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A CLASS OF MULTIPLICATIVE LATTICES

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Abstract. We study the multiplicative lattices L which satisfy the condition $a = (a : (a : b))(a : b)$ for all $a, b \in L$. Call them sharp lattices. We prove that every totally ordered sharp lattice is isomorphic to the ideal lattice of a valuation domain with value group $\mathbb{Z}$ or $\mathbb{R}$. A sharp lattice L localized at its maximal elements are totally ordered sharp lattices. The converse is true if L has finite character.

Keywords: multiplicative lattice; Prüfer lattice; Prüfer integral domain

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1. Introduction

We recall some standard terminology. A *multiplicative lattice* is a complete lattice $(L, \leqslant)$ (with bottom element 0 and top element 1) which is also a commutative monoid with identity 1 (the top element) such that

$$a\Big(\bigvee_{\alpha} b_\alpha\Big) = \bigvee_{\alpha}(ab_\alpha) \quad \text{for each } a, b_\alpha \in L.$$

When $x \leqslant y$ $(x, y \in L)$, we say that x is *below* y or that y is *above* x. An element x of L is *cancellative* if $xy = xz$ $(y, z \in L)$ implies $y = z$. For $x, y \in L$, $(y : x)$ denotes the element $\bigvee\{a \in L;\ ax \leqslant y\}$; so $(y : x)x \leqslant y$.

An element c of L is *compact* if $c \leqslant \bigvee S$, with $S \subseteq L$, implies $c \leqslant \bigvee T$ for some finite subset T of S (here $\bigvee W$ denotes the join of all elements in W). An element in L is *proper* if $x \neq 1$. When 1 is compact, every proper element is below some *maximal* element (i.e., maximal in $L - \{1\}$). Let $\mathrm{Max}(L)$ denote the set of maximal elements of L. By "(L, m) is local", we mean that $\mathrm{Max}(L) = \{m\}$. A proper element p is *prime* if $xy \leqslant p$ (with $x, y \in L$) implies $x \leqslant p$ or $y \leqslant p$. Every maximal element is

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prime, L is a (*lattice*) *domain* if 0 is a prime element. An element x is *meet-principal* (or *weak meet-principal*) if

$$y \wedge zx = ((y : x) \wedge z)x \quad \forall\, y, z \in L \quad (\text{or } (y : x)x = x \wedge y \ \forall\, y \in L).$$

An element x is *join-principal* (or *weak join-principal*) if

$$y \vee (z : x) = ((yx \vee z) : x) \quad \forall\, y, z \in L \quad (\text{or } (xy : x) = y \vee (0 : x) \ \forall\, x \in L).$$

And x is *principal* if it is both meet-principal and join-principal. If x and y are principal elements, then so is xy. The converse is also true if L is a lattice domain and $xy \neq 0$. In a lattice domain, every nonzero principal element is cancellative. The lattice L is *principally generated* if every element is a join of principal elements. Moreover, L is a *C-lattice* if 1 is compact, the set of compact elements is closed under multiplication and every element is a join of compact elements. In a C-lattice, every principal element is compact.

The C-lattices have a well behaved localization theory. Let L be a C-lattice and L^* the set of its compact elements. For $p \in L$ a prime element and $x \in L$, the *localization* of x at p is

$$x_p = \bigvee \{a \in L^*;\ as \leqslant x \text{ for some } s \in L^* \text{ with } s \nleqslant p\}.$$

Then $L_p := \{x_p;\, x \in L\}$ is again a lattice with multiplication $(x, y) \mapsto (xy)_p$, join $\{(b_\alpha)\} \mapsto (\bigvee b_\alpha)_p$ and meet $\{(b_\alpha)\} \mapsto (\bigwedge b_\alpha)_p$. For $x, y \in L$, we have:

- ▷ $x \leqslant x_p$, $(x_p)_p = x_p$, $(x \wedge y)_p = x_p \wedge y_p$, and $x_p = 1$ if and only if $x \nleqslant p$.
- ▷ $x = y$ if and only if $x_m = y_m$ for each $m \in \operatorname{Max}(L)$.
- ▷ $(y : x)_p \leqslant (y_p : x_p)$ with equality if x is compact.
- ▷ The set of compact elements of L_p is $\{c_p\colon\ c \in L^*\}$.
- ▷ A compact element x is principal if and only if x_m is principal for each $m \in \operatorname{Max}(L)$.

In [1] a study of sharp integral domains was done. An integral domain D is a *sharp domain* if whenever $A_1A_2 \subseteq B$ with A_1, A_2, B ideals of D, we have a factorization $B = B_1B_2$ with $B_i \supseteq A_i$ ideals of D, $i = 1, 2$. Moreover, sharp domains and some of their generalizations have been investigated by various authors, see also [8]. In the present paper we extend almost all results in [1] to the setup of multiplicative lattices. Our key definition is the following.

Definition 1. A lattice L is a *sharp lattice* if whenever $a_1a_2 \leqslant b$ with $a_1, a_2, b \in L$, we have a factorization $b = b_1b_2$ with $a_i \leqslant b_i \in L$, $i = 1, 2$.

In Section 2 we work in the setup of C-lattices (simply called lattices). After obtaining some basic facts (see Propositions 2 and 3), we show that if (L, m) is a local sharp lattice and $m = x_1 \vee \ldots \vee x_n$ with $x_1, \ldots, x_n$ join principal elements, then $m = x_i$ for some i, see Theorem 6. While a lattice whose elements are principal is trivially a sharp lattice (see Remark 5), the converse is true in a principally generated lattice whose elements are compact, see Corollary 8.

In Section 3, we work in the setup of C-lattice domains generated by principal elements (simply called lattices). It turns out that every nontrivial totally ordered sharp lattice is isomorphic to the ideal lattice of a valuation domain with value group $\mathbb{Z}$ or $\mathbb{R}$, see Theorem 16. A nontrivial sharp lattice L is Prüfer (i.e., its compacts are principal) of dimension one (see Theorem 17), thus, the localizations at its maximal elements are totally ordered sharp lattices. The converse is true if L has finite character (see Definition 18) because in this case $(a : b)_m = (a_m : b_m)$ for all $a, b \in L - \{0\}$ and $m \in \mathrm{Max}(L)$, see Proposition 19. A countable sharp lattice has all its elements principal, see Corollary 23.

For basic facts or terminology, our reference papers are [2] and [11].

2. Basic results

In this section, the term *lattice* means a C-lattice.

We begin by giving several characterizations for the sharp lattices. As usual, we say that a *divides* b (denoted $a \mid b$) if $b = ac$ for some $c \in L$.

Proposition 2. *For a lattice L the following statements are equivalent:*

(i) *L is sharp.*

(ii) *$a = (a : (a : b))(a : b)$ for all $a, b \in L$.*

(iii) *$(a : b) \mid a$ for all $a, b \in L$.*

(iv) *$(a : b) \mid a$ whenever $a, b \in L$, $0 < a < b < 1$ and a is not a prime.*

Proof. (i) $\Rightarrow$ (iii) Since $(a : (a : b))(a : b) \leqslant a$, and L is sharp, we have a factorization $a = a_1 a_2$ with $a_1 \geqslant (a : (a : b))$ and $a_2 \geqslant (a : b)$. We get

$$a_2 \leqslant (a : a_1) \leqslant (a : (a : (a : b))) = (a : b) \leqslant a_2,$$

where the equality is easy to check, so $(a : b) = a_2$ divides a.

(iii) $\Rightarrow$ (ii) From $a = x(a : b)$ with $x \in L$, we get $x \leqslant (a : (a : b))$, so

$$a = x(a : b) \leqslant (a : (a : b))(a : b) \leqslant a.$$

(ii) $\Rightarrow$ (i) Let $a_1, a_2, b \in L$ with $b \geqslant a_1 a_2$. By (ii) we get $b = (b : (b : a_1))(b : a_1)$ and clearly $a_1 \leqslant (b : (b : a_1))$ and $a_2 \leqslant (b : a_1)$.

(iv) ⇔ (iii) Follows from observing that:
(1) $(a : b) = (a : (a \vee b))$ and (2) $(a : b) \in \{a, 1\}$ if a is a prime. □

Proposition 3. *If L is a sharp lattice and $m \in L$ a maximal element, there is no element properly between m and m^2.*

Proof. If $m^2 < x < m$, then $(x : m) = m$, so $(x : (x : m)) = m$, thus $x = (x : m)(x : (x : m)) = m^2$, a contradiction, see Proposition 2. □

Recall that a ring R is a *special primary ring* if R has a unique maximal ideal M and if each proper ideal of R is a power of M, see [9], page 206.

Corollary 4. *The ideal lattice of a Noetherian commutative unitary ring R is sharp if and only if R is a finite direct product of Dedekind domains and special primary rings.*

Proof. Combine Propositions 2 and 3 and [6], Theorem 39.2, Proposition 39.4. □

Remark 5. Let L be a lattice.

(i) If all elements of L are weak meet principal, then L is sharp (see Proposition 2). In particular, this happens when $a \wedge b = ab$ for all $a, b \in L$.

(ii) If L is sharp, then every $p \in L - \{1\}$ whose only divisors are p and 1 is a prime element because $(p : b) = p$ or 1 for all $b \in L$ (see Proposition 2). The converse is not true. Indeed, let L be the lattice $0 < a < b < c < 1$ with $a^2 = b^2 = ab = 0$, $ac = a$, $bc = b$, $c^2 = c$. Here every $x \in L - \{c, 1\}$ has nontrivial factors, while the lattice is not sharp because $(a : b) = b$ does not divide a.

(iii) A finite lattice $0 < a_1 < \ldots < a_n < 1$, $n \geqslant 2$, is sharp provided $a_{i+1}^2 \geqslant a_i$ for $1 \leqslant i \leqslant n-1$. By Proposition 2 (iv), it suffices to show that whenever $(a_i : a_j) = a_k$ with $1 \leqslant i < j, k \leqslant n$, it follows that a_k divides a_i. Indeed, from $(a_i : a_j) = a_k$ we get $a_j a_k \leqslant a_i \leqslant a_{i+1}^2 \leqslant a_j a_k$, so $a_i = a_j a_k$.

(iv) Using similar arguments, it can be shown that a lattice whose poset is $0 < a < b < c < 1$ is sharp if and only if $c^2 \geqslant b$ and either $b^2 \geqslant a$ or ($b^2 = 0$ and $bc = a$). In this case, a computer search finds 13 sharp lattices out of 22 lattices.

We give the main result of this section.

Theorem 6. *Let L be a sharp lattice.*

(i) *If $x, y \in L$ are join principal elements and $(x : y) \vee (y : x) = x \vee y$, then $x \vee y = 1$.*

(ii) *If (L, m) is local and $m = x_1 \vee \ldots \vee x_n$ with $x_1, \ldots, x_n$ join principal elements, then $m = x_i$ for some i.*

P r o o f. (i) Since L is sharp and $(x \vee y)^2 \leqslant x^2 \vee y$, we can factorize $x^2 \vee y = ab$ with $x \vee y \leqslant a \wedge b$. Since x is join principal and $(y : x) \leqslant x \vee y$, we get

$$x \vee y \leqslant a \leqslant (x^2 \vee y) : b \leqslant (x^2 \vee y) : (x \vee y) = (x^2 \vee y) : x = x \vee (y : x) = x \vee y.$$

Thus $a = x \vee y = b$, as a and b play symmetric roles. So $x^2 \vee y = ab = (x \vee y)^2$. As y is join principal and $(x^2 : y) \leqslant (x : y) \leqslant x \vee y$, we finally get

$$1 = ((x^2 \vee xy \vee y^2) : y) = (x^2 : y) \vee x \vee y = x \vee y.$$

(ii) Suppose that $n \geqslant 2$ and no x_i can be deleted from the given representation $m = x_1 \vee \ldots \vee x_n$. It is straightforward to show that a factor lattice of a sharp lattice is again sharp. Modding out by $x_3 \vee \ldots \vee x_n$, we may assume that $n = 2$. As $(x_1 : x_2) \vee (x_2 : x_1) \leqslant m = x_1 \vee x_2$, we get a contradiction from (i). $\square$

Before giving an application of Theorem 6, we insert a simple lemma.

Lemma 7. *Let L be a sharp lattice and $p \in L$ a prime element. If L is sharp, then so is L_p.*

P r o o f. Let $a_1, a_2, b \in L$ with $(a_1 a_2)_p \leqslant b_p$. As L is sharp, we have $b_p = c_1 c_2$ for some $a_i \leqslant c_i \in L$ $(i = 1, 2)$, so $b_p = (c_1 c_2)_p$ and $(a_i)_p \leqslant (c_i)_p$. $\square$

Following [3], we say that a lattice L is *weak Noetherian* if it is principally generated and each $x \in L$ is compact.

Corollary 8. *Let L be a weak Noetherian lattice. Then L is sharp if and only if its elements are principal.*

P r o o f. The "only if part" is covered by Remark 5 (i). For the converse, pick an arbitrary maximal element $m \in L$. It suffices to prove that m is principal, see [3], Theorem 1.1. As m is compact, we can check this property locally (see [3], Lemma 1.1), so we may assume that L is local (see Lemma 7). Apply Theorem 6 (ii) to complete the proof. $\square$

3. Sharp lattice domains

In this section, the term *lattice* means a C-lattice domain generated by principal elements.

First we introduce an ad-hoc definition.

Definition 9. A lattice L is a *pseudo-Dedekind lattice* if $(x : a)$ is a principal element whenever $x, a \in L$ and x is principal.

Proposition 10. *Every sharp lattice is pseudo-Dedekind.*

Proof. The assertion follows from Proposition 2 because a factor of a nonzero principal element is principal [3], Lemma 2.3. □

Example 11. There exist pseudo-Dedekind lattices which are not sharp. For instance, let M be the (distributive) lattice of all ideals of the multiplicative monoid $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, see [2], page 138. Every $a \in M$ has the form $a = \bigcup\{y\mathbb{N}_0|\ y \in S\}$ for some $S \subseteq \mathbb{N}_0$. If $x \in \mathbb{N}_0$, then $(x\mathbb{N}_0 : a) = \bigcap\{(x\mathbb{N}_0 : y\mathbb{N}_0)\colon\ y \in S\} = z\mathbb{N}_0$ (for some $z \in \mathbb{N}_0$) is a principal element. So M is a pseudo-Dedekind lattice. But M is not sharp because for $a = 4\mathbb{N}_0 \cup 9\mathbb{N}_0$ and $b = 2\mathbb{N}_0 \cup 3\mathbb{N}_0$, we get $(a : b) = b^2$ and $(a : (a : b)) = b$, so $(a : b)(a : (a : b)) = b^3 \neq a$. See also [1], Example 8 for a ring-theoretic example of this kind.

A lattice L is a *Prüfer lattice* if every nonzero compact element of L is principal. It is well known (see [2], Theorem 3.4) that L is a Prüfer lattice if and only if L_m is totally ordered for each maximal element m.

Indeed, the "if part" follows from the fact that a locally principal nonzero compact element is principal. For the converse, we may assume that L is a Prüfer local lattice. Let a, b be principal nonzero elements of L. Then $a \vee b = c$ is compact, hence principal. We get $c = (a : c)c \vee (b : c)c = ((a : c) \vee (b : c))c$, so $1 = (a : c) \vee (b : c)$ since c is cancellative. As L is local, one of the terms, say $(a : c)$, equals 1, hence $b \leqslant c \leqslant a$. So every two principal elements are comparable, thus, L is totally ordered.

We show that a sharp lattice is Prüfer.

Remark 12. If L is a pseudo-Dedekind lattice, then the set P of all principal elements of L is a cancellative GCD monoid in the sense of [7], Section 10.2. Indeed, the LCM of two elements $x, y \in P$ is $x \wedge y = y(x : y)$.

Proposition 13. *Every sharp lattice is Prüfer.*

Proof. As L is principally generated, it suffices to show that $a \vee b$ is a principal element for each pair of nonzero principal elements $a, b \in L$. Dividing a, b by their GCD (see Remark 12), we may assume that $(a : b) = a$ and $(b : a) = b$. Then $a \vee b = 1$ (see Theorem 6). □

Example 14. Let $\mathbb{Z}_-$ denote the set of all integers $\leqslant 0$ together with the symbol $-\infty$. Then $\mathbb{Z}_-$ is a lattice under the usual addition and order. Note that $\mathbb{Z}_-$ is isomorphic to the ideal lattice of a discrete valuation domain, so $\mathbb{Z}_-$ is sharp.

Let $\mathbb{R}_1$ denote the set of all intervals $(r,\infty]$ and $[r,\infty]$ for $r \in \mathbb{R}_{\geqslant 0}$ together with $\{\infty\}$. Then $\mathbb{R}_1$ is a lattice under the usual interval addition and inclusion. To show that $\mathbb{R}_1$ is sharp, it suffices to check that $a = (a : (a : b))(a : b)$ for all $a, b \in \mathbb{R}_1 - \{\{\infty\}\}$ with $a \leqslant b$, see Proposition 2. This is done in the table below.

a	b	$(a : b)$	$(a : (a : b))$
$[r,\infty]$	$[t,\infty]$	$[r-t,\infty]$	$[t,\infty]$
$(r,\infty]$	$(t,\infty]$	$[r-t,\infty]$	$(t,\infty]$
$[r,\infty]$	$(t,\infty]$	$[r-t,\infty]$	$[t,\infty]$
$(r,\infty]$	$[t,\infty]$	$(r-t,\infty]$	$[t,\infty]$

Note that $\mathbb{R}_1$ is isomorphic to the ideal lattice of a valuation domain with value group $\mathbb{R}$.

We embark to show that every nontrivial totally ordered sharp lattice is isomorphic to $\mathbb{Z}_-$ or $\mathbb{R}_1$ above. Although the following lemma is known, we insert a proof for reader's convenience.

Lemma 15. *Let $L \neq \{0,1\}$ be a totally ordered lattice with maximal element m and $p \in L$, $0 \neq p \neq m$, a prime element. Then*

(i) *p is not principal.*
(ii) *$(z : (z : p)) = p$ for each nonzero principal element $z \leqslant p$.*
(iii) *If L is also pseudo-Dedekind, then* $\mathrm{Spec}(L) = \{0, m\}$.

Proof. As $p \neq m$, there exists a principal element $p < y \leqslant m$.

(i) As y is principal, we get $p = y(p : y) = yp$ because p is a prime so $p = (p : y)$. Hence, p is not cancellative, so it is not principal.

(ii) Let $z \leqslant p$ be a nonzero principal element. Note that $(z : (z : p)) \neq 1$, otherwise $zy = (z : p)y \geqslant (z : y)y = z$, so $zy = z$, a contradiction because z is cancellative. Since $p \leqslant (z : (z : p))$, it suffices to show that $x \nleqslant (z : (z : p))$ for each principal $x \nleqslant p$. As p is prime, we have $z \leqslant p < x^2$. If $x \leqslant (z : (z : p))$, then $x(z : p) \leqslant z$, so $z = x^2(z : x^2) \leqslant x^2(z : p) \leqslant zx$, hence $z = zx$, thus $x = 1$, a contradiction. $\square$

Theorem 16. *For a totally ordered lattice $L \neq \{0,1\}$, the following are equivalent:*

(i) *L is sharp.*
(ii) *L is pseudo-Dedekind.*
(iii) *L is isomorphic to $\mathbb{Z}_-$ or $\mathbb{R}_1$ of Example* 14.

P r o o f. (i) ⇒ (ii) follows from Proposition 10.

(ii) ⇒ (iii) Let m be the maximal element of L. Let G be the monoid of nonzero principal elements of L. Then G is a cancellative totally ordered monoid with respect to the opposite of the order induced from L. Let $a, b \in G$. Since L is totally ordered, we get that a divides b or b divides a. Moreover, since $\mathrm{Spec}(L) = \{0, m\}$ (see Lemma 15), a divides some power of b. By [5], Proposition 2.1.1, the quotient group of G can be embedded as an ordered subgroup K of $(\mathbb{R}, +)$; hence K is cyclic or dense in $\mathbb{R}$. If K is cyclic, it follows easily that L is isomorphic to $\mathbb{Z}_-$ of Example 14. Suppose that K is dense in $\mathbb{R}$, so there exists an ordered monoid embedding $v\colon G \to \mathbb{R}_{\geqslant 0}$ with dense image. We claim that v is onto. Deny, so there exists a positive real $g \notin v(G)$. Let $a \in G$ with $v(a) > g$ and set $b := \bigvee\{x \in G\colon v(x) \geqslant g\}$. Since L is pseudo-Dedekind, it follows that $c = (a : b)$ is a principal element. On the other hand, a straightforward computation shows that

$$c = \bigvee\{x \in G\colon v(x) \geqslant v(a) - g\}, \tag{3.1}$$

so $v(c) \geqslant v(a) - g$, in fact $v(c) > v(a) - g$ because $g \notin v(G)$. As K is dense in $\mathbb{R}$, there exists $d \in G$ with $v(c) > v(d) > v(a) - g$, so $c < d$. On the other hand, formula (1) gives $d \leqslant c$, a contradiction. It remains that $v(G) = \mathbb{R}_{\geqslant 0}$. Now it is easy to see that sending $[r, \infty]$ into $v^{-1}(r)$ and $(r, \infty]$ into $\bigvee\{x \in G\colon v(x) \geqslant r\}$ we get a lattice isomorphism from $\mathbb{R}_1$ to L.

(iii) ⇒ (i) follows from Example 14. □

We prove the main result of this paper.

Theorem 17. *Let $L \neq \{0, 1\}$ be a sharp lattice. Then L_m is isomorphic to $\mathbb{Z}_-$ or $\mathbb{R}_1$ (see Example 14) for every $m \in \mathrm{Max}(L)$ and L is a one-dimensional Prüfer lattice.*

P r o o f. As L is a Prüfer lattice (see Proposition 13), we may change L by L_m and thus assume that L is totally ordered and sharp (see Lemma 7). Apply Theorem 16 and Lemma 15 to complete. □

We extend the concepts of "finite character" and "h-local" from integral domains to lattices.

Definition 18. Let L be a lattice.

(i) L has *finite character* if every nonzero element is below only finitely many maximal elements.

(ii) L is *h-local* if it has finite character and every nonzero prime element is below a unique maximal element.

The next result extends [10], Lemma 3.8 to lattices.

Proposition 19. *Let L be an h-local lattice. If $a, b \in L - \{0\}$ and $m \in \operatorname{Max}(L)$, then $(a : b)_m = (a_m : b_m)$.*

P r o o f. We first prove two claims.

Claim 1: If $n \in \operatorname{Max}(L) - \{m\}$, then $a_n \nleqslant m$.

Suppose that $a_n \leqslant m$. Let S be the set of all products bc, where $b, c \in L$ are compact elements with $b \nleqslant m$ and $c \nleqslant n$. Note that S is multiplicatively closed. Moreover, a is not above any member of S. Indeed, if $bc \leqslant a$ and $c \nleqslant n$, then $b \leqslant a_n \leqslant m$. By [2], Theorem 2.2 and its proof, there exits a prime element $p \geqslant a$ such that p is not above any member of S. It follows that $p \leqslant m \wedge n$, which is a contradiction because L is h-local. Indeed, if $p \nleqslant m$, then $b \nleqslant m$ for a compact $b \leqslant p$, so $b = b \cdot 1 \in S$. Thus, Claim 1 is proved.

Claim 2: The element $s := \bigwedge\{a_n \colon n \in \operatorname{Max}(L),\ n \neq m\}$ is not below m.

Indeed, as L is h-local, a is below only finitely many maximal elements $n_1, \ldots, n_k$ distinct from m, hence $s = a_{n_1} \wedge \ldots \wedge a_{n_k}$. By Claim 1, s is not below m, thus Claim 2 is proved. To complete the proof, we use element s in Claim 2 as follows. We have

$$sb(a_m : b_m) \leqslant \bigwedge\{a_q \colon q \in \operatorname{Max}(L)\} = a,$$

so $s(a_m : b_m) \leqslant (a : b)$, hence $(a_m : b_m) \leqslant (a : b)_m$ because $s \nleqslant m$. Since clearly $(a : b)_m \leqslant (a_m : b_m)$, we get the result. □

Theorem 20. *For a finite character lattice $L \neq \{0, 1\}$, the following statements are equivalent:*

(i) *L is sharp.*

(ii) *L_m is isomorphic to $\mathbb{Z}_-$ or $\mathbb{R}_1$ (see Example 14) for every $m \in \operatorname{Max}(L)$.*

P r o o f. (i) ⇒ (ii) is covered by Theorem 17.

(ii) ⇒ (i) From (ii) we derive that L has Krull dimension one, so L is h-local. Let $a, b \in L - \{0\}$. It suffices to check locally the equality $a = (a : (a : b))(a : b)$. But this follows from Theorem 16 and Proposition 19. □

Say that elements a, b of a lattice L are *comaximal* if $a \vee b = 1$. The following result is [4], Lemma 4.

Lemma 21. *Let L be a lattice and $z \in L$ a compact element which is below infinitely many maximal elements. There exists an infinite set $\{a_n \colon n \geqslant 1\}$ of pairwise comaximal proper compact elements such that $z \leqslant a_n$ for each n.*

Proposition 22. *Any countable pseudo-Dedekind Prüfer lattice L has finite character.*

Proof. Suppose on the contrary there exists a nonzero element $z \in L$ which is below infinitely many maximal elements. Since L is principally generated, we may assume that z is principal. By Lemma 21, there exists an infinite set $(a_n)_{n \geqslant 1}$ of proper pairwise comaximal compact elements above z. As L is Prüfer, each a_n is principal. Since L is countable, we get $\tau := \bigwedge_{n \in A} a_n = \bigwedge_{n \in B} a_n$ for two nonempty subsets $B \nsubseteq A$ of $\mathbb{N}$. Pick $k \in B - A$, so $a_k \geqslant \tau$. Since every a_n is above z, we get $z = a_n b_n$ for a nonzero principal element $b_n \in L$ and $(z : b_n) = a_n$. We have

$$\tau = \bigwedge_{n \in A} a_n = \bigwedge_{n \in A} (z : b_n) = \Big(z : \bigvee_{n \in A} b_n\Big),$$

so τ is a principal element because L is pseudo-Dedekind. From $a_k \geqslant \tau$ we get $\tau = a_k c$ for a nonzero principal element $c \in L$. Hence,

$$c \leqslant (\tau : a_k) = \bigwedge_{n \in A} (a_n : a_k) = \bigwedge_{n \in A} a_n = \tau = a_k c$$

because $a_n \vee a_k = 1$ for each $n \in A$. From $a_k c = c$, we get $a_k = 1$, which is a contradiction. □

A lattice L is a *Dedekind lattice* if every element of L is principal.

Corollary 23. *A countable sharp lattice L is a Dedekind lattice.*

Proof. Let $m \in \mathrm{Max}(L)$. As L_m is countable, Theorem 17 implies that L_m is isomorphic to $\mathbb{Z}_-$, so each element of L_m is principal. By Proposition 22, L has finite character. It follows easily that every element of L is compact and locally principal, hence principal. □

Our concluding remark is in the spirit of [11], Remark 4.7.

Remark 24. Let L be a Prüfer lattice. Then L is modular because it is locally totally ordered. By [2], Theorem 3.4, L is isomorphic to the lattice of ideals of some Prüfer integral domain. In particular, it follows that a sharp lattice is isomorphic to the lattice of ideals of some sharp integral domain.

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