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Applied and Engineering Versions of the Theory of Elastoplastic Processes of Active Complex Loading. Part 1: Conditions of Mathematical Well-Posedness and Methods for Solving Boundary Value Problems

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Abstract—In the proposed theory of plasticity, the deviator constitutive relation has a trinomial form (the vectors of stresses, stress rates, and strain rates, which are formed form the deviators, are coplanar) and contains two material functions; one of these functions depends on the modulus of the stress vector, and the other, on the angle between the stress vector and the strain rate, the length of the deformation trajectory arc, and the moduli of the stress and strain vectors. The spherical parts of the stress and strain tensors satisfy the relations of elastic variation in the volume.
We obtain conditions on the material functions of the model which ensure the mathematical well-posedness of the statement of the initial—boundary value problem (i.e., the existence and uniqueness of the generalized solution, and its continuous dependence on the external loads). We also describe the scheme for solving the initial—boundary value problem step by step using the model and present the expression for the Jacobian of the boundary value problem at the time step. These results are formalized as a subprogram for prescribing the mechanical properties of the user material in the finite-element complex ABAQUS, which allows one to calculate the structure deformations on the basis of the proposed theory.

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1. CONSTITUTIVE RELATIONS

In Il'yushin's general mathematical theory of elastoplastic processes [1], a specific relation of trinomial form was developed [2—6], which is based on the assumption that the stress vector $\boldsymbol{\sigma}$, the stress increment vector $d\boldsymbol{\sigma}$, and the strain increment vector $d\boldsymbol{э}$ are coplanar in the five-dimensional space $Э_5$ [1],

$$\frac{d\boldsymbol{\sigma}}{ds} = N\frac{d\boldsymbol{э}}{ds} - (N-P)\cos\vartheta\frac{\boldsymbol{\sigma}}{\sigma}, \tag{1.1}$$

where $\sigma = |\boldsymbol{\sigma}|$, $ds = |d\boldsymbol{э}|$, $\vartheta = \arccos(\boldsymbol{\sigma}/\sigma\cdot d\boldsymbol{э}/ds)$ is the angle between the vectors $\boldsymbol{\sigma}$ and $d\boldsymbol{э}$, s is the arc length of the deformation trajectory $\boldsymbol{э}(s)$, and N and P are functionals of the parameters of the intrinsic geometry of the deformation trajectory, which should be determined from elastoplastic deformation tests with specimens made of the material under study.

In the present paper, we use the vectors $\boldsymbol{\sigma}$ and $\boldsymbol{э}$ related to the stress deviator S_{ij} and the strain deviator $Э_{ij}$ by the formulas

$$\sigma_1 = \tfrac{3}{2}S_{11},\quad \sigma_2 = \sqrt{3}(S_{22}+\tfrac{1}{2}S_{11}),\quad \sigma_3 = \sqrt{3}S_{12},\quad \sigma_4 = \sqrt{3}S_{13},\quad \sigma_5 = \sqrt{3}S_{23}, \tag{1.2}$$

$$э_1 = Э_{11},\quad э_2 = \frac{2Э_{22}+Э_{11}}{\sqrt{3}},\quad э_3 = \frac{2Э_{12}}{\sqrt{3}},\quad э_4 = \frac{2Э_{13}}{\sqrt{3}},\quad э_5 = \frac{2Э_{23}}{\sqrt{3}}. \tag{1.3}$$

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The vectors (1.2) and (1.3) differ from the stress and strain vectors [1] by the factors $\sqrt{3/2}$ and $\sqrt{2/3}$, respectively. The vectors (1.2) and (1.3) are more convenient for analyzing the experimental data and verifying the model.

Relation (1.1) is the general form of the $\boldsymbol{\sigma} \sim \boldsymbol{э}$ relation for plane deformation trajectories and spatial trajectories with small torsion [5]. For deformation trajectories of arbitrary geometry and dimension $n \geq 3$, relation (1.1) is a conjecture [4, 6], which has been confirmed with acceptable accuracy in a series of tests along sufficiently complex deformation trajectories.[1)] In this connection, it is important to study the properties of the functionals N and P and find their approximations in various classes of loading processes. Formulas more general than (1.1) relating the vectors $\boldsymbol{\sigma}$ and $\boldsymbol{э}$ and containing three functionals of the deformation process were considered in [7, 8].

Relation (1.1) was originally proposed to describe a two-link deformation process [9]. In this case, according to the isotropy postulate [1], N and P are functions of the angle ϑ_0 of the curve break, the arc length s_0 of the deformation trajectory at the curve break time, and the arc length increment $\Delta s = s - s_0$. It follows from the isotropy and plasticity postulates [2] that, for an infinitesimal Δs, the function N is independent of ϑ_0, while the function P significantly depends on ϑ_0. (For example, P is of the order of the tangential strengthening modulus for $\vartheta_0 \ll \pi/2$ and of the order of the material shear elasticity modulus G for $\vartheta_0 \geq \pi/2$.) The functions N and P were studied in two-link deformation experiments in [3, 10–15] and by other authors.

Consider the well-known special versions [17–20, 21–22] of relation (1.1) for two loading classes. We rewrite (1.1) as

$$\frac{d\boldsymbol{\sigma}}{ds} = N\frac{d\boldsymbol{э}}{ds} - \left(N\cos\vartheta - \frac{d\sigma}{ds}\right)\frac{\boldsymbol{\sigma}}{\sigma}. \tag{1.4}$$

(The equation $d\sigma/ds = P\cos\vartheta$ can be obtained by multiplying both sides of (1.1) by the vector $\boldsymbol{\sigma}/\sigma$.) Then the functionals N and $d\sigma/ds$ should be determined. The relation of the theory of mean curvature processes [16] can be written in the form (1.4), $N = N_*$, and $d\sigma/ds = \Phi'(s)$, where $N_* = \text{const}$ in the plastic region and $\Phi(s)$ is the function of simple (i.e., active rectilinear) loading. In [17], the following, different relation was proposed for medium curvature processes: $N = \sigma k(s)\vartheta/\sin\vartheta$ and $d\sigma/ds = \Phi'(s)\cos\vartheta$. (Here $k(s)$ is the material function of complex loading.) In [18], the theory [16] was generalized by supplementing the functions N and $d\sigma/ds$ with a dependence on ϑ (of a prescribed form) and two material functions of s (in addition to $\Phi(s)$); the model was intended for deformation trajectories of medium and small curvature with breaks under the assumption that, between neighboring break points, the quantity s increases at least by the delay trace value.[2)] It was shown in [5, 19] that the form of the laws of variation of the vector properties is the same for ideally elastic and ideally plastic materials, and hence it was proposed to assume that N is a relatively slowly varying function of the order of $3G$ [5] or the constant $3G$ [19]. The behavior of N and $d\sigma/ds$ calculated directly from the results of experiments of deformation along polynomial curves was studied in [20]; it was shown that the assumptions $N = \text{const} \approx 0.8 \times 3G$ and $d\sigma/ds = f(\vartheta)$ are acceptable in the first approximation for describing the loading of this type beyond the elasticity region. It was proposed in [7] to describe the deformation along plane trajectories by the functions N and $d\sigma/ds$ which contain the simple loading function $\Phi(s)$, a prescribed function of ϑ, and two material constants of complex loading. In [21, 22], relation (1.4) with the functions $N = g(\sigma)$ and $d\sigma/ds = f(\vartheta, \sigma, s)$ was experimentally and theoretically investigated in the class of active loading processes with possible final unloading; such a version of the theory can be considered as a development of the model [20].

In the present paper, the model considered in [21, 22] is generalized and specified on the basis of a significantly larger array of experimental data. (They are presented in Part 2 of the present paper.) For the functions N and $d\sigma/ds$, we adopt the following hypotheses:

$$N = g(\sigma), \quad d\sigma/ds = f(\vartheta, \sigma, s, э). \tag{1.5}$$

[1)] R. A. Vasin, *Experimental and Theoretical Study of Constitutive Relations in the Theory of Elastoplastic Processes*, Doctoral Dissertation in Physics and Mathematics (MGU, Moscow, 1987) [in Russian].

[2)] S. V. Ermakov, *Analysis of Relations and Boundary Value Problems of the Theory of Elastoplastic Medium and Small Curvature Processes*, Candidate's Dissertation in Physics and Mathematics (Moscow State University, Moscow, 1984) [in Russian].

We include $\text{э} = |\boldsymbol{\text{э}}|$ as an argument of the function f on the basis of an analysis of the results of experiments along cyclic circular deformation trajectories. According to (1.4) and (1.5), we obtain the vector constitutive relation

$$\frac{d\boldsymbol{\sigma}}{ds} = g(\sigma)\frac{d\boldsymbol{\text{э}}}{ds} - [g(\sigma)\cos\vartheta - f(\vartheta, \sigma, s\text{э})]\frac{\boldsymbol{\sigma}}{\sigma}. \tag{1.6}$$

In deviator form, it becomes

$$\frac{dS_{ij}}{ds} = \frac{2}{3}\left\{g(\sigma)\frac{d\text{Э}_{ij}}{ds} - [g(\sigma)\cos\vartheta - f(\vartheta, \sigma, s, \text{э})]\frac{S_{ij}}{\sigma}\right\}, \tag{1.7}$$

where $\sigma = \sqrt{3S_{kl}S_{kl}/2} = \sigma_u$ is the stress intensity, $\text{э} = \sqrt{2\text{Э}_{kl}\text{Э}_{kl}/3} = \varepsilon_u$ is the strain intensity, $ds = \sqrt{2d\text{Э}_{kl}d\text{Э}_{kl}/3}$, and $\vartheta = \arccos(S_{kl}d\text{Э}_{kl}/s/ds)$.

Relation (1.7) should be supplemented with a constraint equation relating the first invariants of stress and strain tensors; for this equation, we use the bulk elasticity law with modulus K,

$$\sigma_{mm} = 3K\varepsilon_{mm}. \tag{1.8}$$

2. CONDITIONS OF MATHEMATICAL WELL-POSEDNESS AND METHODS FOR SOLVING INITIAL BOUNDARY VALUE PROBLEMS

Let a body of volume $\Omega \subset R^3$ with piecewise smooth boundary Γ be subjected to mass and surface forces $\mathbf{X}(t, \mathbf{x})$ and $\mathbf{T}(t, \mathbf{x})$. If the constitutive relations (1.7) and (1.8) are used, then the quasistatic initial–boundary value problem of determining the displacement vector $\mathbf{u}(t, \mathbf{x})$, the stress tensor $\varepsilon_{ij}(t, \mathbf{x})$, and the strain tensor $\sigma_{ij}(t, \mathbf{x})$, $(t, \mathbf{x}) \in [0, T] \times \Omega$, has the form

$$\begin{gathered}\sigma_{ij,j} + X_i = 0,\\ \dot{S}_{ij} = \frac{2}{3}\left\{g(\sigma)\dot{\text{Э}}_{ij} - [g(\sigma)\cos\vartheta - f(\vartheta, \sigma, s, \text{э})]\dot{s}\frac{S_{ij}}{\sigma}\right\}, \quad \sigma_{mm} = 3K\varepsilon_{mm},\\ \varepsilon_{ij} = \frac{u_{i,j} + u_{j,i}}{2}, \quad u_i\big|_{\Gamma_1} = 0, \quad \sigma_{ij}n_j\big|_{\Gamma_2} = T_i.\end{gathered} \tag{2.1}$$

It is also assumed that $\Gamma_1 \cup \Gamma_2 = \Gamma$, $\Gamma_1 \cap \Gamma_2 = \varnothing$, and $\Gamma_1 \neq \varnothing$.

Problem (2.1) is solved by the step method. The time interval $[0, T]$ is divided into N parts by the points t_n, $n = 0, 1, \ldots, N$. (Here $t_0 = 0$, $t_N = T$, and the set $\{t_n\}$ contains all times of breaks of $\mathbf{X}(t, \mathbf{x})$ and $\mathbf{T}(t, \mathbf{x})$ with respect to t). At the times t_n, $n = 0, 1, \ldots, N-1$, we successively solve the boundary value problem of determining the velocity vector field $\mathbf{v}(t_n + 0, \mathbf{x})$, $\mathbf{x} \in \Omega$, for the known fields $S_{ij}(t_n, \mathbf{x})$, $s(t_n, \mathbf{x})$, and $\text{э}(t_n, \mathbf{x})$, which were determined at the preceding step,

$$\begin{gathered}\dot{\sigma}_{ij,j}(\mathbf{v}) + \dot{X}_i = 0,\\ \dot{S}_{ij}(\mathbf{v}) = \frac{2}{3}\left\{g(\sigma)\dot{\text{Э}}_{ij}(\mathbf{v}) - [g(\sigma)\cos\vartheta(\mathbf{v}) - f(\vartheta(\mathbf{v}), \sigma, s, \text{э})]\dot{s}(\mathbf{v})\frac{S_{ij}}{\sigma}\right\},\\ \dot{\sigma}_{mm}(\mathbf{v}) = 3K\dot{\varepsilon}_{mm}(\mathbf{v}), \quad \dot{\varepsilon}_{ij}(\mathbf{v}) = \frac{v_{i,j} + v_{j,i}}{2}, \quad v_i\big|_{\Gamma_1} = 0, \quad \dot{\sigma}_{ij}(\mathbf{v})n_j\big|_{\Gamma_2} = \dot{T}_i.\end{gathered} \tag{2.2}$$

(The dot above a symbol stands for the derivative with respect to t.) We use $\mathbf{v}(t_n + 0, \mathbf{x})$ to determine the approximate solution of problem (2.1) with time step $[t_n, t_{n+1}]$:

$$\mathbf{Y}(t, \mathbf{x}) = \mathbf{Y}(t_n, \mathbf{x}) + (t - t_n)\dot{\mathbf{Y}}(\mathbf{v}(t_n + 0, \mathbf{x})), \quad \forall(t, \mathbf{x}) \in [t_n, t_{n+1}] \times \Omega, \quad \mathbf{Y} = \{u_k, \varepsilon_{ij}, \sigma_{ij}\}.$$

We assume that the functions g and f, together with their first derivatives, are bounded and continuous.[3)] Now consider the generalized statement of the boundary value problem (2.2) in the form of the nonlinear operator equation [23–28]

$$\mathbf{A}\mathbf{v} = \mathbf{f}, \tag{2.3}$$

[3)]Piecewise continuous and piecewise smooth approximations to material functions are often used in practice. If necessary, small neighborhoods of the break points and corners can be replaced by inclined or rounded parts.

$$\mathbf{A}\colon H(\Omega) \to H^*(\Omega), \quad \langle \mathbf{Ac}, \boldsymbol{\psi} \rangle = \int_\Omega \left[\dot{S}_{ij}(\mathbf{v}) + \dot{\sigma}_{mm}(\mathbf{v}) \frac{\delta_{ij}}{3} \right] \dot{\varepsilon}_{ij}(\boldsymbol{\psi})\, d\Omega, \quad \forall \mathbf{v}, \boldsymbol{\psi} \in H(\Omega),$$

$$\mathbf{f} \in H^*(\Omega), \quad \langle \mathbf{f}, \boldsymbol{\psi} \rangle = \int_\Omega \dot{X}_m \psi_m \, d\Omega + \int_{\Gamma_2} \dot{T}_m \psi_m \, d\Gamma_2, \quad \forall \boldsymbol{\psi} \in H(\Omega).$$

Here $H(\Omega)$ is the Hilbert space obtained as the completion of the set of vector functions $\{v_i \in C^2(\Omega), v_i\big|_{\Gamma_1} = 0\}$ with respect to the norm corresponding to the inner product

$$(\mathbf{v}_1, \mathbf{v}_2)_{H(\Omega)} = \int_\Omega [2G \dot{\ni}_{ij}(\mathbf{v}_1) \dot{\ni}_{ij}(\mathbf{v}_2) + K \dot{\varepsilon}_{mm}(\mathbf{v}_1) \dot{\varepsilon}_{mm}(\mathbf{v}_2)]\, d\Omega, \quad \forall \mathbf{v}_1, \mathbf{v}_2 \in H(\Omega).$$

The expressions $\dot{S}_{ij}(\mathbf{v})$, $\dot{\sigma}_{mm}(\mathbf{v})$, and $\dot{\varepsilon}_{ij}(\mathbf{v})$ are defined in (2.2). Taking into account the type of the space $H(\Omega)$, we impose the following constraints [23] on the external loads:

$$\dot{X}_i(t, \mathbf{x}) \in L_p(\Omega), \quad p > 6/5, \quad \dot{T}_i(t, \mathbf{x}) \in L_q(\Gamma_2), \quad q > 4/3, \quad \forall t \in [0, t]. \tag{2.4}$$

The solution $\mathbf{v} \in H(\Omega)$ of the operator equation (2.3) will be called a generalized solution of the boundary value problem (2.2). If, in addition, $v_i \in C^2(\Omega) \cap C^1(\bar{\Omega})$, then $\mathbf{v}$ is a classical solution of problem (2.2). In the case of an inhomogeneous kinematic boundary condition, problem (2.2) can be reduced to the corresponding homogenous problem by a change of variables [28].

Assume that the functions $g(\sigma)$ and $f(\vartheta, \sigma, s, \ni)$ satisfy the following inequalities in their domains:

$$\varphi(\vartheta, \sigma, s, \ni) \equiv g(\sigma) \cos\vartheta - f(\vartheta, \sigma, s, \ni) \geq 0, \tag{2.5}$$

$$\chi(\vartheta, \sigma, s, \ni) \equiv g(\sigma) \sin^2\vartheta + f(\vartheta, \sigma, s, \ni) \cos\vartheta \geq 3G\mu = \text{const} > 0, \tag{2.6}$$

$$\begin{aligned} &2\sqrt{\chi(\vartheta_1, \sigma, s, \ni) - 3G\mu}\sqrt{\chi(\vartheta_2, \sigma, s, \ni) - 3G\mu} \\ &\qquad - 2[g(\sigma) - 3G\mu]\cos(\vartheta_1 - \vartheta_2) + \varphi(\vartheta_1, \sigma, s, \ni)\cos\vartheta_2 + \varphi(\vartheta_2, \sigma, s, \ni)\cos\vartheta_1 \geq 0, \end{aligned} \tag{2.7}$$

$$|\varphi(\vartheta_1, \sigma, s, \ni) - \varphi(\vartheta_2, \sigma, s, \ni)| \leq M_0 |\vartheta_1 - \vartheta_2|, \quad M_0 = \text{const}. \tag{2.8}$$

Then the operator $\mathbf{A}$ given by (2.3) is strongly monotone with constant

$$m_A = \min\{1, \mu\} \tag{2.9}$$

and Lipschitz continuous with constant

$$M_A = \max\left\{1, \frac{\sup g + \sup \varphi + M_0 \pi/\sqrt{2}}{3G}\right\}. \tag{2.10}$$

Indeed, for any $\mathbf{v}_1, \mathbf{v}_2 \in H(\Omega)$ we obtain (omitting some intermediate transformations)

$$\begin{aligned} &\langle \mathbf{Av}_1 - \mathbf{Av}_2, \mathbf{v}_1 - \mathbf{v}_2 \rangle - m_A \|\mathbf{v}_1 - \mathbf{v}_2\|^2_{H(\Omega)} \\ &\quad = \int_\Omega \{\dot{S}_{ij}(\mathbf{v}_1) - \dot{S}_{ij}(\mathbf{v}_2) - 2m_A[\dot{\ni}_{ij}(\mathbf{v}_1) - \dot{\ni}_{ij}(\mathbf{v}_2)]\}[\dot{\ni}_{ij}(\mathbf{v}_1) - \dot{\ni}_{ij}(\mathbf{v}_2)]\, d\Omega \\ &\quad + K(1 - m_A) \int_\Omega [\dot{\varepsilon}_{mm}(\mathbf{v}_1) - \dot{\varepsilon}_{mm}(\mathbf{v}_2)]^2 d\Omega \\ &\quad \geq \int_\Omega \{[\chi(\vartheta(\mathbf{v}_1), \sigma, s, \ni)\cos\vartheta(\mathbf{v}_1) - 3G\mu][\dot{s}(\mathbf{v}_1)]^2 \\ &\quad + [\chi(\vartheta(\mathbf{v}_2), \sigma, s, \ni)\cos\vartheta(\mathbf{v}_2) - 3G\mu][\dot{s}(\mathbf{v}_2)]^2 - [2(g(\sigma) - 3G\mu)\cos(\vartheta(\mathbf{v}_1) - \vartheta(\mathbf{v}_2)) \\ &\quad - \varphi(\vartheta(\mathbf{v}_1), \sigma, s, \ni)\cos\vartheta(\mathbf{v}_2) - \varphi(\vartheta(\mathbf{v}_2), \sigma, s, \ni)\cos\vartheta(\mathbf{v}_1)]\dot{s}(\mathbf{v}_1)\dot{s}(\mathbf{v}_2)\}\, d\Omega \geq 0, \end{aligned}$$

$$\|\mathbf{A}\mathbf{v}_1 - \mathbf{A}\mathbf{v}_2\|^2_{H^*(\Omega)} - M_A^2\|\mathbf{v}_1 - \mathbf{v}_2\|^2_{H(\Omega)} = (2G)^{-1}\int\limits_\Omega [\dot S_{ij}(\mathbf{v}_1) - \dot S_{ij}(\mathbf{v}_2)][\dot S_{ij}(\mathbf{v}_1) - \dot S_{ij}(\mathbf{v}_2)]\,d\Omega$$

$$- 2GM_A^2\int\limits_\Omega [\dot{\text{Э}}_{ij}(\mathbf{v}_1) - \dot{\text{Э}}_{ij}(\mathbf{v}_2)][\dot{\text{Э}}_{ij}(\mathbf{v}_1) - \dot{\text{Э}}_{ij}(\mathbf{v}_2)]\,d\Omega$$

$$- K(1 - M_A^2)\int\limits_\Omega [\dot\varepsilon_{mm}(\mathbf{v}_1) - \dot\varepsilon_{mm}(\mathbf{v}_2)]^2 d\Omega \le (3G)^{-1}\int\limits_\Omega \{[g(\sigma)|\dot{\boldsymbol{\text{э}}}(\mathbf{v}_1) - \dot{\boldsymbol{\text{э}}}(\mathbf{v}_2)|$$

$$+ |\varphi(\vartheta(\mathbf{v}_1), \sigma, s, \text{э})\dot s(\mathbf{v}_1) - \varphi(\vartheta(\mathbf{v}_2), \sigma, s, \text{э})\dot s(\mathbf{v}_2)|]^2 - [3GM_A|\dot{\boldsymbol{\text{э}}}(\mathbf{v}_1) - \dot{\boldsymbol{\text{э}}}(\mathbf{v}_2)|]^2\}\,d\Omega \le 0,$$

as desired. These properties of the operator $\mathbf{A}$ and the Browder–Minty theorem [28, pp. 97 and 104 of the Russian translation] imply the following assertion.

Theorem. *Assume that the functions g and f are bounded and continuous together with their first derivatives and satisfy inequalities* (2.5)–(2.8) *and the external loads satisfy conditions* (2.4). *Then there exists a unique generalized solution $\mathbf{v} \in H(\Omega)$ of the boundary value problem* (2.2)*, and this solution continuously depends on the external load functional,*

$$\|\mathbf{v}_1 - \mathbf{v}_2\|_{H(\Omega)} \equiv \|\mathbf{A}^{-1}\mathbf{f}_1 - \mathbf{A}^{-1}\mathbf{f}_2\|_{H(\Omega)} \le \frac{\|\mathbf{f}_1 - \mathbf{f}_2\|_{H^*(\Omega)}}{m_A}, \quad \forall \mathbf{f}_1 m \mathbf{f}_2 \in H^*(\Omega),$$

and can be obtained by the iteration method

$$\mathbf{J}\mathbf{v}^{(k+1)} = \mathbf{J}\mathbf{v}^{(k)} - \gamma(\mathbf{A}\mathbf{v}^{(k)} - \mathbf{f}), \quad j = 0, 1, \ldots, \tag{2.11}$$

$$\mathbf{J}\colon H(\Omega) \to H^*(\Omega), \quad \langle \mathbf{J}\mathbf{v}_1, \mathbf{v}_2\rangle = (\mathbf{v}_1, \mathbf{v}_2)_{H(\Omega)}, \quad \forall \mathbf{v}_1, \mathbf{v}_2 \in H(\Omega), \tag{2.12}$$

converging for any value of the iteration parameter $\gamma \in (0, 2m_A/M_A^2)$ and any initial approximation $\mathbf{v}^{(0)} \in H(\Omega)$ at the rate of a geometric progression with ratio $\xi = (1 - 2m_A\gamma + M_A^2\gamma^2)^{1/2} < 1$; i.e., the error satisfies the estimate $\|\mathbf{v}^{(k)} - \mathbf{v}\| \le \|\mathbf{A}\mathbf{v}^{(0)} - \mathbf{f}\|_{H^*(\Omega)}\gamma\xi^k/(1 - \xi)$; the values of m_A and M_A are given in (2.9) *and* (2.10).

Some of the constraints imposed on the functions g and f by the assumptions of the theorem can be given physical meaning. Condition (2.6) means that the material must be stable in the sense that the work of stress increments on strain increments is positive, $(d\boldsymbol{\sigma} \cdot d\boldsymbol{\text{э}}) = \chi(\vartheta, \sigma, s, \text{э})|d\text{э}|^2 > 0$. In the case of simple loading from an initial isotropic state, we have $\vartheta = 0$, $ds = d\text{э}$, and $d\sigma/d\text{э} = f(0, \Phi(\text{э}), \text{э}, \text{э})$, and hence, in the special case of $\vartheta = 0$, condition (2.6) implies that the material should have a strictly positive tangential modulus under simple loading,[4] $d\sigma/d\text{э} = \Phi'(\text{э}) \ge 3G\mu = \text{const} > 0$; on the other hand, the condition that the function f is bounded restricts the values of the tangential modulus from above. It follows from (2.5) and (2.6) that $d\boldsymbol{\sigma}$ is the sum of a vector directed along $d\boldsymbol{\text{э}}$ and a vector opposite to $\boldsymbol{\sigma}$; thus, $d\boldsymbol{\sigma}$ tends to rotate $\boldsymbol{\sigma}$ towards $d\boldsymbol{\text{э}}$, which, in particular, agrees with the fact that $\boldsymbol{\sigma}$ and $d\boldsymbol{\text{э}}$ are codirected on deformation trajectories of small curvature and there is a certain angle between $\boldsymbol{\sigma}$ and $d\boldsymbol{\text{э}}$ on trajectories of constant medium and large curvature. Condition (2.5) is close to the inequality

$$f(\vartheta, \sigma, s, \text{э}) \le 3G_u \cos\vartheta \tag{2.13}$$

(G_u is the shear modulus of the material under unloading if the unloading is assumed to be linear), which implies the validity of the thermodynamic condition that the dissipation power is negative,

$$\sigma_{ij}\,d\varepsilon_{ij}^{(P)} = S_{ij}\,d\text{Э}_{ij}^{(P)} = (\boldsymbol{\sigma} \cdot d\boldsymbol{\text{э}}^{(P)}) \equiv \left(\boldsymbol{\sigma} \cdot \left[d\boldsymbol{\text{э}} - \frac{d\boldsymbol{\sigma}}{3G_u}\right]\right) = \left[\cos\vartheta - \frac{f(\vartheta, \sigma, s, \text{э})}{3G_u}\right]\sigma|d\boldsymbol{\text{э}}| \ge 0.$$

When the experimental approximations to the material functions g and f are obtained, one should verify conditions (2.5)–(2.8) and (2.12) (for example, by numerical methods). Any violation of these

[4] If the material has a yield plateau, then, when approximating the simple loading function $\Phi(\text{э})$, the yield plateau can be replaced by a linear strengthening segment with a sufficiently small positive slope.

conditions can lead to erroneous results (divergence, nonuniqueness, absence of physical meaning) when solving boundary value problems.

It follows from the results in [27] that if the conditions of the above-stated theorem (about the generalized solution of the boundary value problem (2.2)) are satisfied, then the initial–boundary value problem (2.1) with external loads in the classes $X_i(t,\mathbf{x}) \in L_2([0,T], L_p(\Omega))$ and $T_i(t,\mathbf{x}) \in L_2([0,T], L_q(\Gamma_2))$ has a unique generalized solution $\mathbf{u} \in L_2([0,T], H(\Omega))$. This solution continuously depends on the external loads (with respect to the norm of the space $L_2([0,T], H^*(\Omega))$), and the sequence of solutions obtained by the step method converges to this solution as the time step tends to zero. Here $L_2([0,T], H(\Omega))$ is the space of measurable functions $\mathbf{u}(t)$ on $[0,T]$ ranging in the space $H(\Omega)$ with inner product

$$(\mathbf{u}_1, \mathbf{u}_2)_{L_2([0,T],H(\Omega))} = \int_0^T (\dot{\mathbf{u}}_1(t), \dot{\mathbf{u}}_2(t))_{H(\Omega)} dt, \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in L_2([0,T], H(\Omega)).$$

When solving the initial–boundary value problems in our theory, it is expedient to use the explicit step scheme

$$\sigma_{ij}(t_{n+1}) = \sigma_{ij}(t_n) + K[\varepsilon_{mm}(t_{n+1}) - \varepsilon_{mm}(t_n)]\delta_{ij} + \frac{2}{3}\Big\{ g(\sigma(t_n))[\text{Э}_{ij}(t_{n+1}) - \text{Э}_{ij}(t_n)]$$
$$- \varphi(\vartheta(t_{n-1}), \sigma(t_n), s(t_n), \text{э}(t_n))\dot{s}(t_{n-1})(t_{n+1} - t_n)\frac{S_{ij}(t_n)}{\sigma(t_n)} \Big\}, \tag{2.14}$$

where

$$\vartheta(t_{n-1}) = \arccos\left\{ \frac{S_{kl}(t_{n-1})[\text{Э}_{kl}(t_n) - \text{Э}_{kl}(t_{n-1})]}{\sigma(t_{n-1})[s(t_n) - s(t_{n-1})]} \right\}, \quad \dot{s}(t_{n-1}) = \frac{s(t_n) - s(t_{n-1})}{t_n - t_{n-1}}.$$

Then, at each time step, one needs to solve a linear (quasi-elastic) boundary value problem instead of solving a nonlinear boundary value problem (for example, by the method of elastic solutions (2.11)) if an implicit step scheme were used; the latter differs from (2.13) by the replacement of $\vartheta(t_{n-1})$, $\dot{s}(t_{n-1})$ by $\vartheta(t_n)$, $\dot{s}(t_n)$. If the explicit scheme (2.13) is used, then the time step on the intervals of sharp variation in the external load should be sufficiently small. If the scheme (2.13) is used, then the problems of convergence and accuracy of the approximate solution of the initial–boundary value problem are solved by comparing the solutions obtained on several time meshes with different mesh spacings (but with the same division of the body into finite elements).

For the scheme (2.13), the tensor of tangential moduli (the material Jacobian) formally coincides with the tensor of moduli of elasticity of an isotropic linearly elastic material with bulk modulus K and shear modulus $g(\sigma)/3$,

$$\frac{\partial \Delta\sigma_{ij}}{\partial \Delta\varepsilon_{kl}} = \left[K - \frac{2g(\sigma)}{9}\right]\delta_{ij}\delta_{kl} + \frac{g(\sigma)}{3}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{2.15}$$

The computations of the components of the stress tensor $\sigma_{ij}(t+\Delta t)$ (2.13) and the material Jacobian (2.14) from the given components of the tensors $\sigma_{ij}(t)$, $\varepsilon_{ij}(t)$, and $\Delta\varepsilon_{ij}(t)$ and known scalar parameters of the state (at the beginning of the current time step) were realized as the subroutine UMAT, which is intended for implementing nonstandard models of the mechanical behavior of materials in the ABAQUS software for computations by the finite element method. The computations of the state parameters $\sigma(t+\Delta t)$, $s(t+\Delta t)$, $\text{э}(t+\Delta t)$, $\vartheta(t)$, and $\dot{s}(t)$ at the end of the current time interval were also implemented in the same subroutine. Thus, a mathematical apparatus was constructed for finite-element computations of the structure deformation according to the scheme (1.7), (1.8). The problems of physical reliability of the proposed theory (and hence of the solutions obtained using this theory) will be discussed in Part 2 of this paper.

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