicare [43] and Sundman [44]. In fact, Sundman solved the general problem in principle by using two transformations of time and neglecting the case of triple collision. Unfortunately, the solutions are represented by infinite series and do not reveal the true character of motions. The authors of [25] believe, which is also doubtful, that the regularization of the three-body problem proposed in [45] is also useful in the general case from the practical standpoint. They also note that the following two requirements can be distinguished in the problem of regularization of the general three-body problem: (1) regularization of all collisions of two bodies by using one "global" transformation; (2) improved treatment of close triple collisions. Several examples of transformations satisfying the first requirement in the plane bounded three-body problems are given in [45–48]. Waldvogel [49] presented a global regularization the plane three-body problem with arbitrary masses. The equations obtained in this case are symmetric with respect to some point masses and satisfy both the above requirements. In [50], the Levi-Civita regularization is generalized to unperturbed plane bounded three-body problem by using the complex Levi-Civita variables and the time transformation containing the product of raised-to-the-third-power distances from the body of negligibly small mass to the two bodies of attraction which have finite masses. We note that, for regularization in the three-body problem, one often uses another time transformation which contains the product of distances from the body of negligibly small mass to the two bodies of finite mass, and, as the author of [25] says, the equations obtained by using the time transformation containing the cubed distances is not regular, because the new time tends to infinity as the three body collide.

In [25], the canonical Hamilton formalism and two KS-transformations were used to regularize the equations of the perturbed spatial unbounded three-body problem. This regularization allows one particle to collide with the other two particles whose relative motion is described by singular equations which still exhibit a good behavior when being solved numerically. This paper present the eight-dimensional regularization of the general three-body problem which is based on the double KS-regularization and has the following properties (by r_0, r_1, and r_{01} we denote the distances between the bodies M_0 and M_2, M_1 and M_2, M_0 and M_1, respectively): (1) the equations of motion are regular in collision of two bodies as $r_0 \to 0$ or as $r_1 \to 0$; (2) under the conditions that $r_{01} \geq r_0$ or $r_{01} \geq r_1$, the equations of motion can be solved well in the nearly triple collisions.

Aarseth and Zare believe that their new method for regularizing the three-body problem, which they proposed in [25], cannot be used to study the three-body problem where one of the particles is massless, i.e., to study the bounded three-body problem. At the same time, they say that this new method is closely related to the standard KS-regularization which allows one of the particles to be weightless.

We note the widely cited book [51], where, in particular, the Levi-Civits and Kustaanheimo–Stiefel regularizations of equations of the plane and spatial two-body problems and the Aarseth–Zare regularization of spatial three-body problem are presented and many aspects of their use (the program, algorithimc, and practical aspects of their use) are discussed.

In the book [10, pp. 34–45], it is proposed to use the double KS-transformation (with reference to [25]) to construct canonical equations of the perturbed bounded three-body problem. The regularized (transformed according to the Aarseth–Zare methodology) Hamiltonian of the problem and the eighteenth-order equations in general standard Hamiltonian form are written. It is noted that the system of these differential equations contains equations determining the time transformation and the law of variation in the system energy. The right-hand sides of differential equations of the problem studied in [10] are not written explicitly, i.e., not written in the form obtained after differentiation of the Hamiltonian with respect to the problem variables (by the way, such equations are also not written in [25]). This does not permit completely estimating the properties of the equations obtained by the method proposed in [10].

At the same time, as was noted in [25], the removal of a singularity in the Hamiltonian need not always eliminate the singular terms in the equations of motion. The authors of [25] believe that if the Hamiltonian is regular but contains terms of the form r_k^α, where $\alpha < 2$, then the final equations of motion are not regular. Moreover, the unbounded and bounded three-body problems are significantly different, namely, the unperturbed unbounded three-body problem contains the energy integral, but such an integral does not exists in the unperturbed bounded three-body problem. The variation in the energy of the system, which is considered as an additional variable in the standard KS-regularization, is described in the perturbed boundary three-body problem, as is shown below, by a differential equation which does not have the property of global regularity in the transition to new independent variables in this equation, which are traditionally used in this problem (this equation is not regular if the following two conditions are satisfied simultaneously: $r_0 \to 0$ and $r_1 \to 0$). Therefore, it is doubtful that the equations of bounded three-body problem, obtained by the method [10] based on the use of the system energy as an additional variable, have the property of global regularity.

We believe that a way out of this situation in the perturbed spatial bounded circular three-body problem is not to use the energy as an additional variable, as is usually done, but to use a variable which is the Jacobi integral in the unperturbed spatial bounded circular three-body problem (more precisely, this is the Jacobi constant of motion in this problem). In this paper, we use precisely this approach to construct the quaternion regular equations of perturbed spatial bounded circular three-body problem.

In this paper, we develop a quaternion method for regularizing the differential equations of perturbed spatial bounded three-body problem based on the use of Kustaanheimo–Stiefel variables and methodologically closely related to the quaternion method for regularizing the differential equations of perturbed spatial two-body problem proposed by the author of this paper in [16, 17] (also see [13–15]). In the first part of this paper, we consider the initial Newtonian equations of perturbed spatial bounded three-body problem and formulate the problem of regularization of these equations. We present the energy relations and differential equations describing the variations in the system energy in the perturbed spatial bounded three-body problem and the first integrals of differential equations of unperturbed spatial bounded circular three-body problem (the Jacobi integrals). We consider the equations of perturbed spatial bounded three-body problem written in rotating coordinates and using the rotation quaternions (Euler (Rodrigues–Hamilton) parameter) to describe the angular motion of these coordinates systems. We also present the differential equations for the angular momenta in the three-body problem under study.

In the same part of the paper, we obtain local regular quaternion differential equations of perturbed spatial bounded circular three-body problem, where, as an additional variable, we use the Kepler energy or the total energy of the system. We also obtain local regular quaternion differential equations of perturbed spatial bounded circular circular three-body problem, where, as an additional variable, we use a quantity which is the Jacobi constant of motion in the unperturbed spatial bounded circular three-body problem.

The obtained equations are systems of nonlinear nonstationary eleventh-order differential equations with respect to the Kustaanheimo–Stiefel variables, their first derivatives, Kepler or total energy or a variable which is the Jacobi constant of integration in the case of unperturbed spatial bounded circular three-body problem, and also with respect to time and an auxiliary time variable. In these equations, as new independent variables, we use the variables related to the time by the differential relations, namely, the derivative with respect to time of the new (first or second) variable is equal to the distance from the body of negligibly small mass to the first or second body of finite mass.

The constructed sets of differential equations of perturbed spatial bounded three-body problem allow us to develop regular analytical and numerical methods for studying the motions of the body of negligibly small mass near the other two bodies of finite mass and also permit developing regular algorithms for integrating these equations, where one of the constructed systems of eleventh-order differential equations is used to study the motion of the body M_2 of negligibly small mass near the body M_0 (when the distances r_0, r_1 between the bodies M_2 and M_0, M_2 and M_1 satisfy the inequality $m_1r_0^2 \leq m_0r_1^2$) and the other system of differential equations of the same order is used to study the motions of the body M_2 near the body M_1 (when the distances r_1 and r_0 satisfy the inequality $m_0r_1^2 < m_1r_0^2$) (in these inequalities, m_0 and m_1 are the masses of the bodies M_0 and M_1).

In the second part of the paper, we study the problem of constructing not only local but also global regular quaternion differential equations of the perturbed spatial bounded three-body problem, i.e., equations are regular if the conditions $r_0 = 1$, $r_1 = 0$ or the conditions $r_0 \to 0$, $r_1 \to 0$ are satisfied simultaneously. The construction of systems of differential equations used to solve this problem is based on the equations and relations presented in the first part of the paper.

2. INITIAL DIFFERENTIAL EQUATIONS OF PERTURBED SPATIAL BOUNDED THREE-BODY PROBLEM

We consider three material points M_0, M_1, and M_2 of masses m_0, m_1, and m_2, which mutually attract each other according to the law of universal gravitation. The unbounded three-body problem consists [52] of determining and studying all possible motions of material points M_0, M_1, and M_2. The bounded three-body problem is the problem [52] about the motion of material point $M_2 = M$ of zero mass $m_2 = 0$ (more precisely, of mass m_2 negligibly small as compared to the masses m_0 and m_1) attracted according to the Newton law by the other two material points M_0 and M_1 of nonzero masses m_0 and m_1.

The bounded three-body problem is [52] the limit version of the unbounded three-body problem. It is widely used in both the classical celestial mechanics (for example, the theory of motion of the Moon) and the mechanics of cosmic flight (the problem of attaining the Moon). The differential equations of the bounded three-body problem are obtained from the equations of the unbounded three-body problem (5.1.04) in [52] if we set $m_2 = 0$ in them. They have the form

$$\begin{gathered}
\frac{d^2\xi_0}{dt^2} = \frac{fm_1(\xi_1-\xi_0)}{\Delta_{01}^3}, \quad \frac{d^2\eta_0}{dt^2} = \frac{fm_1(\eta_1-\eta_0)}{\Delta_{01}^3}, \quad \frac{d^2\zeta_0}{dt^2} = \frac{fm_1(\zeta_1-\zeta_0)}{\Delta_{01}^3}, \\
\frac{d^2\xi_1}{dt^2} = \frac{fm_0(\xi_0-\xi_1)}{\Delta_{01}^3}, \quad \frac{d^2\eta_1}{dt^2} = \frac{fm_0(\eta_0-\eta_1)}{\Delta_{01}^3}, \quad \frac{d^2\zeta_1}{dt^2} = \frac{fm_0(\zeta_0-\zeta_1)}{\Delta_{01}^3}, \\
\frac{d^2\xi_2}{dt^2} = \frac{fm_0(\xi_0-\xi_2)}{\Delta_{02}^3} + \frac{fm_1(\xi_1-\xi_2)}{\Delta_{12}^3} + p_\xi, \\
\frac{d^2\eta_2}{dt^2} = \frac{fm_0(\eta_0-\eta_2)}{\Delta_{02}^3} + \frac{fm_1(\eta_1-\eta_2)}{\Delta_{12}^3} + p_\eta, \\
\frac{d^2\zeta_2}{dt^2} = \frac{fm_0(\zeta_0-\zeta_2)}{\Delta_{02}^3} + \frac{fm_1(\zeta_1-\zeta_2)}{\Delta_{12}^3} + p_\zeta,
\end{gathered} \tag{2.1}$$

$$\begin{gathered}
\Delta_{01}^2 = (\xi_0-\xi_1)^2 + (\eta_0-\eta_1)^2 + (\zeta_0-\zeta_1)^2, \\
\Delta_{02}^2 = (\xi_0-\xi_2)^2 + (\eta_0-\eta_2)^2 + (\zeta_0-\zeta_2)^2, \\
\Delta_{12}^2 = (\xi_1-\xi_2)^2 + (\eta_1-\eta_2)^2 + (\zeta_1-\zeta_2)^2.
\end{gathered} \tag{2.2}$$

Here ξ_0, η_0, ζ_0; ξ_1, η_1, ζ_1, and ξ_2, η_2, ζ_2 are the Cartesian coordinates of material points M_0, M_1, and $M_2 = M$ in the inertial coordinate system $O\xi\eta\zeta$; Δ_{01}, Δ_{02}, and Δ_{12} are the respective mutual distances between points M_0 and M_1, M_0 and M_2, M_1 and M_2, and f is the gravitation constant.

We note that, in contrast to equations (5.1.04) in [52], equations (2.1), (2.2) additionally contain projections p_ξ, p_η, p_ζ on the axes of the inertial coordinate system $O\xi\eta\zeta$ of the perturbing acceleration $\mathbf{p}$ of material point $M_2 = M$ due to the forces acting on point M_2 other than the forces of gravitational attraction on the side of points M_0 and M_1.

We introduce the vectors $\mathbf{r}_0 = (M_0, M)$, $\mathbf{r}_1 = (M_1, M)$, $\mathbf{r}_{01} = (M_0, M_1)$, and $\mathbf{r}_{10} = (M_1, M_0) = -\mathbf{r}_{01}$. The projections of the vectors $\mathbf{r}_0$ and $\mathbf{r}_1$ on the axes of the inertial coordinate system $O\xi\eta\zeta$ are respectively equal to $\xi_2 - \xi_0, \eta_2 - \eta_0, \zeta_2 - \zeta_0$ and $\xi_2 - \xi_1, \eta_2 - \eta_1, \zeta_2 - \zeta_1$.

Using the above-introduced vectors $\mathbf{r}_0$ and $\mathbf{r}_1$ as new vector variables, from differential equations (2.1) we derive the following vector form of differential equations of the perturbed spatial bounded three-body problem:

$$\frac{d^2\mathbf{r}_0}{dt^2} = -\frac{fm_0}{r_0^3}\mathbf{r}_0 - \frac{fm_1}{r_1^3}\mathbf{r}_1 - \frac{fm_1}{r_{01}^3}\mathbf{r}_{01} + \mathbf{p}, \tag{2.3}$$

$$\frac{d^2\mathbf{r}_1}{dt^2} = -\frac{fm_0}{r_0^3}\mathbf{r}_0 - \frac{fm_1}{r_1^3}\mathbf{r}_1 - \frac{fm_0}{r_{01}^3}\mathbf{r}_{10} + \mathbf{p}, \tag{2.4}$$

$$\mathbf{r}_{01} = \mathbf{r}_0 - \mathbf{r}_1, \quad \mathbf{r}_{10} = \mathbf{r}_1 - \mathbf{r}_0 = -\mathbf{r}_{01},$$

$$r_0 = |\mathbf{r}_0| = \Delta_{02}, \quad r_1 = |\mathbf{r}_1| = \Delta_{12}, \quad r_{01} = |\mathbf{r}_{01}| = |\mathbf{r}_{10}| = \Delta_{01}.$$

Differential equation (2.3) describes the motion of point $M_2 = M$ in the coordinate system $M_0X_0Y_0Z_0$ with origin at point M_0 and with coordinates axes M_0X_0, M_0Y_0, M_0Z_0 parallel to the corresponding axes of the inertial coordinate system $O\xi\eta\zeta$, and differential equation (2.4) describes the motion of the same point in the coordinate system $M_1X_1Y_1Z_1$ with origin at point M_1 and with coordinate axes M_1X_1, M_1Y_1, M_1Z_1 also parallel to the corresponding inertial axes $O\xi$, $O\eta$, $O\zeta$.

Differential equation (2.3) can be considered independently of differential equation (2.4) if we set $\mathbf{r}_1 = \mathbf{r}_0 - \mathbf{r}_{01}$ in it and taken into account that the vector $\mathbf{r}_{01}$ satisfies the differential equation

$$\frac{d^2\mathbf{r}_{01}}{dt^2} = -\frac{f(m_0 + m_1)}{r_{01}^3}\mathbf{r}_{01} \tag{2.5}$$

of the unperturbed spatial two-body problem (M_0 and M_1) which, as is known, can be integrated. Therefore, we assume that the vector $\mathbf{r}_{01}$ contained in equation (2.3) is a known function of time: $\mathbf{r}_{01} = \mathbf{r}_{01}(t)$. Similarly, differential equation (2.4) can be considered independently of differential equation (2.3) if we use the relation $\mathbf{r}_0 = \mathbf{r}_1 - \mathbf{r}_{10}$ in it and taken into account that the vector $\mathbf{r}_{10} = -\mathbf{r}_{01}$ is a known function of time.

Equations (2.3) and (2.4) can also be treated as a system of two differential equations with unknown vector variables $\mathbf{r}_0$ and $\mathbf{r}_1$.

We note that the coordinate representation of equation (2.3) coincides (for $\mathbf{p} = 0$) with the equations of the bounded three-body problem (6.1) in [53].

Vector equations (2.3) and (2.4) of the perturbed spatial bounded three-body problem have singular points $r_0 = 0$ and $r_1 = 0$ at which these equations are degenerate. The problem of removing these singularities (the separate removal of one of the singularities and the simultaneous removal of both singularities) is precisely the subject of regularization of differential equations of perturbed bounded three-body problem. We note that the conditions $r_0 = 0$ and $r_1 = 0$ cannot be simultaneously satisfied in the majority of problem in celestial mechanics and astrodynamics. Nevertheless, it is interesting from theoretical and practical standpoints (from the standpoint of construction of effective high-precision algorithms for numerical integration of differential equations of the three-body problem, which are necessary for the high-precision prediction of motion of celestial and cosmic bodies) to obtain regular equations which do not degenerate when both of these conditions are satisfied simultaneously.

3. ENERGY RELATIONS AND EQUATIONS IN THE SPATIAL BOUNDED THREE-BODY PROBLEM

The energy characteristics of motion of bodes and differential equations for these characteristics are used to construct regular equations of celestial mechanics and astrodynamics.

Let us consider the energy h_0 of motion of point $M_2 = M$ in the coordinate system $M_0X_0Y_0Z_0$ and the energy h_1 of motion in the coordinate system $M_1X_1Y_1Z_1$:

$$h_0 = \frac{1}{2}v_0^2 - \frac{fm_0}{r_0} - \frac{fm_1}{r_1}, \quad h_1 = \frac{1}{2}v_1^2 - \frac{fm_0}{r_0} - \frac{fm_1}{r_1},$$
$$v_0 = |\mathbf{v}_0|, \quad \mathbf{v}_0 = \frac{d\mathbf{r}_0}{dt}, \quad v_1 = |\mathbf{v}_1|, \quad \mathbf{v}_1 = \frac{d\mathbf{r}_1}{dt}, \tag{3.1}$$

where $\mathbf{v}_0$ and $\mathbf{v}_1$ are the respective velocity vectors of motion of point M in the coordinate systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$.

Differentiating relations (3.1) with respect to time and taking equations (2.3) and (2.4) into account, we obtain different forms of differential equations for the energies h_0 and h_1:

$$\frac{dh_0}{dt} = -fm_1\left(\frac{\mathbf{v}_0\cdot\mathbf{r}_{01}}{r_{01}^3} + \frac{\mathbf{v}_{01}\cdot\mathbf{r}_1}{r_1^3}\right) + \mathbf{v}_0\cdot\mathbf{p} = fm_1\frac{r_{01}\dot{r}_{01} - \mathbf{v}_{01}\cdot\mathbf{r}_0}{r_1^3} - fm_1\frac{\mathbf{v}_0\cdot\mathbf{r}_{01}}{r_{01}^3} + \mathbf{v}_0\cdot\mathbf{p}, \tag{3.2}$$

$$\frac{dh_1}{dt} = -fm_0\left(\frac{\mathbf{v}_1\cdot\mathbf{r}_{10}}{r_{01}^3} + \frac{\mathbf{v}_{10}\cdot\mathbf{r}_0}{r_0^3}\right) + \mathbf{v}_1\cdot\mathbf{p} = fm_0\left(\frac{\mathbf{v}_1\cdot\mathbf{r}_{01}}{r_{10}^3} + \frac{\mathbf{v}_{01}\cdot\mathbf{r}_0}{r_0^3}\right) + \mathbf{v}_1\cdot\mathbf{p} =$$
$$= fm_0\frac{r_{01}\dot{r}_{01} + \mathbf{v}_{01}\cdot\mathbf{r}_1}{r_0^3} + fm_0\frac{\mathbf{v}_1\cdot\mathbf{r}_{01}}{r_{01}^3} + \mathbf{v}_1\cdot\mathbf{p}, \tag{3.3}$$

$$\mathbf{v}_0 = \frac{d\mathbf{r}_0}{dt}, \quad \mathbf{v}_{01} = \frac{d\mathbf{r}_{01}}{dt}, \quad \mathbf{v}_1 = \frac{d\mathbf{r}_1}{dt}, \quad \mathbf{v}_{10} = \frac{d\mathbf{r}_{10}}{dt}.$$

Here the central dot is the symbol of scalar product of vectors.

We note that the energy equations (3.2) and (3.3) hold for the general perturbed bounded three-body problem and the singularities of these equation arise due to the nonzero velocity $\mathbf{v}_{01}$ of motion of body M_1 with respect to body M_0.

It is well known that, for the equations of unperturbed bounded circular three-body problem (when $\mathbf{p} = 0$), there exist a first integral called the Jacobi integral [52, 53]. To obtain the Jacobi integral by the method proposed in [53], we write equation (2.3) as

$$\frac{d^2\mathbf{r}_0}{dt^2} = -\frac{fm_0}{r_0^3}\mathbf{r}_0 + fm_1\left[\frac{1}{r_1^3}(\mathbf{r}_{01} - \mathbf{r}_0) - \frac{1}{r_{01}^3}\mathbf{r}_{01}\right] + \mathbf{p}. \tag{3.4}$$

By x_0, y_0, z_0 we denote the Cartesian coordinates of point M in the coordinate system $M_0X_0Y_0Z_0$ (the projections of the vector $\mathbf{r}_0$ on the axes of this coordinate system), and by x_{01}, y_{01}, z_{01}, the projections of the vector $\mathbf{r}_{01}$ on the axes of this coordinate system (the coordinates of point M_1 in the coordinate system $M_0X_0Y_0Z_0$). We project equation (3.4) on the axes of the coordinate system $M_0X_0Y_0Z_0$ to obtain the scalar equations of the perturbed bounded three-body problem in the form

$$\frac{d^{x_0}t}{d =^{x_0}} - \frac{fm_0}{r_0^3}x_0 + fm_1\left[\frac{1}{r_1^3}(x_{01} - x_0) - \frac{1}{r_{01}^3}x_{01}\right] + p_x,$$
$$\frac{d^{y_0}t}{d =^{y_0}} - \frac{fm_0}{r_0^3}y_0 + fm_1\left[\frac{1}{r_1^3}(y_{01} - y_0) - \frac{1}{r_{01}^3}y_{01}\right] + p_y, \tag{3.5}$$
$$\frac{d^{z_0}t}{d =^{z_0}} - \frac{fm_0}{r_0^3}z_0 + fm_1\left[\frac{1}{r_1^3}(z_{01} - z_0) - \frac{1}{r_{01}^3}z_{01}\right] + p_z,$$

$$r_0^2 = x_0^2 + y_0^2 + z_0^2, \quad r_{01}^2 = x_{01}^2 + y_{01}^2 + z_{01}^2,$$
$$r_1^2 = (x_{01} - x_0)^2 + (y_{01} - y_0)^2 + (z_{01} - z_0)^2 = x_1^2 + y_1^2 + z_1^2. \tag{3.6}$$

Here x_1, y_1, z_1 and $p_x = p_\xi$, $p_y = p_\eta$, $p_z = p_\zeta$ are the projections of the vectors $\mathbf{r}_1$ and $\mathbf{p}$ on the axes of the coordinate system $M_0X_0Y_0Z_0$ (they are equal to the corresponding projections of these vectors on the axe of the coordinate system $M_1X_1Y_1Z_1$).

Equations (3.5), (3.6) with $p_x = 0$, $p_y = 0$, and $p_z = 0$ coincide with equations (6.1), (6.2) in [53].

We consider a special case of the bounded three-body problem, i.e., the circular bounded three-body problem. We assume that the material point M_0 is the Earth about with the material point M_1, i.e., the Moon, moves in a circular orbit according to the Kepler laws. We also assume that the plane of the Moon circular orbit coincides with the coordinate plane $M_0X_0Y_0$. Then for the Moon coordinates x_{01}, y_{01}, z_{01}, we have the following expressions [53]:

$$x_{01} = a\cos(nt), \quad y_{01} = a\sin(nt), \quad z_{01} = 0. \tag{3.7}$$

In this case, we assume that the axis M_0X_0 passes through the point of the Moon location (this time moment is taken as the initial epoch of time reading).

The radius of the Moon circular orbit $a = |\mathbf{r}_{01}|$ and the angular velocity n of motion in its circular orbit are related by the well-known expression

$$n^2 = \frac{f(m_0 + m_1)}{a^3}, \quad a = |\mathbf{r}_{01}|. \tag{3.8}$$

We obtain the projections of the Moon velocity in the coordinate system $M_0X_0Y_0Z_0$ differentiating relations (3.7) with respect to time:

$$\dot{x}_{01} = -an\sin(nt) = -ny_{01}, \quad \dot{y}_{01} = an\cos(nt) = nx_{01}, \quad \dot{z}_{01} = 0. \tag{3.9}$$

Relations (3.7)–(3.9) can be obtained by integrating equation (2.5) with regard to the above assumptions about the motion of body M_1 (the Moon) and the choice of the coordinate system $M_0X_0Y_0Z_0$.

First, we respectively multiply equations (3.5) by $2dx_0/dt$, $2dy_0/dt$, $2dz_0/dt$ and sum the obtained relations. Then we respectively multiply equations (3.5) by $2ny_0$, $-2nx_0$, 0 and again sum the obtained equations. We obtain the two equations

$$\begin{aligned} &2\dot{x}_0\ddot{x}_0 + 2\dot{y}_0\ddot{y}_0 + 2\dot{z}_0\ddot{z}_0 = 2(\dot{x}_0X + \dot{y}_0Y + \dot{z}_0Z), \\ &2n(y_0\ddot{x}_0 - x_0\ddot{y}_0) = 2n(y_0X - x_0Y). \end{aligned} \tag{3.10}$$

Here X, Y, Z are the right-hand sides of equations (3.5), one dot and two dots over a symbol denote the first and second derivatives with respect to time t, respectively.

We sum the left- and right-hand sides of the obtained relations (3.10) with regard to the expressions for the right-hand sides X, Y, Z of equations (3.5). After several transformations, we obtain the following differential equation for the energy h_0 of motion of point M in the coordinate system $M_0X_0Y_0Z_0$ in the case of perturbed bounded circular three-body problem:

$$\frac{dh_0}{dt} = -\frac{fm_1}{r_{01}^3}\frac{d}{dt}(\mathbf{r}_0 \cdot \mathbf{r}_{01}) - n\frac{d}{dt}(y_0\dot{x}_0 - x_0\dot{y}_0) + \frac{d\mathbf{r}_0}{dt} \cdot \mathbf{p} + n(y_0p_x - x_0p_y) \tag{3.11}$$

with the scalar product

$$\mathbf{r}_0 \cdot \mathbf{r}_{01} = x_0x_{01} + y_0y_{01} = a[\cos(nt)x_0 + \sin(nt)y_0].$$

Similarly, using the scalar form of the vector equation (2.4)

$$\begin{aligned} \frac{d^2x_1}{dt^2} &= -\frac{fm_1}{r_1^3}x_1 + fm_0\left[\frac{1}{r_0^3}(x_{10} - x_1) - \frac{1}{r_{10}^3}x_{10}\right] + p_x, \\ \frac{d^2y_1}{dt^2} &= -\frac{fm_1}{r_1^3}y_1 + fm_0\left[\frac{1}{r_0^3}(y_{10} - y_1) - \frac{1}{r_{10}^3}y_{10}\right] + p_y, \\ \frac{d^2z_1}{dt^2} &= -\frac{fm_1}{r_1^3}z_1 + fm_0\left[\frac{1}{r_0^3}(z_{10} - z_1) - \frac{1}{r_{10}^3}z_{10}\right] + p_z, \\ r_1^2 &= x_1^2 + y_1^2 + z_1^2, \quad r_{10}^2 = x_{10}^2 + y_{10}^2 + z_{10}^2, \\ r_0^2 &= (x_{10} - x_1)^2 + (y_{10} - y_1)^2 + (z_{10} - z_1)^2 = x_0^2 + y_0^2 + z_0^2, \end{aligned}$$

we obtain the following differential equation for the energy h_1 of motion of point M in the coordinate system $M_1X_1Y_1Z_1$ in the case of perturbed bounded circular three-body problem:

$$\frac{dh_1}{dt} = -\frac{fm_0}{r_{01}^3}\frac{d}{dt}(\mathbf{r}_1 \cdot \mathbf{r}_{10}) - n\frac{d}{dt}(y_1\dot{x}_1 - x_1\dot{y}_1) + \frac{d\mathbf{r}_1}{dt} \cdot \mathbf{p} + n(y_1p_x - x_1p_y) \tag{3.12}$$

with the scalar product

$$\mathbf{r}_1 \cdot \mathbf{r}_{10} = x_1x_{10} + y_1y_{10} = -a[\cos(nt)x_1 + \sin(nt)y_1],$$

where x_1, y_1, z_1 are Cartesian coordinates of point M in the coordinate system $M_1X_1Y_1Z_1$ (projections of the vector $\mathbf{r}_1$ on the axes of this coordinate system equal to the projections of this vector on the axes of the coordinate system $M_0X_0Y_0Z_0$) and $x_{10} = -x_{01}$, $y_{10} = -y_{01}$, $z_{10} = 0$ are the projections of the vector $\mathbf{r}_{10} = -\mathbf{r}_{01}$ on the axes of the coordinate system $M_1X_1Y_1Z_1$ (coordinates of point M_0 in the coordinate system $M_1X_1Y_1Z_1$).

Equations (3.11) and (3.12) can be written as

$$\frac{dH_0}{dt} = \frac{d\mathbf{r}_0}{dt} \cdot \mathbf{p} + n(y_0p_x - x_0p_y), \tag{3.13}$$

$$\frac{dH_1}{dt} = \frac{d\mathbf{r}_1}{dt} \cdot \mathbf{p} + n(y_1p_x - x_1p_y), \tag{3.14}$$

$$\begin{aligned} H_0 &= h_0 + \frac{fm_1}{r_{01}^3}(\mathbf{r}_0 \cdot \mathbf{r}_{01}) + n(y_0\dot{x}_0 - x_0\dot{y}_0) \\ &= \frac{1}{2}(\dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2) - \frac{fm_0}{r_0} - \frac{fm_1}{r_1} + \frac{fm_1}{r_{01}^3}(x_0x_{01} + y_0y_{01}) + n(y_0\dot{x}_0 - x_0\dot{y}_0), \end{aligned} \tag{3.15}$$

$$\begin{aligned} H_1 &= h_1 + \frac{fm_0}{r_{01}^3}(\mathbf{r}_1 \cdot \mathbf{r}_{10}) + n(y_1\dot{x}_1 - x_1\dot{y}_1) \\ &= \frac{1}{2}(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) - \frac{fm_0}{r_0} - \frac{fm_1}{r_1} + \frac{fm_0}{r_{01}^3}(x_1x_{10} + y_1y_{10}) + n(y_1\dot{x}_1 - x_1\dot{y}_1). \end{aligned} \tag{3.16}$$

One can show that the quantities (functions of time) H_0 and H_1 differ by a constant:

$$H_1 = H_0 + \frac{1}{2}\frac{f(m_0 - m_1)}{a}, \quad a = |\mathbf{r}_{01}| = \text{const}.$$

We note that the quantities $c_{31} = x_0\dot{y}_0 - y_0\dot{x}_0$ and $c_{1z} = x_1\dot{y}_1 - y_1\dot{x}_1$ contained in (3.15) and (3.16) with opposite signs are moments of the velocity vectors $\mathbf{v}_0$ and $\mathbf{v}_1$ of point M with respect to the coordinate axes M_0Z_0 and M_1Z_1, respectively, and the quantities $x_0p_y - y_0p_x$ and $x_1p_y - y_1p_x$ contained in equations (3.13) and (3.14) also with opposite signs are moments of the perturbed acceleration vector $\mathbf{p}$ with respect to these axes.

The equations of perturbed bounded circular three-body problem are obtained from equations (2.3) and (2.4) with the projections of the vector $\mathbf{r}_{01}$ in the coordinate system $M_0X_0Y_0Z_0$ prescribed by relations (3.7) and the projections of the vector $\mathbf{r}_{10}$ in the coordinate system $M_1X_1Y_1Z_1$ prescribed by the relations $x_{10} = -a\cos(nt)$, $y_{10} = -a\sin(nt)$, and $z_{01} = 0$, respectively. It follows from equations (3.13) and (3.14) that the equations of the unperturbed bounded circular three-body problem have first integral with the perturbing acceleration is equal to $\mathbf{p} = 0$:

$$H_0 = \frac{1}{2}(\dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2) - \frac{fm_0}{r_0} - \frac{fm_1}{r_1} + \frac{fm_1}{r_{01}^3}(x_0x_{01} + y_0y_{01}) + n(y_0\dot{x}_0 - x_0\dot{y}_0) = H_0(t_0) = \text{const}, \tag{3.17}$$

$$H_1 = \frac{1}{2}(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) - \frac{fm_0}{r_0} - \frac{fm_1}{r_1} + \frac{fm_0}{r_{01}^3}(x_1x_{10} + y_1y_{10}) + n(y_1\dot{x}_1 - x_1\dot{y}_1) = H_1(t_0) = \text{const}. \tag{3.18}$$

We note that the integral (3.17) coincides with the integral (6.9) in [53] of the unperturbed bounded circular three-body problem (Jacobi integral).

4. DIFFERENTIAL EQUATIONS OF PERTURBED SPATIAL BOUNDED THREE-BODY PROBLEM WRITTEN IN ROTATING COORDINATE SYSTEMS. ROTATION QUATERNIONS INTRODUCED IN THE EQUATIONS OF MOTION

We introduce two rotating coordinate systems $M_0X_0'Y_0'Z_0'$ and $M_1X_1'Y_1'Z_1'$ whose axes M_0X_0' and M_1X_1' are directed along the radius vectors $\mathbf{r}_0$ and $\mathbf{r}_1$, respectively. By $\boldsymbol{\omega}_0$ and $\boldsymbol{\omega}_1$ we denote the vectors of absolute angular velocities of rotation of the coordinate systems $M_0X_0'Y_0'Z_0'$ and $M_1X_1'Y_1'Z_1'$, and by ω_{0i} and ω_{1i}, the projections of these vectors on the axes of the coordinate systems $M_0X_0'Y_0'Z_0'$ and $M_1X_1'Y_1'Z_1'$, respectively.

To describe the orientations (angular position) of the coordinate system $M_0X_0'Y_0'Z_0'$ in the coordinate system $M_0X_0Y_0Z_0$ (and hence in the inertial coordinate system $O\xi\eta\zeta$), we use the normalized rotation quaternion $\boldsymbol{\lambda}_0$, and to describe the orientation of the coordinate system $M_1X_1'Y_1'Z_1'$ in the coordinate system $M_1X_1Y_1Z_1$ (and hence in the inertial coordinate system $O\xi\eta\zeta$), we use the normalized rotation quaternion $\boldsymbol{\lambda}_1$:

$$\boldsymbol{\lambda}_i = \lambda_{i0} + \lambda_{i1}\mathbf{i} + \lambda_{i2}\mathbf{j} + \lambda_{i3}\mathbf{k}, \quad \|\boldsymbol{\lambda}_i\| = \lambda_{i0}^2 + \lambda_{i1}^2 + \lambda_{i2}^2 + \lambda_{i3}^2 = 1, \quad i = 0, 1,$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the Hamilton vector imaginary units, λ_{ij} $(i, j = 0, 1, 2, 3)$ are components of the orientation quaternion (the (Euler) Rodrigues–Hamilton parameters) characterizing the orientation of the coordinate system $M_iX_i'Y_i'Z_i'$ in the inertial coordinate system.

Let us write the vector differential equation (2.3) in the rotating coordinate system $M_0X_0'Y_0'Z_0'$, and the vector differential equation (2.4), in the rotating coordinate system $M_1X_1'Y_1'Z_1'$. Passing to the scalar problem, we obtain equations (4.1)–(4.3) and (4.4)–(4.6):

$$\begin{aligned}\ddot{r}_0 - r_0(\omega_{02}^2 + \omega_{03}^2) + \frac{fm_0}{r_0^2} &= -\frac{fm_1r_0}{r_{01}^3} + fm_1\left(\frac{1}{r_{01}^3} - \frac{1}{r_1^3}\right)x_1' + p_1' \\ &= -\frac{fm_1r_0}{r_1^3} + fm_1\left(\frac{1}{r_1^3} - \frac{1}{r_{01}^3}\right)x_{01}' + p_1',\end{aligned} \tag{4.1}$$

$$\begin{aligned}2\omega_{03}\dot{r}_0 + r_0\dot{\omega}_{03} + r_0\omega_{01}\omega_{02} &= fm_1\left(\frac{1}{r_{01}^3} - \frac{1}{r_1^3}\right)y_1' + p_2' = fm_1\left(\frac{1}{r_1^3} - \frac{1}{r_{01}^3}\right)y_{01}' + p_2', \\ 2\omega_{02}\dot{r}_0 + r_0\dot{\omega}_{02} - r_0\omega_{01}\omega_{03} &= -fm_1\left(\frac{1}{r_{01}^3} - \frac{1}{r_1^3}\right)z_1' - p_3' = -fm_1\left(\frac{1}{r_1^3} - \frac{1}{r_{01}^3}\right)z_{01}' - p_3',\end{aligned} \tag{4.2}$$

$$\begin{aligned}&2\dot{\lambda}_{00} = -\omega_{01}\lambda_{01} - \omega_{02}\lambda_{02} - \omega_{03}\lambda_{03}, \quad 2\dot{\lambda}_{01} = \omega_{01}\lambda_{00} + \omega_{03}\lambda_{02} - \omega_{02}\lambda_{03}, \\ &2\dot{\lambda}_{02} = \omega_{02}\lambda_{00} - \omega_{03}\lambda_{01} + \omega_{01}\lambda_{03}, \quad 2\dot{\lambda}_{03} = \omega_{03}\lambda_{00} + \omega_{02}\lambda_{01} - \omega_{01}\lambda_{02},\end{aligned} \tag{4.3}$$

$$\begin{aligned}\ddot{r}_1 - r_1(\omega_{12}^2 + \omega_{13}^2) + \frac{fm_1}{r_1^2} &= -\frac{fm_0r_1}{r_{01}^3} + fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)x_0'' + p_1'' \\ &= -\frac{fm_0r_1}{r_0^3} + fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)x_{01}'' + p_1'',\end{aligned} \tag{4.4}$$

$$\begin{aligned}2\omega_{13}\dot{r}_1 + r_1\dot{\omega}_{13} + r_1\omega_{11}\omega_{12} &= fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)y_0'' + p_2'' = fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)y_{01}'' + p_2'', \\ 2\omega_{12}\dot{r}_1 + r_1\dot{\omega}_{12} - r_1\omega_{11}\omega_{13} &= -fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)z_0'' - p_3'' = -fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)z_{01}'' - p_3'',\end{aligned} \tag{4.5}$$

$$\begin{aligned}&2\dot{\lambda}_{10} = -\omega_{11}\lambda_{11} - \omega_{12}\lambda_{12} - \omega_{13}\lambda_{13}, \quad 2\dot{\lambda}_{11} = \omega_{11}\lambda_{10} + \omega_{13}\lambda_{12} - \omega_{12}\lambda_{13}, \\ &2\dot{\lambda}_{12} = \omega_{12}\lambda_{10} - \omega_{13}\lambda_{11} + \omega_{11}\lambda_{13}, \quad 2\dot{\lambda}_{13} = \omega_{13}\lambda_{10} + \omega_{12}\lambda_{11} - \omega_{11}\lambda_{12},\end{aligned} \tag{4.6}$$

As variables in the equations of bounded three-body problem (4.1)–(4.3) and (4.4)–(4.6), we have the distances r_0 and r_1 from point M to points M_0 and M_1, their derivatives $\dot{r}_0$ and $\dot{r}_1$ (the respective projections of the velocity vectors $\mathbf{v}_0$ and $\mathbf{v}_1$ of point M in the coordinates systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$ on the directions of radius vectors $\mathbf{r}_0$ and $\mathbf{r}_1$), the projections ω_{02}, ω_{03} and ω_{12}, ω_{13} of the vectors of absolute angular velocities $\boldsymbol{\omega}_0$ and $\boldsymbol{\omega}_1$ of rotation of the coordinate systems $M_0X_0'Y_0'Z_0'$ and $M_1X_1'Y_1'Z_1'$ on the axes of the same coordinate systems $M_0X_0'Y_0'Z_0'$ and $M_1X_1'Y_1'Z_1'$, respectively, and the Rodrigues–Hamiltonian parameters λ_{0j} and λ_{1j} characterizing the orientation of the coordinate systems $M_0X_0'Y_0'Z_0'$ and $M_1X_1'Y_1'Z_1'$ in the inertial coordinate system $O\xi\eta\zeta$. In the equations, the respective projections ω_{01} and ω_{11} of the vectors of angular velocities $\boldsymbol{\omega}_0$ and $\boldsymbol{\omega}_1$ on the directions of radius vectors $\mathbf{r}_0$ and $\mathbf{r}_1$ are arbitrarily determined parameters. The quantities x_1', y_1', z_1'; x_{01}', y_{01}', z_{01}'; and p_1', p_2', p_3' in these equations are projections of the radius vectors $\mathbf{r}_1$, $\mathbf{r}_{01}$ and the vector of perturbing acceleration $\mathbf{p}$ on the axes of rotating coordinate system $M_0X_0'Y_0'Z_0'$, and the quantities x_0'', y_0'', z_0''; x_{01}'', y_{01}'', z_{01}''; and p_1'', p_2'', p_3'' are projections of the radius vectors $\mathbf{r}_0$, $\mathbf{r}_{01}$, and the acceleration vector $\mathbf{p}$ on the axes of the rotating coordinate system $M_1X_1'Y_1'Z_1'$.

The Cartesian coordinates x_0, y_0, z_0 and x_1, y_1, z_1 of point M in the coordinate systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$ are determined in terms of the above-listed variables by the formulas

$$\begin{aligned} &x_i = r_i(\lambda_{i0}^2 + \lambda_{i1}^2 - \lambda_{i2}^2 - \lambda_{i3}^2), \\ &y_i = 2r_i(\lambda_{i1}\lambda_{i2} + \lambda_{i0}\lambda_{i3}) = 2r_i(\lambda_{i1}\lambda_{i3} - \lambda_{i0}\lambda_{i2}), \quad i = 0, 1, \end{aligned} \tag{4.7}$$

and the projections v'_{0k} and v'_{1k} of the velocity vectors $\mathbf{v}_0$ and $\mathbf{v}_1$ of point M in the coordinate systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$ on the axes of the coordinate systems $M'_0X'_0Y'_0Z'_0$ and $M'_1X'_1Y'_1Z'_1$ are determined by the formulas

$$v'_{i1} = \dot{r}_i, \quad v'_{i2} = r_i\omega_{i3}, \quad v'_{i3} = -r_i\omega_{i2}, \quad i = 0, 1. \tag{4.8}$$

The projections $p_1 = p_x$, $p_2 = p_y$, $p_3 = p_z$ of the vector $\mathbf{p}$ on the axes of the coordinate system $M_0X_0Y_0Z_0$ are related to the projections p'_k on the axes of the coordinate system $M_0X'_0Y'_0Z'_0$ by the reprojection relations

$$\begin{aligned} p_1 &= (\lambda_{00}^2 + \lambda_{01}^2 - \lambda_{02}^2 - \lambda_{03}^2)p'_1 + 2(\lambda_{01}\lambda_{02} - \lambda_{00}\lambda_{03})p'_2 + 2(\lambda_{01}\lambda_{03} + \lambda_{00}\lambda_{02})p'_3, \\ p_2 &= 2(\lambda_{01}\lambda_{02} + \lambda_{00}\lambda_{03})p'_1 + (\lambda_{00}^2 - \lambda_{01}^2 + \lambda_{02}^2 - \lambda_{03}^2)p'_2 + 2(\lambda_{02}\lambda_{03} - \lambda_{00}\lambda_{01})p'_3, \\ p_3 &= 2(\lambda_{01}\lambda_{03} - \lambda_{00}\lambda_{02})p'_1 + 2(\lambda_{02}\lambda_{03} + \lambda_{00}\lambda_{01})p'_2 + (\lambda_{00}^2 - \lambda_{01}^2 - \lambda_{02}^2 + \lambda_{03}^2)p'_3, \\ p'_1 &= (\lambda_{00}^2 + \lambda_{01}^2 - \lambda_{02}^2 - \lambda_{03}^2)p_1 + 2(\lambda_{01}\lambda_{02} + \lambda_{00}\lambda_{03})p_2 + 2(\lambda_{01}\lambda_{03} - \lambda_{00}\lambda_{02})p_3, \\ p'_2 &= 2(\lambda_{01}\lambda_{02} - \lambda_{00}\lambda_{03})p_1 + (\lambda_{00}^2 - \lambda_{01}^2 + \lambda_{02}^2 - \lambda_{03}^2)p_2 + 2(\lambda_{02}\lambda_{03} + \lambda_{00}\lambda_{01})p_3, \\ p'_3 &= 2(\lambda_{01}\lambda_{03} + \lambda_{00}\lambda_{02})p_1 + 2(\lambda_{02}\lambda_{03} - \lambda_{00}\lambda_{01})p_2 + (\lambda_{00}^2 - \lambda_{01}^2 - \lambda_{02}^2 + \lambda_{03}^2)p_3. \end{aligned} \tag{4.9}$$

Similar relations exist between the projections p_k of the vector $\mathbf{p}$ on the axes of the coordinate system $M_0X_0Y_0Z_0$ and the its projections p''_k on the axes of the coordinate system $M_1X'_1Y'_1Z'_1$ (in (4.9), instead of p'_k, it is necessary to take p''_k, and instead of λ_{0j}, to take λ_{1j}).

Differential equations (4.3), (4.6) and relations (4.7), (4.9) in quaternion form become

$$\begin{aligned} &2\frac{d\boldsymbol{\lambda}_i}{dt} = \boldsymbol{\lambda}_i \circ \boldsymbol{\Omega}_i, \quad i = 0, 1, \\ &\boldsymbol{\lambda}_i = \lambda_{i0} + \lambda_{i1}\mathbf{i} + \lambda_{i2}\mathbf{j} + \lambda_{i3}\mathbf{k}, \quad \boldsymbol{\Omega} = \omega_{i1}\mathbf{i} + \omega_{i2}\mathbf{j} + \omega_{i3}\mathbf{k}, \end{aligned} \tag{4.10}$$

$$\mathbf{R}_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k} = r_i\boldsymbol{\lambda}_i \circ \mathbf{i} \circ \bar{\boldsymbol{\lambda}}_i, \quad i = 0, 1, \tag{4.11}$$

$$\begin{aligned} &\mathbf{P} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k} = \boldsymbol{\lambda}_0 \circ \mathbf{P}' \circ \bar{\boldsymbol{\lambda}}_0 = \boldsymbol{\lambda}_1 \circ \mathbf{P}'' \circ \bar{\boldsymbol{\lambda}}_1, \\ &\mathbf{P}' = p'_1\mathbf{i} + p'_2\mathbf{j} + p'_3\mathbf{k} = \bar{\boldsymbol{\lambda}}_0 \circ \mathbf{P} \circ \boldsymbol{\lambda}_0, \quad \mathbf{P}'' = p''_1\mathbf{i} + p''_2\mathbf{j} + p''_3\mathbf{k} = \bar{\boldsymbol{\lambda}}_1 \circ \mathbf{P} \circ \boldsymbol{\lambda}_1. \end{aligned} \tag{4.12}$$

Here and below, the symbol $\circ$ (central circle) denotes the quaternion multiplication, the upper bar denotes the conjugate quaternion, for example, $\bar{\boldsymbol{\lambda}}_0 = \lambda_{00} - \lambda_{01}\mathbf{i} - \lambda_{02}\mathbf{j} - \lambda_{03}\mathbf{k}$; the quaternion is differentiated under the assumption that the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ remain unchanged; and the quaternion $\mathbf{P}''$, which is used below, is additionally introduced.

The projections x'_1, y'_1, z'_1; x''_0, y''_0, z''_0; and x'_{01}, y'_{01}, z'_{01}; x''_{01}, y''_{01}, z''_{01} of the radius vectors $\mathbf{r}_1$, $\mathbf{r}_0$, and $\mathbf{r}_{01}$ contained in equations (4.1), (4.2) and (4.4), (4.5) are determined in terms of new variables by the quaternion relations

$$\mathbf{R}'_1 = x'_1\mathbf{i} + y'_1\mathbf{j} + z'_1\mathbf{k} = \bar{\boldsymbol{\lambda}}_0 \circ \mathbf{R}_1 \circ \boldsymbol{\lambda}_0 = r_1\boldsymbol{\mu} \circ \mathbf{i} \circ \bar{\boldsymbol{\mu}}, \quad \boldsymbol{\mu} = \bar{\boldsymbol{\lambda}}_0 \circ \boldsymbol{\lambda}_1, \tag{4.13}$$

$$\mathbf{R}''_0 = x''_0\mathbf{i} + y''_0\mathbf{j} + z''_0\mathbf{k} = \bar{\boldsymbol{\lambda}}_1 \circ \mathbf{R}_0 \circ \boldsymbol{\lambda}_1 = r_0\boldsymbol{\mu} \circ \mathbf{i} \circ \bar{\boldsymbol{\mu}}, \quad \boldsymbol{\mu} = \bar{\boldsymbol{\lambda}}_0 \circ \boldsymbol{\lambda}_1, \tag{4.14}$$

$$\begin{aligned} &\mathbf{R}'_{01} = x'_{01}\mathbf{i} + y'_{01}\mathbf{j} + z'_{01}\mathbf{k} = \bar{\boldsymbol{\lambda}}_0 \circ \mathbf{R}_{01} \circ \boldsymbol{\lambda}_0, \quad \mathbf{R}_{01} = x_{01}\mathbf{i} + y_{01}\mathbf{j} + z_{01}\mathbf{k}, \\ &\mathbf{R}''_{01} = x''_{01}\mathbf{i} + y''_{01}\mathbf{j} + z''_{01}\mathbf{k} = \bar{\boldsymbol{\lambda}}_1 \circ \mathbf{R}_{01} \circ \boldsymbol{\lambda}_1. \end{aligned} \tag{4.15}$$

It follows from relations (4.13) and (4.14) that

$$\bar{\boldsymbol{\mu}} \circ \mathbf{R}'_1 \circ \boldsymbol{\mu} = r_1\mathbf{i}, \quad \boldsymbol{\mu} \circ \mathbf{R}''_0 \circ \bar{\boldsymbol{\mu}} = r_0\mathbf{i}.$$

The projections v_{0k} and v_{1k} of the velocity vectors $\mathbf{v}_0$ and $\mathbf{v}_1$ of point M in the coordinate systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$ on the axes of the coordinate systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$, which coincide with their projections on the axes of the inertial coordinate system, are determined by the quaternion formulas

$$\mathbf{V}_i = v_{i1}\mathbf{i} + v_{i2}\mathbf{j} + v_{i3}\mathbf{k} = \boldsymbol{\lambda} \circ \mathbf{V}'_i \circ \bar{\boldsymbol{\lambda}}, \quad i = 0, 1, \tag{4.16}$$

$$\mathbf{V}'_i = v'_{i1}\mathbf{i} + v'_{i2}\mathbf{j} + v'_{i3}\mathbf{k} = \dot{r}_i\mathbf{i} + r_i\omega_{i3}\mathbf{j} - r_i\omega_{i2}\mathbf{k}, \quad i = 0, 1. \tag{4.17}$$

5. DIFFERENTIAL EQUATIONS OF PERTURBED SPATIAL BOUNDED THREE-BODY PROBLEM WRITTEN IN NONHOLONOMIC (AZIMUTHALLY FREE) ACCOMPANYING COORDINATE TRIHEDRONS

We introduce the vectors $\mathbf{c}_0$ and $\mathbf{c}_1$ of moments of the velocities $\mathbf{v}_0$ and $\mathbf{v}_1$ of point M in the coordinate systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$ with respect to points M_0 and M_1, respectively:

$$\mathbf{c}_i = \mathbf{r}_i \times \dot{\mathbf{r}}_i = \mathbf{r}_i \times \mathbf{v}_i, \quad i = 0, 1.$$

The projections c_{ik} $(k = 1, 2, 3)$ of the vector $\mathbf{c}_i$ on the axes of the rotating coordinate system $M_iX'_iY'_iZ'_i$ are determine by the relations

$$c_{i1} = 0, \quad c_{i2} = r_i^2\omega_{i2}, \quad c_{i3} = r_i^2\omega_{i3}, \quad i = 0, 1. \tag{5.1}$$

We complete the definition of motion of the coordinate system $M_iX'_iY'_iZ'_i$ by the assumption that the arbitrarily determined projection ω_{i1} of the vector of its absolute angular velocity $\boldsymbol{\omega}_i$ on the direction of the radius vector $\mathbf{r}_i$ (axis $M_iX'_i$) is equal to zero:

$$\omega_{i1} = 2(-\lambda_{i1}\dot{\lambda}_{i0} + \lambda_{i0}\dot{\lambda}_{i1} + \lambda_{i3}\dot{\lambda}_{i2} - \lambda_{i2}\dot{\lambda}_{i3}) = 0, \quad i = 0, 1. \tag{5.2}$$

In this case, as it follows from (5.1) and (5.2), the coordinate system $M_iX'_iY'_iZ'_i$ rotates with absolute angular velocity $\boldsymbol{\omega}_i$ collinear to the vector of the velocity moment $\mathbf{c}_i$:

$$\boldsymbol{\omega}_i = r_i^{-2}\mathbf{c}_i, \quad i = 0, 1. \tag{5.3}$$

Such a coordinate system is nonholonomic (azimuthally free) accompanying coordinate trihedron.

Differential equations of bounded three-body problem (4.1)–(4.6) with regard to (5.2) become

$$\begin{aligned}
&\ddot{r}_0 - r_0(\omega_{02}^2 + \omega_{03}^2) + \frac{fm_0}{r_0^2} = -\frac{fm_1r_0}{r_{01}^3} + fm_1\left(\frac{1}{r_{01}^3} - \frac{1}{r_1^3}\right)x'_1 + p'_1,\\
&\qquad = -\frac{fm_1r_0}{r_1^3} + fm_1\left(\frac{1}{r_1^3} - \frac{1}{r_{01}^3}\right)x'_{01} + p'_1,\\
&2\omega_{03}\dot{r}_0 + r_0\dot{\omega}_{03} = fm_1\left(\frac{1}{r_{01}^3} - \frac{1}{r_1^3}\right)y'_1 + p'_2 = fm_1\left(\frac{1}{r_1^3} - \frac{1}{r_{01}^3}\right)y'_{01} + p'_2,\\
&2\omega_{02}\dot{r}_0 + r_0\dot{\omega}_{02} = -fm_1\left(\frac{1}{r_{01}^3} - \frac{1}{r_1^3}\right)z'_1 - p'_3 = -fm_1\left(\frac{1}{r_1^3} - \frac{1}{r_{01}^3}\right)z'_{01} - p'_3,\\
&2\dot{\lambda}_{00} = -\omega_{02}\lambda_{02} - \omega_{03}\lambda_{03}, \quad 2\dot{\lambda}_{01} = \omega_{03}\lambda_{02} - \omega_{02}\lambda_{03},\\
&2\dot{\lambda}_{02} = \omega_{02}\lambda_{00} - \omega_{03}\lambda_{01}, \quad 2\dot{\lambda}_{03} = \omega_{03}\lambda_{00} + \omega_{02}\lambda_{01},\\
&\ddot{r}_1 - r_1(\omega_{12}^2 + \omega_{13}^2) + \frac{fm_1}{r_1^2} = -\frac{fm_0r_1}{r_{01}^3} + fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)x''_0 + p''_1\\
&\qquad = -\frac{fm_0r_1}{r_0^3} + fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)x''_{01} + p''_1,\\
&2\omega_{13}\dot{r}_1 + r_1\dot{\omega}_{13} = fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)y''_0 + p''_2 = fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)y''_{01} + p''_2,\\
&2\omega_{12}\dot{r}_1 + r_1\dot{\omega}_{12} = -fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)z''_0 - p''_3 = -fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)z''_{01} - p''_3,\\
&2\dot{\lambda}_{10} = -\omega_{12}\lambda_{12} - \omega_{13}\lambda_{13}, \quad 2\dot{\lambda}_{11} = \omega_{13}\lambda_{12} - \omega_{12}\lambda_{13},\\
&2\dot{\lambda}_{12} = \omega_{12}\lambda_{10} - \omega_{13}\lambda_{11}, \quad 2\dot{\lambda}_{13} = \omega_{13}\lambda_{10} + \omega_{12}\lambda_{11}.
\end{aligned}$$

These equations with regard to (5.1) take the form

$$
\begin{aligned}
&\ddot r_0 - \frac{1}{r_0^3}(c_{02}^2 + c_{03}^2) + \frac{fm_0}{r_0^2} = -\frac{fm_1 r_0}{r_{01}^3} + fm_1\left(\frac{1}{r_{01}^3} - \frac{1}{r_1^3}\right)x_1' + p_1',\\
&\qquad = -\frac{fm_1 r_0}{r_1^3} + fm_1\left(\frac{1}{r_1^3} - \frac{1}{r_{01}^3}\right)x_{01}' + p_1',
\end{aligned}
\tag{5.4}
$$

$$
\begin{aligned}
&c_{01} = 0, \quad \dot c_{02} = -fm_1\left(\frac{1}{r_{01}^3} - \frac{1}{r_1^3}\right)r_0 z_1' - r_0 p_3' = -fm_1\left(\frac{1}{r_1^3} - \frac{1}{r_{01}^3}\right)r_0 z_{01}' - r_0 p_3',\\
&\dot c_{03} = fm_1\left(\frac{1}{r_{01}^3} - \frac{1}{r_1^3}\right)r_0 y_1' + r_0 p_2' = fm_1\left(\frac{1}{r_1^3} - \frac{1}{r_{01}^3}\right)r_0 y_{01}' + r_0 p_2',
\end{aligned}
\tag{5.5}
$$

$$
\begin{aligned}
&2\dot\lambda_{00} = -\frac{1}{r_0^2}(c_{02}\lambda_{02} - c_{03}\lambda_{03}), \quad 2\dot\lambda_{01} = \frac{1}{r_0^2}(c_{03}\lambda_{02} - c_{02}\lambda_{03}),\\
&2\dot\lambda_{02} = \frac{1}{r_0^2}(c_{02}\lambda_{00} - c_{03}\lambda_{01}), \quad 2\dot\lambda_{03} = \frac{1}{r_0^2}(c_{03}\lambda_{00} + c_{02}\lambda_{01}),
\end{aligned}
\tag{5.6}
$$

$$
\begin{aligned}
&\ddot r_1 - \frac{1}{r_1^3}(c_{12}^2 + c_{13}^2) + \frac{fm_1}{r_1^2} = -\frac{fm_0 r_1}{r_{01}^3} + fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)x_0'' + p_1''\\
&\qquad = -\frac{fm_0 r_1}{r_0^3} + fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)x_{01}'' + p_1'',
\end{aligned}
\tag{5.7}
$$

$$
\begin{aligned}
&c_{11} = 0, \quad \dot c_{12} = -fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)r_1 z_0'' - r_1 p_3'' = -fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)r_1 z_{01}'' - r_1 p_3'',\\
&\dot c_{13} = fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)r_1 y_0'' + r_1 p_2'' = fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)r_1 y_{01}'' + r_1 p_2'',
\end{aligned}
\tag{5.8}
$$

$$
\begin{aligned}
&2\dot\lambda_{10} = -\frac{1}{r_1^2}(c_{12}\lambda_{12} - c_{13}\lambda_{13}), \quad 2\dot\lambda_{11} = \frac{1}{r_1^2}(c_{13}\lambda_{12} - c_{12}\lambda_{13}),\\
&2\dot\lambda_{12} = \frac{1}{r_1^2}(c_{12}\lambda_{10} - c_{13}\lambda_{11}), \quad 2\dot\lambda_{13} = \frac{1}{r_1^2}(c_{13}\lambda_{10} + c_{12}\lambda_{11}).
\end{aligned}
\tag{5.9}
$$

Let us write subsystems (5.6) and (5.9) in quaternion form

$$
\begin{aligned}
&2\dot{\boldsymbol\lambda}_i = r_i^{-2}\boldsymbol\lambda_i \circ \mathbf{C}_i, \quad i = 0, 1,\\
&\boldsymbol\lambda_i = \lambda_{i0} + \lambda_{i1}\mathbf{i} + \lambda_{i2}\mathbf{j} + \lambda_{i3}\mathbf{k}, \quad \mathbf{C}_i = c_{i2}\mathbf{j} + c_{i3}\mathbf{k}.
\end{aligned}
\tag{5.10}
$$

These equations also follow from quaternion equations (4.10) with regard to (5.1) and (5.2).
We also write equations (5.5) and (5.8) in quaternion form:

$$
\dot{\mathbf{C}}_0 = fm_1\left(\frac{1}{r_{01}^3} - \frac{1}{r_1^3}\right)r_0(-z_1'\mathbf{j} + y_1'\mathbf{k}) + r_0(-p_3'\mathbf{j} + p_2'\mathbf{k}),
\tag{5.11}
$$

$$
\dot{\mathbf{C}}_1 = fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)r_0(-z_0'\mathbf{j} + y_0'\mathbf{k}) + r_1(-p_3'\mathbf{j} + p_2'\mathbf{k}),
\tag{5.12}
$$

or in another quaternion form,

$$
\dot{\mathbf{C}}_0 = fm_1\left(\frac{1}{r_1^3} - \frac{1}{r_{01}^3}\right)r_0(-z_{01}''\mathbf{j} + y_{01}''\mathbf{k}) + r_0(-p_3''\mathbf{j} + p_2''\mathbf{k}),
\tag{5.13}
$$

$$
\dot{\mathbf{C}}_1 = fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)r_0(-z_{01}''\mathbf{j} + y_{01}''\mathbf{k}) + r_1(-p_3''\mathbf{j} + p_2''\mathbf{k}).
\tag{5.14}
$$

As variables in the equations of bounded three-body problem (5.4)–(5.6) and (5.7)–(5.9) written in nonholonomic coordinate trihedrons, we have the distances r_0 and r_1 from point M to points M_0 and M_1, their derivatives $\dot r_0$ and $\dot r_1$ (the projections of velocity vectors $\mathbf{v}_0$ and $\mathbf{v}_1$ of point M in the coordinate systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$, respectively, on the directions of radius vectors $\mathbf{r}_0$ and $\mathbf{r}_1$), the projections c_{02}, c_{03}, and c_{12}, c_{13} of the vectors $\mathbf{c}_0$ and $\mathbf{c}_1$ of moments of the velocities $\mathbf{v}_0$ and $\mathbf{v}_1$ of point M in the coordinate systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$ with respect to points M_0 and M_1 on the axes of the rotating coordinate systems $M_0X_0'Y_0'Z_0'$ and $M_1X_1'Y_1'Z_1'$, respectively, and the Rodrigues–Hamilton parameters λ_{0j} and λ_{1j} characterizing the orientation of the coordinate systems $M_0X_0'Y_0'Z_0'$ and $M_1X_1'Y_1'Z_1'$ in the inertial coordinate system $O\xi\eta\zeta$.

As variables in quaternion equations (5.10)–(5.12) and (5.10), (5.13), (5.14) of the bounded three-body problem, we have the quaternions $\mathbf{C}_0$ and $\mathbf{C}_1$ of velocities moments and the quaternions $\boldsymbol{\lambda}_0$ and $\boldsymbol{\lambda}_1$ characterizing the orientation of rotating coordinate systems $M_0X_0'Y_0'Z_0'$ and $M_1X_1'Y_1'Z_1'$ in the inertial coordinate system $O\xi\eta\zeta$. It is necessary to supplement these quaternion equations with scalar equations (5.4) and (5.7) for the distance r_0 and r_1.

The Cartesian coordinates x_0, y_0, z_0 and x_1, y_1, z_1 of point M in the coordinate systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$ are determined in terms of the above-listed variables by formulas (4.7) and the projections v_{0k}' and v_{1k}' of the velocity vectors $\mathbf{v}_0$ and $\mathbf{v}_1$ of point M in the coordinates systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$ on the axes of the coordinate systems $M_0X_0'Y_0'Z_0'$ and $M_1X_1'Y_1'Z_1'$, respectively, are determined according to (4.8) and (5.1) by the formulas

$$v_{i1}' = \dot{r}_i, \quad v_{i2}' = \frac{1}{r_i}c_{i3}, \quad v_{i3}' = -\frac{1}{r_i}c_{i2}, \quad i = 0, 1,$$

which, in quaternion representation, have the form of relations (4.11) and the relations

$$\mathbf{V}_i' = v_{i1}'\mathbf{i} + v_{i2}'\mathbf{j} + v_{i3}'\mathbf{k} = \dot{r}_i\mathbf{i} + \frac{1}{r_i}c_{i3}\mathbf{j} - \frac{1}{r_i}c_{i2}\mathbf{k}, \quad i = 0, 1. \tag{5.15}$$

The projections v_{0k} and v_{1k} of the velocity vectors $\mathbf{v}_0$ and $\mathbf{v}_1$ of point M in the coordinate systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$ on the axes of the coordinate systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$, respectively, which coincide with their projections on the axes of the coordinate system, are determined by quaternion formulas (4.16) and (5.15).

6. DIFFERENTIAL EQUATIONS FOR THE ANGULAR MOMENTA IN THE BOUNDED THREE-BODY PROBLEM

We use the vector relations $\mathbf{r}_{01} = \mathbf{r}_0 - \mathbf{r}_1$, $\mathbf{r}_{10} = \mathbf{r}_1 - \mathbf{r}_0$ and $\mathbf{v}_0 d\mathbf{r}_0/dt$, $\mathbf{v}_1 = d\mathbf{r}_1/dt$ to write the differential equations of the perturbed bounded three-body problem (2.3), (2.4) in the form

$$\frac{d\mathbf{v}_0}{dt} = -\left(\frac{fm_0}{r_0^3} + \frac{fm_1}{r_{01}^3}\right)\mathbf{r}_0 + fm_1\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)\mathbf{r}_1 + \mathbf{p}, \tag{6.1}$$

$$\frac{d\mathbf{v}_1}{dt} = fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)\mathbf{r}_0 - \left(\frac{fm_1}{r_1^3} + \frac{fm_0}{r_{01}^3}\right)\mathbf{r}_1 + \mathbf{p}. \tag{6.2}$$

Taking the vector product from of equation (6.1) from the left by $\mathbf{r}_0$ and the vector product of equation (6.2) from the left by $\mathbf{r}_1$, we obtain differential equations for the vectors $\mathbf{c}_0 = \mathbf{r}_0 \times \mathbf{v}_0$ and $\mathbf{c}_1 = \mathbf{r}_1 \times \mathbf{v}_1$ of moments of the velocities $\mathbf{v}_0$ and $\mathbf{v}_1$ of point M in the coordinate systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$ with respect to points M_0 and M_1, which, respectively, have the form

$$\frac{d\mathbf{c}_0}{dt} = fm_1\left(\frac{1}{r_{01}} - \frac{1}{r_1}\right)\mathbf{r}_0 \times \mathbf{r}_1 + \mathbf{r}_0 \times \mathbf{p}, \tag{6.3}$$

$$\frac{d\mathbf{c}_1}{dt} = fm_0\left(\frac{1}{r_{01}} - \frac{1}{r_1}\right)\mathbf{r}_1 \times \mathbf{r}_0 + \mathbf{r}_1 \times \mathbf{p}. \tag{6.4}$$

From these equations, we derive the differential equations

$$\frac{d(m_0\mathbf{c}_0 + m_1\mathbf{c}_1)}{dt} = fm_0m_1\left(\frac{1}{r_0^3} - \frac{1}{r_1^3}\right)\mathbf{r}_0 \times \mathbf{r}_1 + (m_0\mathbf{r}_0 + m_1\mathbf{r}_1) \times \mathbf{p},$$

whose right-hand side does not contain a term with multiplier r_{01}^{-3}.

We use the vector relations $\mathbf{r}_{01} = \mathbf{r}_0 - \mathbf{r}_1 = -\mathbf{r}_{10}$ and $\mathbf{v}_0 = d\mathbf{r}_0/dt$, $\mathbf{v}_1 = d\mathbf{r}_1/dt$ to write equations (2.3) and (2.4) in different form

$$\frac{d\mathbf{v}_0}{dt} = -\left(\frac{fm_0}{r_0^3} + \frac{fm_1}{r_1^3}\right)\mathbf{r}_0 + fm_1\left(\frac{1}{r_1^3} - \frac{1}{r_{01}^3}\right)\mathbf{r}_{01} + \mathbf{p}, \tag{6.5}$$

$$\frac{d\mathbf{v}_1}{dt} = -\left(\frac{fm_0}{r_1^3} + \frac{fm_1}{r_0^3}\right)\mathbf{r}_1 + fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)\mathbf{r}_{01} + \mathbf{p}. \tag{6.6}$$

Taking the vector product from of equation (6.5) from the left by $\mathbf{r}_0$ and the vector product of equation (6.6) from the left by $\mathbf{r}_1$, we obtain different differential equations for the vectors $\mathbf{c}_0$ and $\mathbf{c}_1$ of moments of the velocities $\mathbf{v}_0$ and $\mathbf{v}_1$ of point M which have the form

$$\frac{d\mathbf{c}_0}{dt} = fm_1\left(\frac{1}{r_1^3} - \frac{1}{r_{01}^3}\right)\mathbf{r}_0 \times \mathbf{r}_{01} + \mathbf{r}_0 \times \mathbf{p}, \tag{6.7}$$

$$\frac{d\mathbf{c}_1}{dt} = fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)\mathbf{r}_1 \times \mathbf{r}_{01} + \mathbf{r}_1 \times \mathbf{p}. \tag{6.8}$$

We write the vector differential equations (6.3) and (6.7) in the rotating coordinates system $M_0X_0'Y_0'Z_0'$ and the vector differential equations (6.4) and (6.8) in the rotating coordinate system $M_1X_1'Y_1'Z_1'$ passing in these equations from absolute to local (relative) derivatives. According to (5.3), the coordinate system $M_iX_i'Y_i'Z_i'$ rotates with the absolute angular velocity $\boldsymbol{\omega}_i$ collinear to the vector of velocity moment $\mathbf{c}_i$: $\boldsymbol{\omega}_i = r_i^{-2}\mathbf{c}_i$, $i = 0, 1$. Therefore, the absolute and local derivatives of the vector $\mathbf{c}_i$ (the derivatives in the inertial and rotating coordinate systems $M_iX_i'Y_i'Z_i'$) coincide and the above-listed differential equations in the rotating coordinate systems $M_0X_0'Y_0'Z_0'$ and $M_1X_1'Y_1'Z_1'$ become

$$\dot{c}_{01}\mathbf{x}_0' + \dot{c}_{02}\mathbf{y}_0' + \dot{c}_{01}\mathbf{z}_0' = fm_1\left(\frac{1}{r_{01}^3} - \frac{1}{r_1^3}\right)r_0\mathbf{x}_0' \times (x_1'\mathbf{x}_0' + y_1'\mathbf{y}_0' + z_1'\mathbf{z}_0') + r_0\mathbf{x}_0' \times (p_1'\mathbf{x}_0' + p_2'\mathbf{y}_0' + p_3'\mathbf{z}_0')$$

$$= fm_1\left(\frac{1}{r_1^3} - \frac{1}{r_{01}^3}\right)r_0\mathbf{x}_0' \times (x_{01}'\mathbf{x}_0' + y_{01}'\mathbf{y}_0' + z_{01}'\mathbf{z}_0') + r_0\mathbf{x}_0' \times (p_1'\mathbf{x}_0' + p_2'\mathbf{y}_0' + p_3'\mathbf{z}_0'),$$

$$\dot{c}_{11}\mathbf{x}_0' + \dot{c}_{12}\mathbf{y}_0' + \dot{c}_{11}\mathbf{z}_0' = fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)r_1\mathbf{x}_1' \times (x_0''\mathbf{x}_1' + y_0''\mathbf{y}_1' + z_0''\mathbf{z}_1') + r_1\mathbf{x}_1' \times (p_1''\mathbf{x}_1' + p_2''\mathbf{y}_1' + p_3''\mathbf{z}_1')$$

$$= fm_0\left(\frac{1}{r_{01}^3} - \frac{1}{r_0^3}\right)r_1\mathbf{x}_1' \times (x_{01}''\mathbf{x}_1' + y_{01}''\mathbf{y}_1' + z_{01}''\mathbf{z}_1') + r_1\mathbf{x}_1' \times (p_1''\mathbf{x}_1' + p_2''\mathbf{y}_1' + p_3''\mathbf{z}_1'),$$

where $\mathbf{x}_0'$, $\mathbf{y}_0'$, $\mathbf{z}_0'$ and $\mathbf{x}_1'$, $\mathbf{y}_1'$, $\mathbf{z}_1'$ are unit vectors of the coordinates axes M_0X_0', M_0Y_0', M_0Z_0' and M_1X_1', M_1Y_1', M_1Z_1', respectively.

Passing to the scalar form, from these equations we obtain differential equations (5.5) and (5.8) for the components c_{ik} of the vector $\mathbf{c}_i$ $(i = 0, 1)$ of velocity moment in the rotating coordinate system $M_iX_i'Y_i'Z_i'$ which, in quaternion representation, have the form of equations (5.11), (5.12) or (5.13), (5.14).

7. QUATERNION REGULARIZATION OF DIFFERENTIAL EQUATIONS OF PERTURBED BOUNDED THREE-BODY PROBLEM

In quaternion equations (5.10), we pass from the Rodrigues–Hamiltonian parameters λ_{ij} $(i = 0, 1,$ $j = 0, 1, 2, 3)$ to the Kustaanheimo–Stiefel variables u_{ij} [7] by the formulas [16, 17]

$$\lambda_{i0} = r_i^{-1/2}u_{i0}, \quad \lambda_{ik} = -r_i^{-1/2}u_{ik}, \quad i = 0, 1, \quad k = 1, 2, 3. \tag{7.1}$$

In quaternion representation, formulas (7.1) become

$$\bar{\boldsymbol{\lambda}}_i = r_i^{-1/2}\mathbf{u}_i, \quad i = 0, 1; \qquad \bar{\boldsymbol{\lambda}}_i = \lambda_{i0} - \lambda_{i1}\mathbf{i} - \lambda_{i2}\mathbf{j} - \lambda_{i3}\mathbf{k}, \quad \mathbf{u}_i = u_{i0} - u_{i1}\mathbf{i} - u_{i2}\mathbf{j} - u_{i3}\mathbf{k}. \tag{7.2}$$

Substituting relations (7.2) into equations (5.10), we obtain

$$2\dot{\mathbf{u}}_i = r_i^{-1}(\dot{r}_i - r_i^{-1}\mathbf{C}_i) \circ \mathbf{u}_i, \quad i = 0, 1. \tag{7.3}$$

Differentiating the left- and right-hand sides of equations (7.3) with respect to time t, after transformations based on the initial equations (7.3), we obtain

$$
\begin{aligned}
&2\ddot{\mathbf{u}}_i + 2r_i^{-1}\dot{r}_i\dot{\mathbf{u}}_i - r_i^{-1}\Big(\ddot{r}_i + \tfrac{1}{2}r_i^{-1}\dot{r}_i^2 - \tfrac{1}{2}r_i^{-3}c_i^2\Big)\mathbf{u}_i = -r_i^2\dot{\mathbf{C}}_i \circ \mathbf{u}_i,\\
&c_i^2 = c_{i2}^2 + c_{i3}^2, \quad i = 0, 1.
\end{aligned} \tag{7.4}
$$

In the first two terms in the left-hand side of equations (7.4), we pas from the independent variable t to a new variable τ_i by the formulas

$$
dt = t_i\,d\tau_i, \quad \frac{d^2}{dt^2} = r_i^{-2}\frac{d^2}{d\tau_i^2} - r_i^{-3}\frac{dr_i}{d\tau_i}\frac{d}{d\tau_i}. \tag{7.5}
$$

We obtain

$$
2\frac{d^2\mathbf{u}_i}{d\tau_i^2} - r_i\Big(\ddot{r}_i + \tfrac{1}{2}r_i^{-1}\dot{r}_i^2 - \tfrac{1}{2}r_i^{-3}c_i^2\Big)\mathbf{u}_i = -\dot{\mathbf{C}}_i \circ \mathbf{u}_i, \quad i = 0, 1. \tag{7.6}
$$

One can see that, the terms containing the first derivatives of the quaternion variable $\mathbf{u}_i$ with respect to the independent variable τ_i are eliminated in this transition.

We substitute the expressions for the derivatives $\ddot{r}_i$ and $\dot{\mathbf{C}}_i$, which follow from equations (5.4), (5.7) and (5.13), (5.14), into equations (7.6) and obtain the equations

$$
\begin{aligned}
2\frac{d^2\mathbf{u}_0}{d\tau_0^2} &- \Big(\tfrac{1}{2}\dot{r}_0^2 + \tfrac{1}{2}r_0^{-2}c_0^2 - fm_0r_0^{-1}\Big)\mathbf{u}_0\\
&= r_0[-fm_1r_0r_1^{-3} + fm_1(r_1^{-3} - r_{01}^{-3})(x'_{01} + z'_{01}\mathbf{j} - y'_{01}\mathbf{k}) + (p'_1 + p'_3\mathbf{j} - p'_2\mathbf{k})] \circ \mathbf{u}_0,
\end{aligned} \tag{7.7}
$$

$$
\begin{aligned}
2\frac{d^2\mathbf{u}_1}{d\tau_1^2} &- \Big(\tfrac{1}{2}\dot{r}_1^2 + \tfrac{1}{2}r_1^{-2}c_1^2 - fm_1r_1^{-1}\Big)\mathbf{u}_1\\
&= r_1[-fm_0r_1r_0^{-3} + fm_0(r_{01}^{-3} - r_0^{-3})(x''_{01} + z''_{01}\mathbf{j} - y''_{01}\mathbf{k}) + (p''_1 + p''_3\mathbf{j} - p''_2\mathbf{k})] \circ \mathbf{u}_1.
\end{aligned} \tag{7.8}
$$

We introduce the notation (Kepler energies)

$$
h_i^* = \tfrac{1}{2}\dot{r}_i^2 + \tfrac{1}{2}r_i^{-2}c_i^{-2} - fm_ir_i^{-1} = \tfrac{1}{2}v_i^2 - fm_ir_i^{-1}, \quad i = 0, 1. \tag{7.9}
$$

Moreover, we take into account the fact that, according to (4.15) and (4.12),

$$
\begin{aligned}
&x'_{01} + z'_{01}\mathbf{j} - y'_{01}\mathbf{k} = -\mathbf{i} \circ (x'_{01}\mathbf{i} + y'_{01}\mathbf{j} + z'_{01}\mathbf{k}) = -\mathbf{i} \circ \mathbf{R}'_{01}, \quad \mathbf{R}'_{01} \circ \mathbf{u}_0 = \mathbf{u}_0 \circ \mathbf{R}_{01},\\
&x''_{01} + z''_{01}\mathbf{j} - y''_{01}\mathbf{k} = -\mathbf{i} \circ (x''_{01}\mathbf{i} + y''_{01}\mathbf{j} + z''_{01}\mathbf{k}) = -\mathbf{i} \circ \mathbf{R}''_{01}, \quad \mathbf{R}''_{01} \circ \mathbf{u}_1 = \mathbf{u}_1 \circ \mathbf{R}_{01},\\
&p'_1 + p'_3\mathbf{j} - p'_2\mathbf{k} = -\mathbf{i} \circ (p'_1\mathbf{i} + p'_2\mathbf{j} + p'_3\mathbf{k}) = -\mathbf{i} \circ \mathbf{P}', \quad \mathbf{P}' \circ \mathbf{u}_0 = \mathbf{u}_0 \circ \mathbf{P},\\
&p''_1 + p''_3\mathbf{j} - p''_2\mathbf{k} = -\mathbf{i} \circ (p''_1\mathbf{i} + p''_2\mathbf{j} + p''_3\mathbf{k}) = -\mathbf{i} \circ \mathbf{P}'', \quad \mathbf{P}'' \circ \mathbf{u}_1 = \mathbf{u}_1 \circ \mathbf{P}.
\end{aligned}
$$

The equations (7.7) and (7.8) become

$$
\frac{d^2\mathbf{u}_0}{d\tau_0^2} - \frac{1}{2}h_0^*\mathbf{u}_0 = -\frac{1}{2}r_0\{fm_1r_0r_1^{-3}\mathbf{u}_0 + \mathbf{i} \circ \mathbf{u}_0 \circ [fm_1(r_1^{-3} - r_{01}^{-3})\mathbf{R}_{01} + \mathbf{P}]\}, \tag{7.10}
$$

$$
\frac{d^2\mathbf{u}_1}{d\tau_1^2} - \frac{1}{2}h_1^*\mathbf{u}_1 = -\frac{1}{2}r_1\{fm_0r_1r_0^{-3}\mathbf{u}_1 + \mathbf{i} \circ \mathbf{u}_1 \circ [fm_0(r_{01}^{-3} - r_0^{-3})\mathbf{R}_{01} + \mathbf{P}]\}, \tag{7.11}
$$

where the quaternions $\mathbf{R}_{01}$ and $\mathbf{P}$ are respectively defined by the second relation in (4.15) and the first relation in (4.12):

$$
\mathbf{R}_{01} = x_{01}\mathbf{i} + y_{01}\mathbf{j} + z_{01}\mathbf{k}, \quad \mathbf{P} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k} = p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k}. \tag{7.12}
$$

The quantities h_0^* and h_1^* (Kepler energies) defined by (7.9) and contained in equations (7.10) and (7.11) are related to total energies h_0 and h_1 of the motion of point M in the coordinate systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$ according to (3.1) by the relations

$$
h_0^* = h_0 + \frac{fm_1}{r_1}, \quad h_1^* = h_1 + \frac{fm_0}{r_0}. \tag{7.13}
$$

We consider the quantities h_0^* and h_1^* as additional variables. We use relations (7.13) and differentiable equations (3.2) and (3.3) to show that these variables satisfy the differential equations

$$\frac{dh_0^*}{dt} = -fm_1r_1^{-3}(\mathbf{v}_0\cdot\mathbf{r}_0) + fm_1(r_1^{-3} - r_{01}^{-3})(\mathbf{v}_0\cdot\mathbf{r}_{01}) + \mathbf{v}_0\cdot\mathbf{p}, \tag{7.14}$$

$$\frac{dh_1^*}{dt} = -fm_0r_0^{-3}(\mathbf{v}_1\cdot\mathbf{r}_1) + fm_0(r_{01}^{-3} - r_0^{-3})(\mathbf{v}_1\cdot\mathbf{r}_{01}) + \mathbf{v}_1\cdot\mathbf{p}, \tag{7.15}$$

where $\mathbf{v}_0 = d\mathbf{r}_0/dt$ and $\mathbf{v}_1 = d\mathbf{r}_1/dt$ are velocity vectors of motion of point M in the coordinate systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$, respectively.

We note that equations (7.14) and (7.15) contain the scalar products $\mathbf{v}_i\cdot\mathbf{r}_i = r_i\dot{r}_i$ $(i = 0, 1)$.

Passing in equations (7.14) and (7.15) to new independent variables τ_0 and τ_1 according to (7.5), we obtain the equations

$$\frac{dh_0^*}{d\tau_0} = -fm_1r_1^{-3}r_0\frac{d\mathbf{r}_0}{d\tau_0} + fm_1(r_1^{-3} - r_{01}^{-3})\left(\frac{d\mathbf{r}_0}{d\tau_0}\cdot\mathbf{r}_{01}\right) + \frac{d\mathbf{r}_0}{d\tau_0}\cdot\mathbf{p}, \tag{7.16}$$

$$\frac{dh_1^*}{d\tau_1} = -fm_0r_0^{-3}r_1\frac{d\mathbf{r}_1}{d\tau_1} + fm_0(r_{01}^{-3} - r_0^{-3})\left(\frac{d\mathbf{r}_1}{d\tau_1}\cdot\mathbf{r}_{01}\right) + \frac{d\mathbf{r}_1}{d\tau_1}\cdot\mathbf{p}. \tag{7.17}$$

In equations (7.10), (7.11) and (7.16), (7.17), we have

$$\begin{aligned}
&r_i = u_{i0}^2 + u_{i1}^2 + u_{i2}^2 + u_{i3}^2, \quad r_{01}^2 = x_{01}^2 + y_{01}^2 + z_{01}^2,\\
&\frac{dr_i}{d\tau_i} = 2\left(u_{i0}\frac{du_{i0}}{d\tau_i} + u_{i1}\frac{du_{i1}}{d\tau_i} + u_{i2}\frac{du_{i2}}{d\tau_i} + u_{i3}\frac{du_{i3}}{d\tau_i}\right),\\
&\frac{d\mathbf{r}_i}{d\tau_i}\cdot\mathbf{r}_{01} = 2x_{01}\left(u_{i0}\frac{du_{i0}}{d\tau_i} + u_{i1}\frac{du_{i1}}{d\tau_i} - u_{i2}\frac{du_{i2}}{d\tau_i} - u_{i3}\frac{du_{i3}}{d\tau_i}\right)\\
&\qquad + 2y_{01}\left(u_{i2}\frac{du_{i1}}{d\tau_i} + u_{i1}\frac{du_{i2}}{d\tau_i} - u_{i3}\frac{du_{i0}}{d\tau_i} - u_{i0}\frac{du_{i3}}{d\tau_i}\right)\\
&\qquad + 2z_{01}\left(u_{i3}\frac{du_{i1}}{d\tau_i} + u_{i1}\frac{du_{i3}}{d\tau_i} + u_{i2}\frac{du_{i0}}{d\tau_i} + u_{i0}\frac{du_{i2}}{d\tau_i}\right), \quad i = 0, 1.
\end{aligned} \tag{7.18}$$

The scalar product $(d\mathbf{r}_i/d\tau_i)\cdot\mathbf{p}$ has the form of the third relation in (7.18), where, instead of x_{01}, y_{01}, and z_{01}, one can take p_1, p_2, and p_3, respectively.

Differential equations (7.10), (7.16) supplemented with the differential equations

$$\frac{dt}{d\tau_0} = r_0, \quad \frac{d\tau_1}{d\tau_0} = r_0r_1^{-1} \tag{7.19}$$

and the relations

$$r_0 = u_{00}^2 + u_{01}^2 + u_{02}^2 + u_{03}^2, \quad r_1^2 = (x_{01} - x_0)^2 + (y_{01} - y_0)^2 + (z_{01} - z_0)^2, \tag{7.20}$$

$$x_0 = u_{00}^2 + u_{01}^2 - u_{02}^2 - u_{03}^2, \quad y_0 = 2(u_{01}u_{02} - u_{00}u_{03}), \quad z_0 = 2(u_{01}u_{03} + u_{00}u_{02}) \tag{7.21}$$

form the system of differential equations of motion of point M which are regular near point M_0. They are a system of nonlinear nonstationary eleventh-order differential equations for the Kustaanheimo–Stiefel variables u_{0j} $(j = 0, 1, 2, 3)$, their first derivatives $du_{0j}/d\tau_0$, the energy variable h_0^*, the time t, and the variable τ_1.

For $m_1 = 0$, equations (7.10) and (7.16) and the first equation in (7.19) imply the quaternion regular equations of perturbed spatial problem of two bodies M and M_0, one of which (M) has a negligibly small mass. These equations coincide with the well-known quaternion regular equations (1.9) of perturbed spatial two-body problem.

Differential equations (7.11), (7.16) supplemented with the differential equations

$$\frac{dt}{d\tau_1} = r_1, \quad \frac{d\tau_0}{d\tau_1} = r_1r_0^{-1} \tag{7.22}$$

and the relations

$$r_1 = u_{10}^2 + u_{11}^2 + u_{12}^2 + u_{13}^2, \quad r_0^2 = (x_{01} - x_1)^2 + (y_{01} - y_1)^2 + (z_{01} - z_1)^2, \tag{7.23}$$

$$x_1 = u_{10}^2 + u_{11}^2 - u_{12}^2 - u_{13}^2, \quad y_1 = 2(u_{11}u_{12} - u_{10}u_{13}), \quad z_1 = 2(u_{11}u_{13} + u_{10}u_{12}) \tag{7.24}$$

form the system of differential equations of motion of point M which are regular near point M_1. They are a system of nonlinear nonstationary eleventh-order differential equations for the Kustaanheimo–Stiefel variables u_{1j} ($j = 0, 1, 2, 3$), their first derivatives $du_{1j}/d\tau_1$, the energy variable h_1^*, the time t, and the variable τ_0.

For $m_0 = 0$, equations (7.11) and (7.17) and the first equation in (7.22) imply the quaternion regular equations of perturbed spatial problem of two bodies M and M_0, one of which (M) has a negligibly small mass. These equations coincide with the well-known quaternion regular equations (1.9) of perturbed spatial two-body problem. These equations coincide with the well-known quaternion regular equations (1.9) of perturbed spatial two-body problem.

The obtained systems of differential equations of perturbed spatial bounded three-body problem permit developing regular analytical and numerical methods for studying the motion of a body of negligibly small mass near two other bodies of finite masses and also permit constructing a regular algorithm for integrating these equations, were equations (7.10), (7.16), (7.19)–(7.21) of this problem supplemented with relations (7.18) (for $i = 0$) are used to study the motion of point M near point M_0 (when the distances r_0 and r_1 satisfy the inequality $m_1 r_0^2 \le m_0 r_1^2$) and equations (7.11), (7.17), (7.22)–(7.24) of this problem supplemented with relations (7.18) (for $i = 0$) are used to study the motion of point M near point M_1 (the distances r_1 and r_0 satisfy the inequality $m_0 r_1^2 < m_1 r_0^2$).

Remark 1. The above-described algorithm for integration of the constructed regular equations of bounded three-body problem is based on the assumption that the projections x_{01}, y_{01}, z_{01} of the vector $\mathbf{r}_{01}$ on the axis of the inertial coordinate system (coordinates of point M_1 in the coordinate system $M_0X_0Y_0Z_0$) contained in this algorithm are true functions of the time t. This, in particular, holds for the bounded circular three-body problem. In the general case, to determine the projections x_{01}, y_{01}, z_{01}, it is necessary to supplement the systems of differential equations (7.10), (7.16) and (7.11), (7.17) with the vector differential equation (2.5) where we first pass to the new independent variable τ_0 or τ_1 by formulas (7.5).

To use the above-constructed regular differential equations of perturbed spatial bounded three-body problem, it is necessary to determine their initial conditions of integration, i.e., it is necessary to determine the initial values of Kustaanheimo–Stiefel variables u_{ij} ($j = 0, 1, 2, 3$) and their first derivative $du_{ij}/d\tau$ in terms of the given initial values of Cartesian coordinates x_i, y_i, z_i of point M in the coordinate system $M_iX_iY_iZ_i$ and the initial value of the projections $\dot{x}_i$, $\dot{y}_i$, $\dot{z}_i$ of the velocity vector $\mathbf{v}_i$ of motion of point M in the coordinate system $M_iX_iY_iZ_i$ on the axes of this coordinate system. The nonunique algorithms for solving this problem (problem with initial conditions) were proposed in [7, 16]. Let us consider a unique algorithm for solving the problem with initial conditions which is proposed by the author of this paper in [22] (also see [13–15]). This algorithm is formed by the relations (where $i = 0, 1$)

$$c_{ix} = y_i\dot{z}_i - z_i\dot{y}_i, \quad c_{iy} = z_i\dot{x}_i - x_i\dot{z}_i, \quad c_{iz} = x_i\dot{y}_i - y_i\dot{x}_i, \tag{7.25}$$

$$r_i = (x_i^2 + y_i^2 + z_i^2)^{1/2}, \quad c_i = (c_{ix}^2 + c_{iy}^2 + c_{iz}^2)^{1/2}, \tag{7.26}$$

$$\boldsymbol{\vartheta} = \vartheta_{i1}\mathbf{i} + \vartheta_{i2}\mathbf{j} + \vartheta_{i3}\mathbf{k} = (r_ic_{ix} - c_iz_i)^{-1}[c_{ix}\mathbf{i} + c_{iy}\mathbf{j} + (c_{iz} - c_i)\mathbf{k}][(x_i - r_i)\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}], \tag{7.27}$$

$$\lambda_{i0} = (1 + \vartheta_i^2)^{-1/2}, \quad \lambda_{ik} = \lambda_{i0}\vartheta_{ik}, \quad k = 1, 2, 3, \quad \vartheta_i^2 = \vartheta_{i1}^2 + \vartheta_{i2}^2 + \vartheta_{i3}^2, \tag{7.28}$$

$$u_{i0} = r_i^{1/2}\lambda_{i0}, \quad u_{ik} = -r_i^{1/2}\lambda_{ik}, \quad k = 1, 2, 3, \tag{7.29}$$

$$\frac{d\mathbf{u}_i}{d\tau_i} = -\frac{1}{2}\mathbf{i} \circ \mathbf{u}_i \circ (\vartheta_{i1}\mathbf{i} + \vartheta_{i2}\mathbf{j} + \vartheta_{i3}\mathbf{k}). \tag{7.30}$$

Relations (7.25) are used to determine the projections c_{ix}, c_{iy}, c_{iz} of vector of velocity moment $\mathbf{c}_i$ of point M on the axes of the coordinate system $M_iX_iY_iZ_i$, relations (7.26) are used to determine the moduli of the radius vector $\mathbf{r}_i$ of point M and the vector of velocity moment $\mathbf{c}_i$ of this point, relations (7.27) are used to determine the vector of finite rotation $\boldsymbol{\vartheta}_i$ of the coordinate system $M_iX_i'Y_i'Z_i'$ with respect to the coordinate system $M_iX_iY_iZ_i$ (in this case, the condition $r_ic_{ix} - c_iz_i \ne 0$ must be satisfied), relations (7.28) are used to determine the Rodrigues–Hamilton parameters λ_{ij} characterizing the orientation of the coordinate system $M_iX_i'Y_i'Z_i'$ in the coordinate system $M_iX_iY_iZ_i$, and finally, relations (7.29) and (7.30) are used to determine the sought variables: the Kustaanheimo–Stiefel variables u_{ij} ($j = 0, 1, 2, 3$) and their first derivatives $du_{ij}/d\tau_i$.

The radius vector $\mathbf{r}_i$ characterizing the position of point M in the coordinate system $M_iX_iY_iZ_i$, its modulus, and the velocity vector $\mathbf{v}_i$ of motion of point M in this coordinate system are determined in terms of the variables $\mathbf{u}_i$ and $d\mathbf{u}_i/d\tau_i$ according to the quaternion formulas [17] (also see [13–15]):

$$\mathbf{R}_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k} = \bar{\mathbf{u}}_i \circ \mathbf{i} \circ \mathbf{u}_i, \quad r_i = \mathbf{u}_i \circ \bar{\mathbf{u}}_i = u_{i0}^2 + u_{i1}^2 + u_{i2}^2 + u_{i3}^2, \quad i = 0, 1, \tag{7.31}$$

$$\mathbf{V}_i = v_{i1}\mathbf{i} + v_{i2}\mathbf{j} + v_{i3}\mathbf{k} = \frac{d\mathbf{R}_i}{dt} = 2\bar{\mathbf{u}}_i \circ \mathbf{i} \circ \frac{d\mathbf{u}_i}{dt} = 2r_i^{-1}\bar{\mathbf{u}}_i \circ \mathbf{i} \circ \frac{d\mathbf{u}_i}{d\tau_i}, \quad i = 0, 1. \tag{7.32}$$

In scalar form, (7.31) and (7.32) become

$$x_i = u_{i0}^2 + u_{i1}^2 - u_{i2}^2 - u_{i3}^2, \quad y_i = 2(u_{i1}u_{i2} - u_{i0}u_{i3}), \quad z_i = 2(u_{i1}u_{i3} + u_{i0}u_{i2}), \quad i = 0, 1,$$

$$v_{i1} = \dot{x}_i = 2(u_{i0}\dot{u}_{i0} + u_{i1}\dot{u}_{i1} - u_{i2}\dot{u}_{i2} - u_{i3}\dot{u}_{i3}) = 2r_i^{-1}\left[u_{i0}\frac{du_{i0}}{d\tau_i} + u_{i1}\frac{du_{i1}}{d\tau_i} - u_{i2}\frac{du_{i2}}{d\tau_i} - u_{i3}\frac{du_{i3}}{d\tau_i}\right],$$

$$v_{i2} = \dot{y}_i = 2(u_{i2}\dot{u}_{i1} + u_{i1}\dot{u}_{i2} - u_{i3}\dot{u}_{i0} - u_{i0}\dot{u}_{i3}) = 2r_i^{-1}\left[u_{i2}\frac{du_{i1}}{d\tau_i} + u_{i1}\frac{du_{i2}}{d\tau_i} - u_{i3}\frac{du_{i0}}{d\tau_i} - u_{i0}\frac{du_{i3}}{d\tau_i}\right],$$

$$v_{i3} = \dot{z}_i = 2(u_{i3}\dot{u}_{i1} + u_{i1}\dot{u}_{i3} + u_{i2}\dot{u}_{i0} + u_{i0}\dot{u}_{i2}) = 2r_i^{-1}\left[u_{i3}\frac{du_{i1}}{d\tau_i} + u_{i1}\frac{du_{i3}}{d\tau_i} + u_{i2}\frac{du_{i0}}{d\tau_i} + u_{i0}\frac{du_{i2}}{d\tau_i}\right].$$

These formulas permit determining the Cartesian coordinates x_i, y_i, z_i of point M in the coordinate system $M_iX_iY_iZ_i$ and the projections of the velocity of point M in the coordinate system $M_iX_iY_iZ_i$ on the axes of this coordinate system in terms of the variables u_{ij} and their derivatives $\dot{u}_{ij}$ or $du_{ij}/d\tau_i$.

8. DIFFERENTIAL QUATERNION REGULAR EQUATIONS OF PERTURBED SPATIAL BOUNDED THREE-BODY PROBLEM WITH THE TOTAL ENERGY OR JACOBI VARIABLE USED AS AN ADDITIONAL VARIABLE

As additional variables, we introduce the total energies h_0 and h_1 of motion of point M in the respective coordinate systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$ into the equations of bounded three-body problem. For this, we substitute expressions (7.13) into equations (7.10) and (7.11) and supplement the obtained equations with differential equations (3.2) and (3.3) for the energies h_0 and h_1 (after passing in these equations to new independent variables τ_0 and τ_1 by formulas (7.5)). We obtain the following differential equations of perturbed spatial bounded three-body problem:

$$\frac{d^2\mathbf{u}_0}{d\tau_0^2} - \frac{1}{2}h_0\mathbf{u}_0 = \frac{1}{2}fm_1r_1^{-1}(1 - r_0^2r_1^{-2})\mathbf{u}_0 - \frac{1}{2}r_0\mathbf{i} \circ \mathbf{u}_0 \circ [fm_1(r_1^{-3} - r_{01}^{-3})\mathbf{R}_{01} + \mathbf{P}], \tag{8.1}$$

$$\begin{aligned}\frac{dh_0}{d\tau_0} &= fm_1r_1^{-3}\left(r_{01}\frac{dr_{01}}{d\tau_0} - \frac{d\mathbf{r}_{01}}{d\tau_0} \cdot \mathbf{r}_0\right) - fm_1r_{01}^{-3}\left(\frac{d\mathbf{r}_0}{d\tau_0} \cdot \mathbf{r}_{01}\right) + \frac{d\mathbf{r}_0}{d\tau_0} \cdot \mathbf{p},\\ \frac{dh_0}{d\tau_0} &= fm_1r_0r_1^{-3}\left(r_{01}\frac{dr_{01}}{dt} - \frac{d\mathbf{r}_{01}}{dt} \cdot \mathbf{r}_0\right) - fm_1r_{01}^{-3}\left(\frac{d\mathbf{r}_0}{d\tau_0} \cdot \mathbf{r}_{01}\right) + \frac{d\mathbf{r}_0}{d\tau_0} \cdot \mathbf{p},\end{aligned} \tag{8.2}$$

$$\frac{d^2\mathbf{u}_1}{d\tau_1^2} - \frac{1}{2}h_1\mathbf{u}_1 = \frac{1}{2}fm_0r_0^{-1}(1 - r_1^2r_0^{-2})\mathbf{u}_1 - \frac{1}{2}r_1\mathbf{i} \circ \mathbf{u}_1 \circ [fm_0(r_{01}^{-3} - r_0^{-3})\mathbf{R}_{01} + \mathbf{P}], \tag{8.3}$$

$$\begin{aligned}\frac{dh_1}{d\tau_1} &= fm_0r_0^{-3}\left(r_{01}\frac{dr_{01}}{d\tau_1} + \frac{d\mathbf{r}_{01}}{d\tau_1} \cdot \mathbf{r}_1\right) + fm_0r_{01}^{-3}\left(\frac{d\mathbf{r}_1}{d\tau_1} \cdot \mathbf{r}_{01}\right) + \frac{d\mathbf{r}_1}{d\tau_1} \cdot \mathbf{p},\\ \frac{dh_1}{d\tau_1} &= fm_0r_1r_0^{-3}\left(r_{01}\frac{dr_{01}}{dt} + \frac{d\mathbf{r}_{01}}{dt} \cdot \mathbf{r}_1\right) + fm_0r_{01}^{-3}\left(\frac{d\mathbf{r}_1}{d\tau_1} \cdot \mathbf{r}_{01}\right) + \frac{d\mathbf{r}_1}{d\tau_1} \cdot \mathbf{p}.\end{aligned} \tag{8.4}$$

Equations (8.1), (8.2) (the first or the second equation) supplemented with equations (7.19)–(7.21) and equations (8.3), (8.4) (the first or the second equation) supplemented with equations (7.22)–(7.24) are a different form of regular equations of perturbed spatial bounded three-body problem, where the total energies h_0 and h_1 are used as additional variables. These equations can be used to study the motion of point M near point M_0 or near point M_1 by the same methodology as equations (7.10), (7.16), (7.19)–(7.21) or (7.11), (7.17), (7.22)–(7.24) containing the energies h_0^* and h_1^*.

By setting $m_1 = 0$ in equations (8.1), (8.2) (the first or the the second equation) and $m_0 = 0$ in equations (8.3), (8.4) (the first or the second equation), from these equations we obtain the quaternion regular equations of perturbed spatial two-body problem (bodies M and M_0 or M and M_1) one of which (M) has a negligibly small mass. These equations coincide with the known quaternion regular equations (1.9) of perturbed spatial two-body problem, because the energies h_0 and h_1 coincide in this case with the Kepler energies h_0^* and h_1^*.

Remark 2. The use of the above-described regular equations of bounded three-body problem is based on the assumption that the projections x_{01}, y_{01}, z_{01} of the vector $\mathbf{r}_{01}$ on the axes of inertial coordinate system (coordinates of point M_1 in the coordinate system $M_0X_0Y_0Z_0$) and their first derivatives $\dot{x}_{01}$, $\dot{y}_{01}$, $\dot{z}_{01}$ with respect to time t, contained in these equations, are known functions of time t. In particular, this holds for the bounded circular three-body problem. In the general case, to determine the projections x_{01}, y_{01}, z_{01} and their first derivatives with respect to the independent variable τ_0 or τ_1, it is necessary additionally to supplement systems of differential equations (8.1), (8.2) and (8.3), (8.4) with the vector differential equation (2.5) after passing in it to the new independent variables τ_0 and τ_1 by formulas (7.5).

In the case of perturbed spatial circular bounded three-body problem, it is expedient to replace the energies h_0 and h_1 of motion of point M in the coordinate systems $M_0X_0Y_0Z_0$ and $M_1X_1Y_1Z_1$, which are used in regular equations of motion as additional variables, by the variables H_0 and H_1 determined by relations (3.15) and (3.16). These variables satisfy differential equations (3.13), (3.14) and are the Jacobi constants of motion of the unperturbed spatial bounded circular three-body problem. Then equations (8.1), (8.2) and (8.3), (8.4) become

$$\frac{d^2\mathbf{u}_0}{d\tau_0^2} - \frac{1}{2}[H_0 - fm_1r_{01}^{-1}(\mathbf{r}_0\cdot\mathbf{r}_{01}) - n(y_0\dot{x}_0 - x_0\dot{y}_0)]\mathbf{u}_0$$
$$= \frac{1}{2}fm_1r_1^{-1}(1 - r_0^2r_1^{-2})\mathbf{u}_0 - \frac{1}{2}r_0\mathbf{i}\circ\mathbf{u}_0\circ[fm_1(r_1^{-3} - r_{01}^{-3})\mathbf{R}_{01} + \mathbf{P}], \tag{8.5}$$

$$\frac{dH_0}{d\tau_0} = \frac{d\mathbf{r}_0}{d\tau_0}\cdot\mathbf{p} + nr_0(y_0p_x - x_0p_y), \tag{8.6}$$

$$\frac{d^2\mathbf{u}_1}{d\tau_0^2} - \frac{1}{2}[H_1 - fm_0r_{01}^{-1}(\mathbf{r}_1\cdot\mathbf{r}_{10}) - n(y_1\dot{x}_1 - x_1\dot{y}_1)]\mathbf{u}_1$$
$$= \frac{1}{2}fm_0r_0^{-1}(1 - r_1^2r_0^{-2})\mathbf{u}_1 - \frac{1}{2}r_1\mathbf{i}\circ\mathbf{u}_1\circ[fm_0(r_{01}^{-3} - r_0^{-3})\mathbf{R}_{01} + \mathbf{P}], \tag{8.7}$$

$$\frac{dH_1}{d\tau_1} = \frac{d\mathbf{r}_1}{d\tau_1}\cdot\mathbf{p} + nr_1(y_1p_x - x_1p_y), \tag{8.8}$$

$$\begin{aligned}
&\mathbf{r}_0\cdot\mathbf{r}_{01} = a[\cos(nt)x_0 + \sin(nt)y_0], \quad \mathbf{r}_1\cdot\mathbf{r}_{10} = -a[\cos(nt)x_1 + \sin(nt)y_1],\\
&x_i = u_{i0}^2 + u_{i1}^2 - u_{i2}^2 - u_{i3}^2, \quad y_i = 2(u_{i1}u_{i2} - u_{i0}u_{i3}), \quad i = 0, 1,\\
&-(y_i\dot{x}_i - x_i\dot{y}_i) = c_{iz} = 2\left(u_{i3}\frac{du_{i0}}{d\tau_i} - u_{i2}\frac{du_{i1}}{d\tau_i} + u_{i1}\frac{du_{i2}}{d\tau_i} - u_{i0}\frac{du_{i3}}{d\tau_i}\right),
\end{aligned} \tag{8.9}$$

the quaternions $\mathbf{R}_{01}$ and $\mathbf{P}$ are determined by relations (7.12), and the distance is $r_{01} = r_{10} = a = \text{const}$.

This, in the case where the distances r_0 and r_1 satisfy the inequality $m_1r_0^2 \leq m_0r_1^2$, the regular equations of perturbed spatial bounded circular three-body problem have the form of equations (8.5), (8.6), (7.19) supplemented with relations (7.20), (7.21), (8.9) (for $i = 0$), and in the case where the distances r_1 and r_0 satisfy the inequality $m_0r_1^2 < m_1r_0^2$, they have the form of equations (8.7), (8.8), (7.22) supplemented with relations (7.23), (7.24), (8.9) (for $i = 1$).

For the unperturbed spatial bounded circular three-body problem, when the perturbing acceleration $\mathbf{p} = 0$ and the quaternion $\mathbf{P} = 0$, these equations become significantly simpler. Equations (8.6), (8.8) imply the Jacobi integrals $H_i = H_{i0} = \text{const}$, $i = 0, 1$, and hence these equations are eliminated from the set of systems of regular equations, Moreover, in equations (8.5) and (8.7), it is necessary to set $H_0 = H_{00} = \text{const}$ and $H_1 = H_{10} = \text{const}$, respectively.

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