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Generalized Theory of Elasticity

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Abstract—We obtain elasticity equations of higher (in the general case, infinite) order than the equations of the classical theory. In contrast to the numerous known versions of the nonclassical theory (Cosserat, nonsymmetric, microstructure, micropolar, multipolar, and gradient), which also result in higher-order equations and contain elasticity relations for traditional and couple stresses with a large number of elastic constants, our theory, regardless of the order of the equations, contains only one additional constant, which can be expressed in terms of the microstructure parameter of the medium. The basic equations of the generalized theory are presented for one-, two-, and three-dimensional problems; these equations take into account the stress gradients and can be written in terms of generalized stresses, strains, and displacements. A boundary value problem that does not require the introduction of couple stresses is stated for the generalized theory of elasticity.

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1. INTRODUCTION

To show that one has to generalize the classical theory of elasticity, consider the problem proposed by Reissner [1] and studied earlier in [2]. We study the plane stress state of a fixed quarter-space loaded by constant tangential stresses τ_0 (Fig. 1). The exact solution of this problem has the form [1, 2]

$$\sigma_x = \tau_0\left(\frac{xy}{x^2+y^2} + \arctan\frac{y}{x} - \frac{\pi}{2}\right), \quad \sigma_y = -\tau_0\left(\frac{xy}{x^2+y^2} - \arctan\frac{y}{x} - \frac{\pi}{2}\right),$$
$$\tau_{xy} = \tau_{yx} = \tau_0\frac{y^2}{x^2+y^2}. \tag{1.1}$$

The stresses (1.1) satisfy the equilibrium equations

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} = 0, \quad \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{yx}}{\partial x} = 0, \quad \tau_{xy} = \tau_{yx}, \tag{1.2}$$

the strain consistency equation, and the boundary conditions $\sigma_x = 0$, $\tau_{xy} = \tau_0$ for $x = 0$ and $\sigma_y = 0$, $\tau_{yx} = 0$ for $y = 0$. It is important that the stresses (1.1) are finite functions in the entire domain, that is, for $0 \le x < \infty$ and $0 \le y < \infty$. A specific feature of this problem is that the stress tensor symmetry is violated at the origin $x = 0$, $y = 0$, because $\tau_{xy}(x = 0) \ne \tau_{yx}(y = 0)$. This specific feature leads to the following effects. First, the stress derivatives, which have the form

$$\frac{\partial\sigma_x}{\partial x} = -\frac{\partial\tau_{yx}}{\partial y} = -\frac{2\tau_0x^2y}{(x^2+y^2)^2}, \quad \frac{\partial\sigma_y}{\partial y} = -\frac{\partial\tau_{yx}}{\partial x} = -\frac{2\tau_0xy^2}{(x^2+y^2)^2},$$

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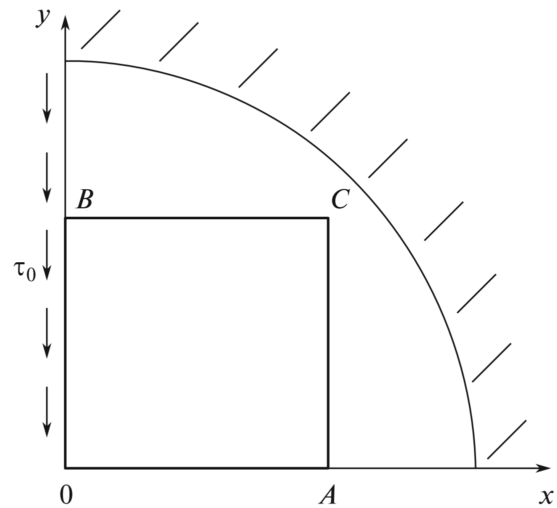

Fig. 1.

satisfy the equilibrium equations (1.2) but are infinite at the origin. We have $\sigma_x = 0$ at point O and $\sigma_x = -\pi\tau_0/2$ at point A (Fig. 1). Thus, the stresses at points O and A differ by a finite value regardless of the distance OA, which can be infinitely small. Second, the moment equation for the element $OACB$ in Fig. 1 is satisfied not owing to the stress tensor symmetry but because the normal and tangential stresses are nonuniformly distributed on the sides AC and BC of this element. Thus, the stresses at point O are not differentiable functions and do not satisfy the symmetry conditions, and hence the equilibrium equations (1.2), obtained under the assumption that the stress distributions are uniform on the sides of an infinitely small element and that the stress tensor is symmetric, do not hold in a neighborhood of the origin.

The problem under study is well posed in the framework of the Cosserat theory of elasticity [3]. A survey of the Cosserat and other versions of the nonclassical theory of elasticity and the corresponding references can be found in [4–11]. A specific feature of these theories is that the elasticity relations contain a system of additional elastic constants whose experimental determination encounters certain difficulties. In particular, the classical theory of elasticity of isotropic bodies contains two independent constants, but the Cosserat theory contains four constants, the Cosserat theory with microrotations taken into account contains six constants, the microstructure theory contains 15 constants, and the number of such constants in the multipolar and gradient theories depends on the level of the theory.

In the present paper, we propose a nonclassical theory that contains one additional constant regardless of the order of the equations. We introduce generalized stresses taking into account the gradients of traditional stresses; this determines the specific characteristics of the proposed theory compared with the known versions of gradient theories based on the strain gradients [8].

2. UNIAXIAL STRESS STATE

To demonstrate the specific features of the proposed theory, consider the uniaxial stress state. We single out a small but finite element $-a/2 \le x \le a/2$ (Fig. 2) whose sides $x = \pm a/2$ are subjected to the stress σ_x. We represent σ_x by a Maclaurin series to obtain

$$\sigma_x^{\pm} = \sigma_x \pm \frac{a}{2}\sigma_x' + \frac{a^2}{2!2^2}\sigma_x'' \pm \frac{a^3}{3!2^3}\sigma_x^{(3)} + \frac{a^4}{4!2^4}\sigma_x^{(4)} \pm \frac{a^5}{5!2^5}\sigma_x^{(5)} + \cdots . \tag{2.1}$$

Here $(*)' = d(*)/dx$ and $\sigma_x^{\pm} = \sigma_x(x = \pm a/2)$. The equilibrium equation for the element shown in Fig. 2 has the form

$$\sigma_x^{+} - \sigma_x^{-} = \sigma_x' + \frac{a^2}{3!2^2}\sigma_x^{(3)} + \frac{a^4}{5!2^4}\sigma_x^{(5)} + \cdots = 0. \tag{2.2}$$

If the derivatives of the stresses are finite and a is small, then it suffices to preserve only the first term in Eq. (2.2), and Eq. (2.2) degenerates into the equilibrium equation of the classical elasticity. But we

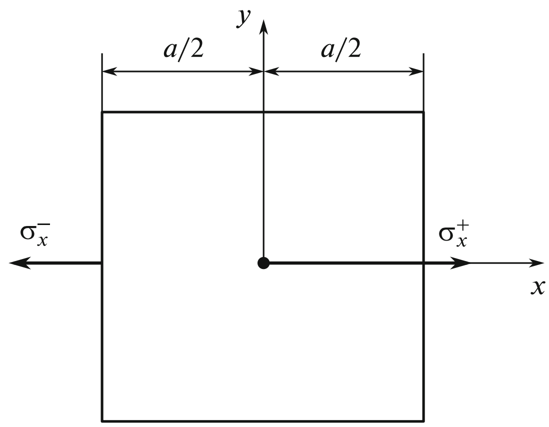

Fig. 2.

shall consider Eq. (2.2) as an equation which generally has infinitely high order and contains a small parameter a^2.

We introduce the following important hypothesis. Since the small parameter a^2 occurs in Eq. (2.2) as a coefficient multiplying the higher-order derivatives, the structure of the solution of this equation is known. It consists of a penetrating solution which follows from Eq. (2.2) for $a^2 = 0$ and a system of boundary layer type solutions with increasing variability factors. It is also known that the solutions of the second type must be exponential. However the structure of Eq. (2.2) shows that these solutions have trigonometric form, which contradicts the physical meaning of the problem. It is important that the proposed theory is phenomenological; i.e., it assumes that the medium characteristics should be determined experimentally for macroscopic specimens. In this case, the parameter A (Fig. 1) is determined experimentally as well, but there are grounds to believe that this parameter is a microstructure parameter of the medium which cannot be determined in principle by using a phenomenological theory and macroscopic experiments. To eliminate the possibility of determining the parameter a in the framework of the proposed theory, we assume

$$a = 2\sqrt{6}ir, \tag{2.3}$$

where i is the imaginary unit and r is an experimentally determinable constant. Thus, the parameter a is imaginary and, in contrast to the parameter r, cannot be determined by experiments traditional in the phenomenological theory. As a result, the equilibrium equation (2.2) acquires the definitive form

$$L(\sigma_x) = \sigma_x' - r^2\sigma_x^{(3)} + \frac{r^4}{80}\sigma_x^{(5)} - \cdots = 0. \tag{2.4}$$

This equation has a traditional penetrating solution obtained for $r = 0$ and a system of exponentially decaying solutions of the boundary layer type.

The stress is related to the strain by the traditional Hooke law containing one elastic constant in the case of uniaxial extension, i.e., the modulus of elasticity; namely,

$$\sigma_x = E\varepsilon_x, \quad \varepsilon_x = u'. \tag{2.5}$$

We substitute relations (2.5) into Eq. (2.4) and obtain

$$u'' - r^2u^{(4)} + \frac{r^4}{80}u^{(6)} - \cdots = 0. \tag{2.6}$$

To obtain the boundary conditions, we use the virtual work principle, which yields

$$\int L(\sigma_x)\delta u\, dx = 0.$$

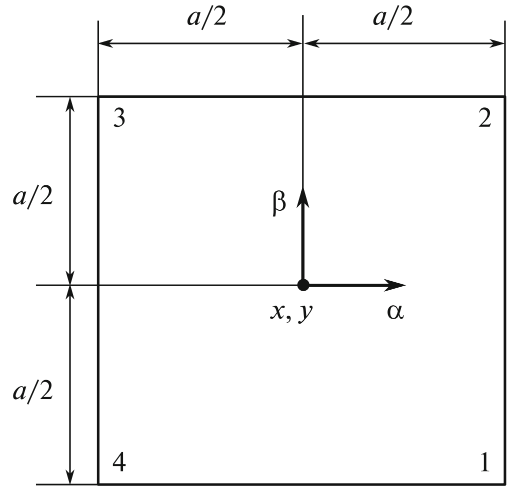

Fig. 3.

By substituting $L(\sigma_x)$ from Eq. (2.4) into this relation and by integrating by parts, we obtain the following boundary conditions:

$$\left(\sigma_x - r^2\sigma_x'' + \frac{r^4}{80}\sigma_x^{(4)} - \cdots\right)\delta u = 0, \quad \left(r^2\sigma_x' - \frac{r^4}{80}\sigma_x^{(3)} + \cdots\right)\delta u' = 0, \quad \left(\frac{r^4}{80}\sigma_x'' + \cdots\right)\delta u'' = 0.$$

Equation (2.6) and relations (2.5) allows us to write out the elastic energy variation

$$\delta U = \int\left(\sigma_x\delta\varepsilon_x + r^2\sigma_x'\delta\varepsilon_x' + \frac{r^4}{80}\sigma_x''\delta\varepsilon_x'' + \cdots\right)dx.$$

In conclusion, consider an alternative method for deriving the equilibrium equation (2.4), which deals with the plane and spatial problems and which we use in our subsequent considerations. Consider the element shown in Fig. 3 and introduce a local coordinate α such that $-a/2 \le \alpha \le a/2$ near the point (x, y). The stress acting at the point x is defined as

$$\bar{\sigma}_x(x) = \frac{1}{a}\int\limits_{-a/2}^{a/2}\sigma_x(x,\alpha)\,d\alpha. \tag{2.7}$$

We represent the stress $\sigma_x(x,\alpha)$ acting inside the element shown in Fig. 3 by a Taylor series around the point x; i.e.,

$$\sigma_x(x,\alpha) = \sigma_x(x) + \alpha\frac{\partial\sigma_x}{\partial x} + \frac{1}{2!}\alpha^2\frac{\partial^2\sigma_x}{\partial x^2} + \frac{1}{3!}\alpha^3\frac{\partial^3\sigma_x}{\partial x^3} + \frac{1}{4!}\alpha^4\frac{\partial^4\sigma_x}{\partial x^4} + \cdots. \tag{2.8}$$

By substituting the expansion (2.8) into (2.7) and by integrating, we obtain

$$\bar{\sigma}_x(x) = \sigma_x + \frac{a^2}{3!2^2}\sigma_x'' + \frac{a^4}{5!2^4}\sigma_x^{(4)} + \cdots. \tag{2.9}$$

In the absence of bulk forces, a condition for the conservation of a tensor field is the condition that the divergence of the field is zero [12], which has the form $\bar{\sigma}_x' = 0$ in the uniaxial stress state. By substituting the expansion (2.9) into this relation and by taking into account condition (2.3), we obtain Eq. (2.4). Note that the stresses (2.9) can be treated as generalized stresses taking into account the traditional stress gradients.

3. ANDREEV'S EXPERIMENTS

A series of Andreev's experiments with extendable plates whose width varies step by step (Fig. 4) was described in [13]. The stress was measured in the plate cross-sections AB and CD, and then Hooke's

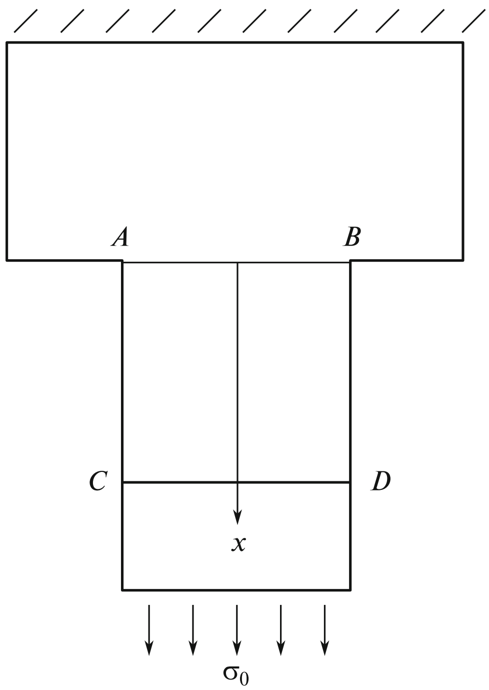

Fig. 4.

law was used to determine the stresses and the resultant stresses in these cross-sections. Numerous experiments discovered a paradoxical effect: the resultant stress in cross-section CD turned out to be equal to the applied force, and the resultant stress in cross-section AB turned out to be by 30% less than the applied force. In response to the remarks of opponents (one of whom was one of the authors of the present paper), Andreev carried out a complex of experimental studies to justify the discovered effect. In particular, the accuracy of strain measurements by the resistance transducers was confirmed by tests with plates with the distance AB equal to 300 mm. The strains were measured by 120 transducers with base 5 mm and with equal resistance. The strains were measured by an alternative method based on the use of a grid placed on the plate and by an optical method for measuring grid node displacements. In this case, the resultant stress in cross-section AB turned out to be 45% less than the applied force. The influence of the transverse strain was investigated; i.e., the variations in the plate thickness were measured in the process of experiment. The effect was observed for two limit forms of Hooke's law which correspond to the plane stress state and the plane strain. Finally, the effect was the same for various values of the load, which eliminates the influence of nonlinearity.

To obtain a qualitative explanation of Andreev's effect, we use Eq. (2.6) where we only preserve the first two terms; i.e.,

$$r^2 u^{\mathrm{IV}} - u^{\mathrm{II}} = 0.$$

The solution of this equation in the domain $x \geq 0$ in Fig. 4 corresponds to the stress

$$\sigma_x = \sigma_0 - rCe^{-x/r}, \tag{3.1}$$

where C is a constant of integration. The one-dimensional solution (3.1) only qualitatively describes the stress state of part $ABCD$ of the plate shown in Fig. 4. But it follows from this solution that $\sigma_x \approx \sigma_0$ in cross-section CD located at a certain distance from the cross-section $x = 0$, while we have $\sigma_x = \sigma_0 - rC$ in cross-section AB, which qualitatively agrees with the results of Andreev's experiments. Note that the constant C contained in the solution (3.1) can be found from the boundary conditions, and the constant r can in principle be determined from experiments similar to Andreev's experiment if the stress gradient is measured on segment AC (Fig. 4).

4. PLANE STRESS STATE

Consider the plane stress state described in the classical theory of elasticity by the equilibrium equations (1.2), the elasticity relations

$$\sigma_x = \bar{E}(\varepsilon_x + \nu\varepsilon_y), \quad \sigma_y = \bar{E}(\varepsilon_y + \nu\varepsilon_x), \quad \bar{E} = E(1-\nu^2), \tag{4.1}$$

$$\tau_{xy} = \tau_{yx} = G\gamma, \quad G = \frac{E}{2(1+\nu)}, \tag{4.2}$$

and the geometric relations

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \quad \omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{4.3}$$

determining the strains ε and γ and the rotation angle ω. Note that, with (4.3) taken into account, the elasticity relations (4.2) for the tangential stresses can be written as

$$\tau_{xy} = 2G\left(\frac{\partial v}{\partial x} - \omega\right), \quad \tau_{yx} = 2G\left(\frac{\partial u}{\partial y} - \omega\right). \tag{4.4}$$

To derive the equilibrium equations for the plane problem of generalized elasticity, we introduce the stress tensor $t(\sigma_x, \sigma_y, \tau_{xy}, \tau_{yx})$ at the point (x, y) and represent it by a Taylor series around this point (Fig. 3); i.e.,

$$\begin{aligned} t(x,y,\alpha,\beta) &= t(x,y) + \alpha\frac{\partial t}{\partial x} + \beta\frac{\partial t}{\partial y} + \frac{1}{2!}\left(\alpha^2\frac{\partial^2 t}{\partial x^2} + 2\alpha\beta\frac{\partial^2 t}{\partial x\partial y} + \beta^2\frac{\partial^2 t}{\partial y^2}\right) \\ &+ \frac{1}{3!}\left(\alpha^3\frac{\partial^3 t}{\partial x^3} + 3\alpha^2\beta\frac{\partial^3 t}{\partial x^2\partial y} + 3\beta^2\alpha\frac{\partial^3 t}{\partial y^2\partial x} + \beta^3\frac{\partial^3 t}{\partial y^3}\right) + \cdots . \end{aligned} \tag{4.5}$$

When constructing the two-dimensional theory, we preserve only the terms represented in Eq. (4.5). For example, we consider edges 1–2 and 3–4 of the element shown in Fig. 3. We set $\alpha = \pm a/2$ and determine the resultant force R and the moments M created by the stresses t acting on these edges,

$$\begin{aligned} R^{1-2}_{3-4}(t) &= \int_{-a/2}^{a/2} t\,d\beta = ta \pm \frac{a^2}{2}\frac{\partial t}{\partial x} + \frac{a^3}{8}\left(\frac{\partial^2 t}{\partial x^2} + \frac{1}{3}\frac{\partial^2 t}{\partial y^2}\right) \pm \frac{a^4}{48}\left(\frac{\partial^3 t}{\partial x^3} + \frac{1}{3}\frac{\partial^3 t}{\partial x\partial y^2}\right), \\ M^{1-2}_{3-4}(t) &= \int_{-a/2}^{a/2} t\beta\,d\beta = \frac{a^3}{12}\frac{\partial t}{\partial y} \pm \frac{a^4}{24}\frac{\partial^2 t}{\partial x\partial y} \pm \frac{a^5}{96}\left(\frac{\partial^3 t}{\partial x^2\partial y} + \frac{1}{5}\frac{\partial^3 t}{\partial y^3}\right). \end{aligned} \tag{4.6}$$

The expressions for the resultant force and the moment created by the stresses acting on edges 1–2 and 3–4 of the element shown in Fig. 3 can be obtained from relations (4.6) with x replaced by y and y replaced by x.

The equilibrium equations for the element in Fig. 3 have the form

$$\begin{aligned} &R_{1-2}(\sigma_x) - R_{3-4}(\sigma_x) + R_{2-3}(\tau_{yx}) - R_{1-4}(\tau_{xy}) = 0, \\ &R_{2-3}(\sigma_y) - R_{1-4}(\sigma_y) + R_{1-2}(\tau_{xy}) - R_{3-4}(\tau_{xy}) = 0, \\ &[R_{1-2}(\tau_{xy}) - R_{3-4}(\tau_{xy}) + R_{2-3}(\tau_{yx}) - R_{1-4}(\tau_{yx})]a \\ &\qquad + M_{1-2}(\sigma_x) - M_{3-4}(\sigma_x) + M_{2-3}(\sigma_y) - M_{1-4}(\sigma_y) = 0. \end{aligned} \tag{4.7}$$

By substituting (4.6) into (4.7), we obtain

$$\begin{aligned} &\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{a^2}{24}\frac{\partial}{\partial x}\Delta(\sigma_x + \tau_{yx}) = 0, \\ &\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{a^2}{24}\frac{\partial}{\partial y}\Delta(\sigma_y + \tau_{xy}) = 0, \end{aligned} \tag{4.8}$$

$$\tau_{xy} - \tau_{yx} + \frac{a^2}{24}\Delta(\tau_{xy} - \tau_{yx}) + \frac{a^2}{12}\left(\frac{\partial^2 \tau_{xy}}{\partial x^2} - \frac{\partial^2 \tau_{yx}}{\partial y^2}\right) + \frac{a^2}{12}\frac{\partial^2}{\partial x \partial y}(\sigma_x - \sigma_y) = 0,$$
$$\Delta(\ldots) = \frac{\partial^2(\ldots)}{\partial x^2} + \frac{\partial^2(\ldots)}{\partial y^2}. \tag{4.9}$$

We use Eqs. (4.8) to eliminate the last term from the right-hand side of Eq. (4.9). As a result, we obtain

$$\tau_{xy} - \tau_{yx} + \frac{a^2}{24}\Delta(\tau_{xy} - \tau_{yx}) + \frac{a^4}{288}\left[\frac{\partial^2}{\partial x \partial y}(\sigma_x - \sigma_y) + \frac{\partial^2}{\partial y^2}\Delta\tau_{yx} - \frac{\partial^2}{\partial x^2}\Delta\tau_{xy}\right] = 0. \tag{4.10}$$

Since Eqs. (4.8) contain terms with coefficients 1 and a^2, we can neglect the last term in Eq. (4.10) containing a^4 and rewrite this equation as

$$\tau_{xy} - \tau_{yx} + \frac{a^2}{24}\Delta(\tau_{xy} - \tau_{yx}) = 0. \tag{4.11}$$

Equations (4.8) and (4.11) can also be obtained by using the approach described in the concluding part of Section 2. We introduce the generalized stress $\bar{t}$ by analogy with the expression (2.7); i.e.,

$$\bar{t}(x, y) = \frac{1}{a^2}\int\limits_{-a/2}^{a/2} d\alpha \int\limits_{-a/2}^{a/2} t(x, y, \alpha, \beta)\, d\beta.$$

By substituting the expansion (4.5) into this relation, we obtain

$$\bar{t}(x, y) = t + \frac{a^2}{24}\Delta t. \tag{4.12}$$

According to the general conditions for the conservation of a tensor field [7], the divergence of the tensor $\bar{t}(x, y)$ must be zero, which corresponds to the equilibrium equations for the element shown in Fig. 3 in the direction of the x- and y-axes. Moreover, the tensor $\bar{t}(x, y)$ should satisfy the symmetry condition, which ensures that the moment equation is satisfied. As a result, we obtain the equilibrium equations (4.8) and (4.11).

We take into account condition (2.3) and introduce the generalized stress by the formulas

$$\bar{\sigma}_{x,y} = \sigma_{x,y} - r^2\Delta\sigma_{x,y}, \quad \bar{\tau}_{xy,yx} = \tau_{xy,yx} - r^2\Delta\tau_{xy,yx}. \tag{4.13}$$

Then the equilibrium equations (4.8) and (4.11) finally become

$$\frac{\partial \bar{\sigma}_x}{\partial x} + \frac{\partial \bar{\tau}_{yx}}{\partial y} = 0, \quad \frac{\partial \bar{\sigma}_y}{\partial y} + \frac{\partial \bar{\tau}_{xy}}{\partial x} = 0, \quad \bar{\tau}_{xy} - \bar{\tau}_{yx} = 0, \tag{4.14}$$

and coincide in form with the equilibrium equations of the classical theory of elasticity.

Consider the elasticity relations. The considered theory preserves relation (4.1) between the normal stresses and strains, which was obtained in tension experiments with macrospecimens. But it is necessary to generalize the elasticity relation (4.2) for the tangential stress, because $\tau_{xy} \neq \tau_{yx}$ in the proposed theory in the general case. To generalize relations (4.4), instead of ω, we introduce an unknown angle of rotation θ independent of displacements. Then

$$\tau_{xy} = 2G\varepsilon_{xy}, \quad \tau_{yx} = 2G\varepsilon_{yx}, \tag{4.15}$$

$$\varepsilon_{xy} = \frac{\partial v}{\partial x} - \theta, \quad \varepsilon_{yx} = \frac{\partial u}{\partial y} - \theta. \tag{4.16}$$

We represent the tangential stresses as the sum of symmetric and antisymmetric components; i.e.,

$$\tau_{xy} = \tau_{\mathrm{s}} + \tau_{\mathrm{a}}, \quad \tau_{yx} = \tau_{\mathrm{a}} + \tau_{\mathrm{s}}, \quad \tau_{\mathrm{s}} = G\gamma, \quad \tau_{\mathrm{a}} = 2G(\omega - \theta), \tag{4.17}$$

where γ and ω are given by formulas (4.3). Thus, the elasticity relations in the proposed generalized theory are determined by equations (4.1) and (4.15).

As above in Section 2, to write out the boundary conditions, we use the virtual work principle, which yields

$$\iint\left[\left(\frac{\partial\bar\sigma_x}{\partial x}+\frac{\partial\bar\tau_{yx}}{\partial y}\right)\delta u+\left(\frac{\partial\bar\sigma_y}{\partial y}+\frac{\partial\bar\tau_{xy}}{\partial x}\right)\delta v+(\bar\tau_{xy}-\bar\tau_{yx})\delta\theta\right]dx\,dy=0.$$

By using the geometric relations (4.3) and (4.16) and by integrating by parts, we obtain an expression for the variation in the potential deformation energy,

$$\begin{aligned}\delta U=\iint\bigg\{&\sigma_x\delta\varepsilon_x+\sigma_y\delta\varepsilon_y+\tau_{xy}\delta\varepsilon_{xy}+\tau_{yx}\delta\varepsilon_{yx}\\&+\frac{c^2}{24}\left[\frac{\partial\sigma_x}{\partial x}\delta\left(\frac{\partial\varepsilon_x}{\partial x}\right)+\frac{\partial\sigma_x}{\partial y}\delta\left(\frac{\partial\varepsilon_x}{\partial y}\right)+\frac{\partial\sigma_y}{\partial y}\delta\left(\frac{\partial\varepsilon_y}{\partial y}\right)+\frac{\partial\sigma_y}{\partial x}\delta\left(\frac{\partial\varepsilon_y}{\partial x}\right)\right.\\&\left.+\frac{\partial\tau_{xy}}{\partial x}\delta\left(\frac{\partial\varepsilon_{xy}}{\partial x}\right)+\frac{\partial\tau_{xy}}{\partial y}\delta\left(\frac{\partial\varepsilon_{xy}}{\partial y}\right)+\frac{\partial\tau_{yx}}{\partial x}\delta\left(\frac{\partial\varepsilon_{yx}}{\partial x}\right)+\frac{\partial\tau_{yx}}{\partial y}\delta\left(\frac{\partial\varepsilon_{yx}}{\partial y}\right)\right]\bigg\}\,dx\,dy,\end{aligned}$$

the boundary conditions

$$\begin{aligned}&\bar\sigma_x\delta u=0,\quad \bar\tau_{xy}\delta v=0\quad \text{for } x=\text{const},\\&\bar\sigma_y\delta v=0,\quad \bar\tau_{yx}\delta u=0\quad \text{for } y=\text{const}\end{aligned}\tag{4.18}$$

similar to the traditional ones, and the additional conditions

$$\begin{aligned}&r^2\frac{\partial\sigma_x}{\partial x}\delta\varepsilon_x=0,\quad r^2\frac{\partial\tau_{xy}}{\partial x}\delta\varepsilon_{xy}=0,\quad r^2\frac{\partial\tau_{yx}}{\partial x}\delta\varepsilon_{yx}=0\quad\text{for } x=\text{const},\\&r^2\frac{\partial\sigma_y}{\partial y}\delta\varepsilon_y=0,\quad r^2\frac{\partial\tau_{xy}}{\partial y}\delta\varepsilon_{xy}=0,\quad r^2\frac{\partial\tau_{yx}}{\partial y}\delta\varepsilon_{yx}=0\quad\text{for } y=\text{const}.\end{aligned}\tag{4.19}$$

We introduce the generalized displacements and angles of rotation

$$(\bar u,\bar v)=(u,v)-r^2\Delta(u,v),\quad (\bar\omega,\bar\theta)=(\omega,\theta)-r^2\Delta(\omega,\theta).\tag{4.20}$$

By using the elasticity relations (4.1), (4.15) and the geometric relations (4.3), (4.16), we can obtain the following relations between the generalized stresses (4.13) and displacements (3.20):

$$\begin{aligned}&\bar\sigma_x=\bar E\left(\frac{\partial\bar u}{\partial x}+\nu\frac{\partial\bar v}{\partial y}\right),\quad \bar\sigma_y=\bar E\left(\frac{\partial\bar v}{\partial y}+\nu\frac{\partial\bar u}{\partial x}\right),\\&\bar\tau_{xy}=2G\left(\frac{\partial\bar v}{\partial x}-\bar\omega\right),\quad \bar\tau_{yx}=2G\left(\frac{\partial\bar u}{\partial y}+\bar\omega\right).\end{aligned}\tag{4.21}$$

Together with the equilibrium equations (4.14), relations (4.21) form a complete system of equations coinciding with the equations of the classical theory of elasticity. If the boundary conditions are static, then the boundary value problem of the considered theory coincides with the first basic problem of the classical theory of elasticity. But this is not true in the general case. In contrast to the classical theory, the solution of the problem in the generalized theory determines the generalized stresses and displacements (4.13) and (4.20) rather than the traditional ones. By integrating Eqs. (4.13) and (4.21) for known generalized stresses and displacements with the boundary conditions (4.19), we can obtain traditional stresses and displacements. With (4.13) and (4.17) taken into account, we can rewrite the boundary conditions (4.18) and (4.19) for the boundary $x=\text{const}$ as follows:

$$\begin{aligned}&(\sigma_x-r^2\Delta\sigma_x)\delta u=0,\quad(\tau_{xy}-r^2\Delta\tau_{xy})\delta v=0,\\&r\frac{\partial\sigma_x}{\partial x}\delta\varepsilon_x=0,\quad r^2\frac{\partial\tau_\text{s}}{\partial x}\delta\gamma=0,\quad r^2\frac{\partial\tau_\text{a}}{\partial x}\delta(\omega-\theta)=0,\end{aligned}\tag{4.22}$$

where τ_s and τ_a are determined by formulas (4.17).

5. GENERAL CASE OF STRESS STATE

In the general case, the equilibrium equations (4.7) of the plane problem can be generalized as

$$\begin{aligned} L_i &= \sigma_{ij,j} - r^2\Delta\sigma_{ij,j} + \cdots = 0, \\ M_k &= \sigma_{ij}\varepsilon_{ijk} - r^2\Delta(\sigma_{ij}\varepsilon_{ijk}) + \cdots = 0. \end{aligned} \tag{5.1}$$

Here ε_{ijk} is the Levi-Civita tensor. To state the boundary value problem in displacements and write out the boundary conditions, just as earlier in Sections 2 and 4, we use the virtual work principle. Consider the variational form

$$\iiint \{L_i\delta u_i + M_i\delta\theta_i\}\, dV = \iiint \big\{[\sigma_{ij,j} - r^2\Delta\sigma_{ij,j}]\delta u_i + [\sigma_{mn}\varepsilon_{mni} - r^2\Delta(\sigma_{mn}\varepsilon_{mni})]\delta\theta_i\big\}\, dV, \tag{5.2}$$

where u_i is the displacement vector, θ_k is the vector of rotations unrelated to the displacements u_i, and V is the volume of the body. By integrating by parts in (5.2), we obtain

$$\begin{aligned} &\iiint \{L_i\delta u_i + M_i\delta\theta_i\}\, dV \\ &\quad = -\iiint \big\{\sigma_{ij}\delta u_{i,j} + r^2\sigma_{ij,k}\delta u_{i,jk} + (\sigma_{mn}\varepsilon_{mni})\delta\theta_i + r^2(\sigma_{mn}\varepsilon_{mni})_{,k}\delta\theta_{i,k}\big\}\, dV \\ &\quad + \oint \big\{[\sigma_{ij} - r^2\sigma_{ij,kk}]n_j\delta u_i + [r^2\sigma_{ij,k}n_k]\delta u_{i,j} + r^2(\sigma_{mn}\varepsilon_{mni})_{,k}n_k\delta\theta_i\big\}\, dF. \end{aligned} \tag{5.3}$$

Here n_j is the unit normal vector to the smooth part of the surface F bounding the body volume V. The first term on the right-hand side in (5.3) is the expression for the potential energy

$$\delta U = \iiint \big\{\sigma_{ij}\delta u_{i,j} + r^2\sigma_{ij,k}\delta u_{i,jk} + (\sigma_{mn}\varepsilon_{mni})\delta\theta_i + r^2(\sigma_{mn}\varepsilon_{mni})_{,k}\delta\theta_{i,k}\big\}\, dV, \tag{5.4}$$

and the second term allows us to write out the natural boundary conditions. The expression for the virtual work principle can also be written as

$$\iiint \{L_i\delta u_i + M_i\delta(\theta_i - \omega_i)\}\, dV = 0, \quad \omega_k = -\frac{1}{2}u_{i,j}\varepsilon_{ijk},$$

where ω_k is the vector of rotations in the classical theory of elasticity. Then the boundary conditions can be represented in the form completely coinciding with the boundary conditions (4.22) written earlier for the plane problem. For the smooth surface with the normal n_i, which is the boundary of the body, we have

$$[\sigma_{ij} - r^2\sigma_{ij,kk}]n_j\delta u_i = 0, \quad [\sigma_{ij,k}n_k]\delta u_{i,j} = 0, \quad [(\sigma_{mn}\varepsilon_{mni})_{,k}n_k]\delta(\theta_i - \omega_i) = 0. \tag{5.5}$$

Consider formula (5.4) determining the potential energy variation. The latter can be written as

$$\delta U_V(u_{i,j}, u_{i,jk}, \theta_k, \theta_{k,p}) = \iiint \left[\frac{\partial U_V}{\partial u_{i,j}}\delta u_{i,j} + \frac{\partial U_V}{\partial u_{i,jk}}u_{i,jk} + \frac{\partial U_V}{\partial \theta_k}\delta\theta_k + \frac{\partial U_V}{\partial \theta_{k,p}}\delta\theta_{k,p}\right] dV. \tag{5.6}$$

Formula (5.6) allows us to determine the elasticity relations. With (5.4) taken into account, we obtain

$$\sigma_{ij} = \frac{\partial U_V}{\partial u_{i,j}}, \quad \sigma_{ij,k} = \frac{1}{r^2}\frac{\partial U_V}{\partial u_{i,jk}}, \quad \sigma_{ij}\ni_{ijk} = \frac{\partial U_V}{\partial \theta_k}, \quad \sigma_{ij,p}\ni_{ijk} = \frac{1}{r^2}\frac{\partial U_V}{\partial \theta_{k,p}}. \tag{5.7}$$

In the proposed model, the variational form (5.2) dictates a special structure of the potential energy and the respective special form of physical relations (5.7). One can see that relations (5.7) are consistent and imply consistent physical relations if the potential energy has the form

$$U = \frac{1}{2}\iiint \big\{C_{ijmn}\varepsilon_{ij}\varepsilon_{mn} + C_{ijmn}r^2\delta_{kl}\varepsilon_{ij,k}\varepsilon_{mn,l} + 2G(\omega_i - \theta_i) + 2Gr^2(\omega_{i,j} - \theta_{i,j})(\omega_{i,j} - \theta_{i,j})\big\}\, dV,$$

where $C_{ijmn} = \lambda\delta_{ij}\delta_{mn} + \mu(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})$, λ is the Lamé coefficient, and δ_{ij} is the Kronecker tensor. In this case, Green's formulas (5.7) for the stress tensor σ_{ij} become

$$\sigma_{ij} = [\lambda\delta_{ij}\delta_{mn} + G(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})]u_{m,n} + G(\theta_k - \omega_k)\varepsilon_{ijk}, \quad \sigma_{mn}\varepsilon_{mni} = 2G(\theta_i - \omega_i). \tag{5.8}$$

Physical relations (5.8) completely correspond to the constitutive equations (4.20) written earlier for the plane problem.

System (5.1) of equilibrium equations, the elasticity relations (5.8), and the boundary conditions (5.5) completely determine the mathematical statement of the problem of generalized theory of elasticity.

6. CONCLUSION

In this paper, we constructed a new version of the classical theory of elasticity whose equations contain the generalized stresses which take into account the gradients of the traditional stresses and contain one additional (to the moduli E and G) constant which needs to be determined experimentally. For problems with static boundary conditions, the corresponding solutions coincide with the classical ones if the stress tensor symmetry conditions are not violated in the boundary conditions. For example, in the problem illustrated in Fig. 1, these conditions are violated at point O, and to solve this problem, one has to use the equations of the generalized theory. In problems with mixed and kinematic boundary conditions, the generalized theory should be used if the solutions of the corresponding problems of the classical theory of elasticity lead to a stress state with greater stress gradients. Such a problem is shown, for example, in Fig. 4; namely, near points A and B, the stress state rapidly varies with the coordinates. Finally, it seems expedient to use the generalized theory in problems whose classical solution is singular [2, 14]. In particular, numerous problems of the body stress state near cracks belong to this class.

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