FERMIONIC NOVIKOV ALGEBRAS ADMITTING INVARIANT NON-DEGENERATE SYMMETRIC BILINEAR FORMS

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Abstract. Novikov algebras were introduced in connection with the Poisson brackets of hydrodynamic type and Hamiltonian operators in the formal variational calculus. Fermionic Novikov algebras correspond to a certain Hamiltonian superoperator in a supervariable. In this paper, we show that fermionic Novikov algebras equipped with invariant non-degenerate symmetric bilinear forms are Novikov algebras.

Keywords: Novikov algebra; fermionic Novikov algebra; invariant bilinear form

MSC 2020: 17B60, 17A30, 17D25

1. Introduction

Gel'fand and Dikii gave a bosonic formal variational calculus in [5], [6] and Xu provided a fermionic formal variational calculus in [12]. By combining the bosonic theory of Gel'fand-Dikii and the fermionic theory, a formal variational calculus of supervariables was given by Xu in [13]. Fermionic Novikov algebras are related to the Hamiltonian superoperator in terms of this theory. A fermionic Novikov algebra is a finite-dimensional vector space A over a field $\mathbb{F}$ with a bilinear product $(x,y)\mapsto xy$ satisfying

$$(xy)z - x(yz) = (yx)z - y(xz), \tag{1.1}$$

$$(xy)z = -(xz)y \tag{1.2}$$

for any $x,y,z\in A$. As described in [13], this algebra corresponds to the Hamiltonian operator H of type 0, i.e., $H^0_{\alpha,\beta}=\sum\limits_{\gamma\in I}(a^{\gamma}_{\alpha,\beta}\Phi_\gamma(2)+b^{\gamma}_{\alpha,\beta}\Phi_\gamma D)$, where $a^{\gamma}_{\alpha,\beta}, b^{\gamma}_{\alpha,\beta}\in\mathbb{R}$.

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According to the identity (1.1), fermionic Novikov algebras are a class of left-symmetric algebras, which are a class of non-associative algebras arising from the study of affine manifolds, affine structures and convex homogeneous cones, see [2], [10]. Novikov algebras, introduced in connection with the Poisson brackets of hydrodynamic type, see [1], [3], [4] and Hamiltonian operators in the formal variational calculus, see [5], [6], [7], [11], [12], are another class of left-symmetric algebras A satisfying

$$(xy)z = (xz)y \quad \text{for any } x, y, z \in A. \tag{1.3}$$

The commutator $[x, y] = xy - yx$ for any x and y in a left-symmetric algebra A defines a Lie algebra, which is called the underlying Lie algebra of A. A bilinear form $\langle \cdot, \cdot \rangle$ on a left-symmetric algebra A is invariant if

$$\langle yx, z \rangle = \langle y, zx \rangle \tag{1.4}$$

for any $x, y, z \in A$.

Zelmanov in [14] classified real Novikov algebras with invariant positive definite symmetric bilinear forms. In [8], Guediri gave the classification for the Lorentzian case. This paper studies real fermionic Novikov algebras admitting invariant non-degenerate symmetric bilinear forms. Our main result is the following theorem.

Theorem 1.1. *Any finite dimensional real fermionic Novikov algebra admitting an invariant non-degenerate symmetric bilinear form is a Novikov algebra.*

2. The proof of Theorem 1.1

Let A be a fermionic Novikov algebra. Given any element $x \in A$, we denote the left and right multiplication operator by L_x and R_x, respectively, i.e., $L_x(y) = xy$ and $R_x(y) = yx$ for any $y \in A$. According to identity (1.2), it follows immediately that for any $x, y \in A$, $R_x R_y = -R_y R_x$. In particular, we have that $R_x^2 = 0$ for any $x \in A$.

Definition 2.1. A non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on a vector space V is of type $(n-p, p)$ if there is a basis $\{e_1, \ldots, e_n\}$ of V such that $\langle e_i, e_i \rangle = -1$ for $1 \leqslant i \leqslant p$, $\langle e_i, e_i \rangle = 1$ for $p+1 \leqslant i \leqslant n$, and $\langle e_i, e_j \rangle = 0$ otherwise. Note that the bilinear form is positive definite if $p = 0$ and is Lorentzian if $p = 1$.

A linear operator σ of $(V, \langle \cdot, \cdot \rangle)$ is self-adjoint if $\langle \sigma(x), y \rangle = \langle x, \sigma(y) \rangle$ for any $x, y \in V$.

Lemma 2.1 ([9], pages 260–261)**.** *Let $\langle\cdot,\cdot\rangle$ be a non-degenerate symmetric bilinear form of type $(n-p,p)$ on $V=\mathbb{R}^n$, then a linear operator σ on V is self-adjoint if and only if V can be expressed as a direct sum of V_k that are mutually orthogonal (hence non-degenerate), σ-invariant, and each $\sigma|_{V_k}$ has an $r\times r$ matrix form either*

$$\begin{pmatrix} \lambda & 0 & \dots & 0 \\ 1 & \lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 1 & \lambda \end{pmatrix}$$

relative to a basis $\alpha_1,\dots,\alpha_r$ $(r\geqslant 1)$ with all scalar products zero except $\langle\alpha_i,\alpha_j\rangle=\pm 1$ when $i+j=r+1$, or

$$\begin{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} & & & & 0 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} a & b \\ -b & a \end{pmatrix} & & & \\ & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} a & b \\ -b & a \end{pmatrix} & & \\ 0 & & \ddots & \ddots & \\ & & & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \end{pmatrix},$$

where $b\neq 0$ relative to a basis $\beta_1,\alpha_1,\dots,\beta_m,\alpha_m$ with all scalar products zero except $\langle\beta_i,\beta_j\rangle=1=-\langle\alpha_i,\alpha_j\rangle$ when $i+j=m+1$.

If the algebra A admits an invariant non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ of type $(n-p,p)$, then $-\langle\cdot,\cdot\rangle$ is an invariant non-degenerate symmetric bilinear form on A of type $(p,n-p)$. Therefore we can assume that $p\leqslant n-p$.

Lemma 2.2. *Let A be a fermionic Novikov algebra admitting an invariant non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ of type $(n-p,p)$, then $\dim\operatorname{Im}R_x\leqslant p$ for any $x\in A$.*

P r o o f. Recall that $R_x^2=0$; it follows that $\operatorname{Im}R_x\subseteq\operatorname{Ker}R_x$. By the invariance of $\langle\cdot,\cdot\rangle$, we have $\langle R_xy,R_xz\rangle=\langle y,R_x^2z\rangle=0$, which yields $\langle\operatorname{Im}R_x,\operatorname{Im}R_x\rangle=0$. Hence $\dim\operatorname{Im}R_x\leqslant p$. □

Let $x_0\in A$ such that $\dim\operatorname{Im}R_x\leqslant\dim\operatorname{Im}R_{x_0}$ for any $x\in A$. By Lemma 2.2, $\dim\operatorname{Im}R_{x_0}\leqslant p$. For convenience, assume that $\dim\operatorname{Im}R_{x_0}=k$. By Lemma 2.1 and $R_{x_0}^2=0$, there exists a basis $\{e_1,\dots,e_n\}$ of A such that the operator R_{x_0} relative to this basis has the matrix of the form

$$\begin{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{pmatrix}_{2k\times 2k} & 0_{2k\times(n-2k)} \\ 0_{(n-2k)\times 2k} & 0_{(n-2k)\times(n-2k)} \end{pmatrix},$$

and the matrix of the metric $\langle\cdot,\cdot\rangle$ with respect to $\{e_1,\ldots,e_n\}$ has the form

$$\begin{pmatrix} C_{2k} & 0 & 0 \\ 0 & -I_{p-k} & 0 \\ 0 & 0 & I_{n-p-k} \end{pmatrix},$$

where $C_{2k} = \operatorname{diag}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ldots,\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$ and I_s denotes the $s\times s$ identity matrix. For any $x\in A$, the matrix of the operator R_x relative to this basis has the form

$$\begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{pmatrix}$$

whose blocks are the same as those of the metric matrix with respect to the basis $\{e_1,\ldots,e_n\}$.

First we can prove that $\begin{pmatrix} A_5 & A_6 \\ A_8 & A_9 \end{pmatrix} = 0_{(n-2k)\times(n-2k)}$. In fact, assume that there exists some nonzero entry d of $\begin{pmatrix} A_5 & A_6 \\ A_8 & A_9 \end{pmatrix}$. Consider the matrix form of the operator $R_x + lR_{x_0}$ with $l\in\mathbb{R}$. For any $l\in\mathbb{R}$, according to the choice of x_0, we know that $\dim\operatorname{Im}(R_x+lR_{x_0}) = \dim\operatorname{Im}(R_{x+lx_0}) \leqslant k$. By taking the 2nd through the $2k$th row, the 1st through the $(2k-1)$th column, and the row and column containing the element d in the matrix of $R_x+lR_{x_0}$, we have the $(k+1)\times(k+1)$ matrix $\begin{pmatrix} B+lI_k & \alpha \\ \beta & d \end{pmatrix}$ with the determinant being a polynomial of degree k in a single indeterminate l. Therefore we can choose an $l'\in\mathbb{R}$ such that the above determinant is nonzero. It follows that

$$\dim\operatorname{Im}(R_x + l'R_{x_0}) = \dim\operatorname{Im}(R_{x+l'x_0}) \geqslant k+1,$$

which is a contradiction.

Secondly, since $R_xR_{x_0} + R_{x_0}R_x = 0$, we have that $A_1 = (M_{ij})_{1\leqslant i,j\leqslant k}$ with $M_{ij} = \begin{pmatrix} b_{ij} & 0 \\ d_{ij} & -b_{ij} \end{pmatrix}$,

$$A_2 = \begin{pmatrix} 0 & \ldots & 0 \\ a_{2,1} & \ldots & a_{2,p-k} \\ \vdots & & \vdots \\ 0 & \ldots & 0 \\ a_{2k,1} & \ldots & a_{2k,p-k} \end{pmatrix} \quad\text{and}\quad A_3 = \begin{pmatrix} 0 & \ldots & 0 \\ c_{2,1} & \ldots & c_{2,n-p-k} \\ \vdots & & \vdots \\ 0 & \ldots & 0 \\ c_{2k,1} & \ldots & c_{2k,n-p-k} \end{pmatrix}.$$

Furthermore, since $\langle R_x y, z\rangle = \langle y, R_x z\rangle$ according to (1.4), we obtain that

$$M_{ij} = \begin{pmatrix} b_{ij} & 0 \\ d_{ij} & -b_{ij} \end{pmatrix}, \quad M_{ji} = \begin{pmatrix} -b_{ij} & 0 \\ d_{ij} & b_{ij} \end{pmatrix},$$

where $b_{ii} = 0$ for any $1 \leqslant i \leqslant k$, and

$$A_4 = -\begin{pmatrix} a_{2,1} & 0 & \dots & a_{2k,1} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{2,p-k} & 0 & \dots & a_{2k,p-k} & 0 \end{pmatrix},$$

$$A_7 = \begin{pmatrix} c_{2,1} & 0 & \dots & c_{2k,1} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ c_{2,n-p-k} & 0 & \dots & c_{2k,n-p-k} & 0 \end{pmatrix}.$$

Since $R_x^2 = 0$, we have that $A_1^2 + A_2A_4 + A_3A_7 = 0_{2k\times 2k}$. Note that for any $1 \leqslant i \leqslant k$,

$$(A_1^2)_{i,i} = (A_1^2 + A_2A_4 + A_3A_7)_{i,i} = 0.$$

It follows that $b_{ij} = 0$ for any i, j. Then we have that $M_{ij} = M_{ji} = \begin{pmatrix} 0 & 0 \\ d_{ij} & 0 \end{pmatrix}$.

Finally, we claim that A_2, A_3, A_4 and A_7 are zero matrices. In the following, we only prove $A_2 = 0_{2k\times(p-k)}$, the proofs of the others are similar. Assume that there exists a nonzero entry d of A_2. Consider the matrix of the operator $R_x + lR_{x_0}$. Similarly to the proof of $\begin{pmatrix} A_5 & A_6 \\ A_8 & A_9 \end{pmatrix} = 0_{(n-2k)\times(n-2k)}$, we consider the matrix $\begin{pmatrix} A_1'+lI_k & \alpha^{\mathrm{T}} \\ -\alpha & 0 \end{pmatrix}$, where d is an entry in the vector α and $A_1' = (d_{ij})_{1\leqslant i,j\leqslant k}$ is a symmetric matrix. Therefore there exists an orthogonal matrix P such that $P^{\mathrm{T}}A_1'P = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix}$. We can choose an $l > \max\{|\lambda_1|, \dots, |\lambda_k|\}$. It follows that the matrix $A_1' + lI_k$ is invertible. We have

$$\begin{aligned} \begin{vmatrix} A_1' + lI_k & \alpha^{\mathrm{T}} \\ -\alpha & 0 \end{vmatrix} &= \left|\begin{pmatrix} P^{\mathrm{T}} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A_1' + lI_k & \alpha^{\mathrm{T}} \\ -\alpha & 0 \end{pmatrix}\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}\right| \\ &= \begin{vmatrix} \begin{pmatrix} \lambda_1 + l & & 0 \\ & \ddots & \\ 0 & & \lambda_k + l \end{pmatrix} & \beta^{\mathrm{T}} \\ -\beta & 0 \end{vmatrix} = \left(\prod_{i=1}^{k}(\lambda_i + l)\right)\sum_{i=1}^{k}\frac{1}{\lambda_i + l}b_i^2 \neq 0, \end{aligned}$$

where $\beta = \alpha P = (b_1, \dots, b_k)$ is a nonzero vector. It follows that

$$\dim \operatorname{Im}(R_x + lR_{x_0}) = \dim \operatorname{Im}(R_{x+lx_0}) \geqslant k + 1,$$

which is a contradiction. Therefore we proved that $A_2 = 0_{2k\times(p-k)}$.

Now, we know that the matrix of R_x has the form

$$\begin{pmatrix} A_1 & 0_{2k\times(n-2k)} \\ 0_{(n-2k)\times 2k} & 0_{(n-2k)\times(n-2k)} \end{pmatrix},$$

where $A_1 = (M_{ij})_{1\leqslant i,j\leqslant k}$ with $M_{ij} = M_{ji} = \left(\begin{smallmatrix} 0 & 0 \\ d_{ij}(x) & 0 \end{smallmatrix}\right)$. Hence $R_xR_y = 0$ for any $x, y \in A$, which implies Theorem 1.1.

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