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GENERALIZED TILTING MODULES OVER RING EXTENSION

Zhen Zhang, Jinan

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Abstract. Let Γ be a ring extension of R. We show the left Γ-module $U = \Gamma \otimes_R C$ with the endmorphism ring $\mathrm{End}_\Gamma U = \Delta$ is a generalized tilting module when ${}_RC$ is a generalized tilting module under some conditions.

Keywords: ring extension; generalized tilting module; faithfully balanced bimodule

MSC 2010: 13D02; 13D07; 13D05

1. Introduction

Let $\xi\colon R \to \Gamma$ be a ring homomorphism, then the ring Γ is called a ring extension of R. So every Γ-module has a natural structure as R-module through ξ; and there are two canonical ways to obtain a Γ-module from an R-module V, namely, $\Gamma \otimes_R V$ and $\mathrm{Hom}_R(\Gamma, V)$. Many properties of V which involve the functor $-\otimes_R V$ are known to be inherited by $\Gamma \otimes_R V$: some of these, as projective, generator, are inherited without further conditions, others, for example, tilting module, $*$-module and quasi-progenerator, require various conditions on V and/or $\Gamma \otimes_R V$, cf. [1], [5], [10]. Dually, many properties of V which involve the functor $\mathrm{Hom}_R(-, V)$ are known to be inherited by $\mathrm{Hom}_R(\Gamma, V)$: some of these, as injective, cogenerator, are inherited without further conditions, others, for example, cotilting module and quasi-duality module require various conditions on V and/or $\mathrm{Hom}_R(\Gamma, V)$, cf. [6].

The generalized tilting module is a kind of generalization of the tilting module (see [7], Definition 5.1.1), and it was first introduced by Wakamatsu in [17] over an Artin algebra. In [18], the notion of the generalized tilting module was generalized

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to any associative rings and many results on generalized tilting modules over Artin algebras were naturally generalized. Note that (generalized) tilting modules in [18] are known as Wakamatsu tilting modules now. Recently, the relations between the tilting modules of finite projective dimensions and extensions of rings were investigated by several authors (cf. [14], [19]). In particular, very recently, Tonolo [14] investigated more generally when an n-tilting module ${}_RV$ extends to an n-tilting module $\operatorname{Tor}^R_{m\geqslant 0}(\Gamma, V)$ over a ring extension Γ of R, but with a restriction on the projective dimension of the module.

Naturally, we want to consider the tilting modules with infinite projective dimensions over the ring extension $\xi\colon R\to\Gamma$.

For every left R-module C, we denote by $\operatorname{Add} C$ ($\operatorname{add} C$) the class of modules isomorphic to direct summands of (finite) direct sums of copies of C and by $\operatorname{Res}({}_RC)$ the class of left R-modules M which admit a proper $\operatorname{Add} C$-projective resolution, i.e., there exists an exact sequence $\ldots\to C_1\to C_0\to M\to 0$ with each $C_i\in\operatorname{Add} C$ and the sequence stays exact under the functor $\operatorname{Hom}_R(C,-)$.

For a left R-module X, we denote by $X^{\perp}$ the subcategory of left R-modules M such that $\operatorname{Ext}^i_R(X,M)=0$ for all $i\geqslant 1$ and similarly, ${}^{\perp}X$ denotes the subcategory of left R-modules N such that $\operatorname{Ext}^i_R(N,X)=0$ for all $i\geqslant 1$.

Note that the proper $\operatorname{Add} C$-projective resolution was called a dominant right ${}_RC$-resolution in [18] and was called an ${}_RC$-resolution in [17] if $C\in C^{\perp}$.

We denote by ${}_C\mathcal{X}$ the class of left R-modules in $C^{\perp}$ which admit the proper $\operatorname{Add} C$-projective resolutions, i.e., ${}_C\mathcal{X}=C^{\perp}\cap\operatorname{Res}({}_RC)$.

In the third section, we prove our main results after we prove several important lemmas which generalize [19], Theorem 4.6 and [14], Corollary 2.5 in case ${}_R\Gamma^*$ has C-grade 0, where ${}_R\Gamma^*$ is the left R-module $\operatorname{Hom}_R(\Gamma, D)$, where ${}_RD$ is an injective cogenerator in the category of all left R-modules (see Theorem 3.6).

Let ${}_RC_S$ be a generalized tilting module with $\operatorname{End}({}_RC)=S$ and Γ a ring extension of R. Denote by U the left Γ-module $\Gamma\otimes_R C$ such that $\operatorname{End}({}_\Gamma U)=\Delta$ and U_Δ has degreewise finitely generated projective resolution. Assuming that ${}_RU\in\operatorname{Res}({}_RC)$ and $\operatorname{Tor}^R_{\geqslant 1}(C,\Gamma)=0$, we prove that ${}_\Gamma U$ is a generalized tilting module if and only if ${}_RU\in{}_C\mathcal{X}$ and ${}_R\operatorname{Hom}_R(\Gamma, D)\in{}_C\mathcal{X}$, where D is an injective cogenerator of $\operatorname{Mod} R$.

To the end, we give applications of our results to split extension rings.

On the other hand, semidualizing modules are common generalizations of dualizing modules and free modules of rank one over commutative Noetherian local rings. This module was first defined by Foxby in [4] as a generalization of a projective module and a Gorenstein module, while Vasconcelos [15] (using spherical modules) and Golod [8] (using suitable modules) initiated the study of semidualizing modules under different names. Recently, Holm and White [9] defined the semidualizing (R,S)-bimodule ${}_RC_S$ over a pair of associative rings R and S. Relative algebra with respect to a semidualizing module has caught many authors' attention. For this topic, we refer the readers to Holm and White's work [9], but also to [11], [12], [13]. In Section 3, we investigate the relation between semidualizing bimodules and ring extensions, which is a non-commutative module version of [2], Theorem 5.1 and Proposition 5.3.

2. Preliminaries

In this section, we introduce a number of notions and results which will be used throughout this work. For unexplained concepts and notation, we refer the readers to [16], [17], [18].

Let R be any ring. We denote by $\mathrm{gen}^*(R)$ the class of left (right) R-modules which admit degreewise finitely generated left (right) R-projective resolutions, i.e., $\mathrm{gen}^*(R) = \{M\colon$ there exists an exact resolution $\ldots \to P_1 \to P_0 \to M \to 0$, where P_i is finitely generated projective left (right) R-module for each $i\}$.

Definition 2.1. Let R and S be any rings. An (R,S)-bimodule ${}_RC_S$ is called faithfully balanced if $\mathrm{End}({}_RC) = S$ and $\mathrm{End}(C_S) = R$. ${}_RC_S$ is called self-orthogonal if $\mathrm{Ext}^i_R(C,C) = 0 = \mathrm{Ext}^i_S(C,C)$ for each $i \geqslant 1$.

Definition 2.2 ([18])**.** A left R-module C is called a (*generalized*) *tilting module* if

(1) ${}_RC \in \mathrm{gen}^*(R)$;
(2) $\mathrm{Ext}^i_R(C,C) = 0$ for all $i \geqslant 1$;
(3) there exists an exact sequence: $0 \to R \xrightarrow{f_0} C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \ldots$, where $C_i \in \mathrm{add}\, C$ and $\mathrm{coker} f_i \in {}^{\perp}C$ for any $i \geqslant 0$.

By [18], Corollary 3.2, we have the following equivalent characterization of the generalized tilting modules.

Lemma 2.3. *The following conditions are equivalent for a bimodule* ${}_RC_S$*:*

(1) ${}_RC$ *is a generalized tilting module with* $\mathrm{End}({}_RC) = S$*;*
(2) C_S *is a generalized tilting module with* $\mathrm{End}(C_S) = R$*;*
(3) ${}_RC \in \mathrm{gen}^*(R)$ *and* $C_S \in \mathrm{gen}^*(S)$ *and* ${}_RC_S$ *is a faithfully balanced, self-orthogonal bimodule.*

3. Generalized tilting modules over ring extensions

In this section, we prove our main results. First, we give some characterization of the class $\mathrm{Res}({}_RC)$ for some left R-module C. Note that the class $\mathrm{Res}({}_RC)$ is the class of left R-modules which admit proper $\mathrm{Add}\,{}_RC$-resolutions.

Let R and S be two associative rings, the following result generalizes [17], Proposition 2.4.

Lemma 3.1. *Let ${}_RC$ be a generalized tilting module with* $\mathrm{End}({}_RC) = S$. *The following two statements are equivalent.*

(1) $M \in \mathrm{Res}({}_RC)$;

(2) $\mathrm{Tor}^S_{\geqslant 1}(C, \mathrm{Hom}_R(C, M)) = 0$ *and* $\nu_M\colon C \otimes_S \mathrm{Hom}_R(C, M) \to M$ *is an isomorphism.*

P r o o f. (1) $\Rightarrow$ (2): Since $M \in \mathrm{Res}({}_RC)$, there exists an exact sequence of left R-modules: $\ldots \to C_1 \to C_0 \to M \to 0$ with $C_i \in \mathrm{Add}_R\, C$. Applying the functor $\mathrm{Hom}_R(C, -)$, we get another exact sequence of left S-modules: $(*) = \ldots \to \mathrm{Hom}_R(C, C_1) \to \mathrm{Hom}_R(C, C_0) \to \mathrm{Hom}_R(C, M) \to 0$. Since $\mathrm{Hom}_R(C, C) \cong S$ and ${}_RC$ is finitely generated, $\mathrm{Hom}_R(C, C_i)$ is a projective S-module for each $i \geqslant 0$. Thus $(*)$ is a projective resolution of the left S-module $\mathrm{Hom}_R(C, M)$. Note that $C \otimes_S \mathrm{Hom}_R(C, C_i) \cong C_i$ for all $i \geqslant 0$. Hence applying the functor $C \otimes_S -$ to $(*)$, we obtain that $C \otimes_S \mathrm{Hom}_R(C, M) \cong M$ and $\mathrm{Tor}^S_{\geqslant 1}(C, \mathrm{Hom}_R(C, M)) = 0$.

(2) $\Rightarrow$ (1): Consider the S-projective resolution of $\mathrm{Hom}_R(C, M)$, $\mathbb{X} = \ldots \to P_1 \to P_0 \to \mathrm{Hom}_R(C, M) \to 0$. By (2), $\mathrm{Tor}^S_{\geqslant 1}(C, \mathrm{Hom}_R(C, M)) = 0$. So $C \otimes_S \mathbb{X}$ is exact. Also $C \otimes_S \mathrm{Hom}_R(C, M) \cong M$, and we get an exact sequence $\ldots \to C \otimes_S P_1 \to C \otimes_S P_0 \to C \otimes_S \mathrm{Hom}_R(C, M) \cong M \to 0$. Clearly, $C \otimes_S P_i \in \mathrm{Add}\, {}_RC$. Apply the functor $\mathrm{Hom}_R(C, -)$ to $C \otimes_S \mathbb{X}$. Since $P_i \cong \mathrm{Hom}_R(C, C \otimes_S P_i)$ for each i, we deduce the sequence $\mathrm{Hom}_R(C, C \otimes_S \mathbb{X})$ is exact. So $M \in \mathrm{Res}({}_RC)$. □

Remark 3.2. When ${}_RC$ is a generalized tilting module, it is easy to see the class $\mathrm{Res}({}_RC)$ is closed under direct sums by the definition. By Lemma 3.1, we can show the class $\mathrm{Res}({}_RC)$ is closed under direct summands. In fact, let $M \in \mathrm{Res}({}_RC)$ and let M' be a direct summand for M, then $\mathrm{Tor}^S_{\geqslant 1}(C, \mathrm{Hom}_R(C, M)) = 0$ and $\nu_M\colon C \otimes_S \mathrm{Hom}_R(C, M) \to M$ is an isomorphism by Lemma 3.1. So $\mathrm{Tor}^S_{\geqslant 1}(C, \mathrm{Hom}_R(C, M')) = 0$. Consider the two split exact sequences $0 \to M' \to M \to M'' \to 0$ and $0 \to M'' \to M \to M' \to 0$. Then $\nu_{M'}\colon C \otimes_S \mathrm{Hom}_R(C, M') \to M$ is an isomorphism by the snake lemma.

Lemma 3.3. *Let D be an injective cogenerator of R-*Mod *and ${}_RC$ a finitely generated left R-module with* $\mathrm{End}({}_RC) = S$. *If $D \in \mathrm{Res}({}_RC)$ and $C_S \in \mathrm{gen}^*(S)$, then ${}_RC_S$ is a faithfully balanced bimodule and* $\mathrm{Ext}^{\geqslant 1}_S(C, C) = 0$.

P r o o f. By Lemma 3.1, $C \otimes_S \mathrm{Hom}_R(C, D) \cong D$ and $\mathrm{Tor}^S_{\geqslant 1}(C, \mathrm{Hom}_R(C, D)) = 0$. So we have the following isomorphisms:

$$\mathrm{Hom}_R(R, D) \cong D \cong C \otimes_S \mathrm{Hom}_R(C, D) \cong \mathrm{Hom}_R(\mathrm{Hom}_S(C, C), D)$$

and

$$0 = \operatorname{Tor}_i^S(C, \operatorname{Hom}_R(C, D)) \cong \operatorname{Hom}_R(\operatorname{Ext}_S^i(C, C), D)$$

by [3], Theorems 3.2.11 and 3.2.13. Since D is an injective cogenerator of R-modules, $\operatorname{End}(C_S) = \operatorname{Hom}_S(C, C) \cong R$ and $\operatorname{Ext}_S^i(C, C) = 0$ for $i \geqslant 1$. □

In the following, in order to facilitate the writing, we use U to denote $\Gamma \otimes_R C$ and Δ to denote $\operatorname{End}({}_\Gamma U)$. We always assume that $U_\Delta \in \operatorname{gen}^*(\Delta)$. Given an $R-S$-bimodule H, we denote by H^* the left S-module $\operatorname{Hom}_R(H, D)$, where ${}_RD$ is an injective cogenerator in the category of all left R-modules.

Similarly to [5], Lemmas 1.2 and 1.3, we have the following lemmas:

Lemma 3.4. *Let $\xi\colon\ R \to \Gamma$ be a ring homomorphism and ${}_RC$ a generalized tilting module with $\operatorname{End}({}_RC) = S$. Then ${}_RU \in \operatorname{Res}({}_RC)$ if and only if $\operatorname{Res}({}_\Gamma U) = \{{}_\Gamma M\colon\ {}_RM \in \operatorname{Res}({}_RC)\}$.*

Proof. ($\Leftarrow$): Clearly ${}_\Gamma U \in \operatorname{Res}({}_\Gamma U)$, so ${}_RU \in \operatorname{Res}({}_RC)$.

($\Rightarrow$): For any ${}_\Gamma M \in \operatorname{Res}({}_\Gamma U)$, there exists an exact sequence of Γ-modules, $\ldots \to U_1 \to U_0 \to M \to 0$ with ${}_\Gamma U_i \in \operatorname{Add}{}_\Gamma U$ for each i, which is $\operatorname{Hom}_\Gamma(U, -)$-exact. Then we obtain an exact sequence of induced R-modules $\mathbb{X} = \ldots \to U_1 \to U_0 \to M \to 0$, which stays exact under the functor $\operatorname{Hom}_R(C, -)$ by [5], Lemma 1.2. Note that ${}_RU_i \in \operatorname{Add}{}_R(\Gamma \otimes_R C)$. As ${}_RU \in \operatorname{Res}({}_RC)$, we get ${}_RU_i \in \operatorname{Res}({}_RC)$ by Remark 3.2. We have the following commutative diagram, where the first row is obtained by applying the functor $C \otimes_S -$ to the exact sequence $\operatorname{Hom}_R(C, \mathbb{X})$:

$$\begin{array}{ccccccc}
C \otimes_S \operatorname{Hom}_R(C, U_1) & \longrightarrow & C \otimes_S \operatorname{Hom}_R(C, U_0) & \longrightarrow & C \otimes_S \operatorname{Hom}_R(C, M) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \nu_{U_1}} & & \downarrow{\scriptstyle \nu_{U_0}} & & \downarrow{\scriptstyle \nu_M} & & \\
U_1 & \longrightarrow & U_0 & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

Since ${}_RC$ is a generalized tilting module, both ν_{U_1} and ν_{U_0} are isomorphisms and $\operatorname{Tor}_{\geqslant 1}^S(C, \operatorname{Hom}_R(C, U_i)) = 0$ for each $i \geqslant 0$ by Lemma 3.1. So ν_M is an isomorphism and the sequence $C \otimes_S \operatorname{Hom}_R(C, \mathbb{X})$ is exact. Moreover, $\operatorname{Tor}_{\geqslant 1}^S(C, \operatorname{Hom}_R(C, M)) = 0$. In fact, let $K_i = \operatorname{Coker}(\operatorname{Hom}_R(C, U_{i+2}), \operatorname{Hom}_R(C, U_{i+1}))$ for each $i \geqslant 0$ and $K_{-1} = \operatorname{Hom}_R(C, M)$. Consider the short exact sequences

$$0 \to K_i \to \operatorname{Hom}_R(C, U_i) \to K_{i-1} \to 0.$$

Applying the functor $C \otimes_S -$, we get a long exact sequence

$$\operatorname{Tor}_1^S(C, \operatorname{Hom}_R(C, U_i)) \to \operatorname{Tor}_1^S(C, K_{i-1}) \to C \otimes_S K_i \to C \otimes_S \operatorname{Hom}_R(C, U_i).$$

Since $C \otimes_S \operatorname{Hom}_R(C, \mathbb{X})$ is exact, $C \otimes_S K_i \to C \otimes_S \operatorname{Hom}_R(C, U_i)$ is a monomorphism. Thus $\operatorname{Tor}_1^S(C, K_{i-1}) = 0$ for each $i \geqslant 0$. But $\operatorname{Tor}_{i+1}^S(C, K_{-1}) \cong \operatorname{Tor}_i^S(C, K_0) \cong \ldots \cong \operatorname{Tor}_1^S(C, K_{i-1}) = 0$ for each $i \geqslant 1$. Hence ${}_RM \in \operatorname{Res}({}_RC)$ also by Lemma 3.1.

On the other hand, take any ${}_\Gamma M$ such that ${}_RM \in \operatorname{Res}({}_RC)$. By the assumption, ${}_RU \in \operatorname{Res}({}_RC)$ and $\operatorname{Res}({}_RC) \subseteq \operatorname{Gen}({}_RC)$. We deduce that ${}_\Gamma M \in \operatorname{Gen}({}_\Gamma U)$ by [5], Lemma 1.3. So there exists an exact sequence of Γ-modules $0 \to M_1 \to U_0 \to M \to 0$ which stays exact under the functor $\operatorname{Hom}_\Gamma(U, -)$, where ${}_\Gamma U_0 \in \operatorname{Add}_\Gamma U$. We obtain the exact sequence of induced R-modules $(\dagger) = 0 \to M_1 \to U_0 \to M \to 0$, which stays exact under the functor $\operatorname{Hom}_R(C, -)$ by [5], Lemma 1.2. Since ${}_RM \in \operatorname{Res}({}_RC)$, $C \otimes_S \operatorname{Hom}_R(C, M) \cong M$ and $\operatorname{Tor}_{\geqslant 1}^S(C, \operatorname{Hom}_R(C, M)) = 0$ by Lemma 3.1. So the sequence $C \otimes_S \operatorname{Hom}_R(C, (\dagger))$ is exact. Also ${}_RU_0 \in \operatorname{Res}({}_RC)$ by Remark 3.2, and we can deduce $C \otimes_S \operatorname{Hom}_R(C, M_1) \cong M_1$ and $\operatorname{Tor}_{\geqslant 1}^S(C, \operatorname{Hom}_R(C, M_1)) = 0$. Thus ${}_RM_1 \in \operatorname{Res}({}_RC)$ by Lemma 3.1. It follows that M_1 is also an Γ-module such that ${}_RM_1 \in \operatorname{Res}({}_RC)$. Now by repeating the process with ${}_\Gamma M_1$, and so on, we get that ${}_\Gamma M \in \operatorname{Res}({}_\Gamma U)$. □

Lemma 3.5. *Let $\xi\colon R \to \Gamma$ be a ring homomorphism and ${}_RC$ a left R-module. Then ${}_R\Gamma^* \in C^\perp$ if and only if $\operatorname{Ext}_\Gamma^{\geqslant 1}(U, N) \cong \operatorname{Ext}_R^{\geqslant 1}(C, N)$ for any Γ-module N.*

Proof. ($\Leftarrow$): By [18], Lemma 4.2, ${}_\Gamma\Gamma^*$ is injective. Thus ${}_\Gamma\Gamma^* \in_\Gamma U^\perp$. So ${}_R\Gamma^* \in C^\perp$.

($\Rightarrow$): By [3], Theorem 3.2.1, ${}_R\Gamma^* \in C^\perp$ if and only if $\operatorname{Tor}_{\geqslant 1}^R(\Gamma, C) = 0$. Take an R-projective resolution of C, $\mathbb{P} = \ldots \to P_1 \to P_0 \to C \to 0$ with each P_i finitely generated projective. Then the sequence $\Gamma \otimes_R \mathbb{P}$ is exact. Note that $\Gamma \otimes_R P_i$ is a finitely generated projective Γ-module for each i. Hence $\Gamma \otimes_R \mathbb{P}$ is a Γ-projective resolution of U. By [5], Lemma 1.2, for each $i \geqslant 1$, we have $\operatorname{Ext}_\Gamma^i(U, N) = H_{-i}(\operatorname{Hom}_\Gamma(\Gamma \otimes_R \mathbb{P}, N)) \cong H_{-i}(\operatorname{Hom}_R(\mathbb{P}, N)) = \operatorname{Ext}_R^i(C, N)$. □

Now, we investigate the generalized tilting modules over a ring extension Γ of R using the above lemmas.

Theorem 3.6. *Let $\xi\colon R \to \Gamma$ be a ring homomorphism and ${}_RC$ a generalized tilting module with $\operatorname{End}({}_RC) = S$. The following are equivalent.*

(1) *${}_R\Gamma^* \in {}_C\mathcal{X}$ and ${}_RU \in {}_C\mathcal{X}$;*

(2) *${}_\Gamma U$ is a generalized tilting module and ${}_RU \in \operatorname{Res}_R C$ and $\operatorname{Tor}_{\geqslant 1}^R(\Gamma, C) = 0$.*

When they are satisfied, for any Γ-module M and Δ-module N, we have

$$ {}_\Gamma M \in {}_U\mathcal{X} \Leftrightarrow {}_RM \in {}_C\mathcal{X}. $$

P r o o f. (1) $\Rightarrow$ (2): Since ${}_C\mathcal{X} = C^{\perp} \cap \operatorname{Res}({}_RC)$, ${}_R\Gamma^* \in C^{\perp}$. By [3], Theorem 3.2.1, $\operatorname{Tor}^R_{\geqslant 1}(\Gamma, C) = 0$. Since ${}_RC$ is a generalized tilting module, ${}_RC \in \operatorname{gen}^*(R)$. So ${}_\Gamma U = {}_\Gamma(\Gamma \otimes_R C) \in \operatorname{gen}^*(\Gamma)$. On the other hand, as ${}_R\Gamma^* \in \operatorname{Res}({}_RC)$ and ${}_RU \in \operatorname{Res}({}_RC)$, we have ${}_\Gamma\Gamma^* \in \operatorname{Res}({}_\Gamma U)$ by Lemma 3.4. Note that ${}_\Gamma\Gamma^*$ is an injective cogenerator of Γ-Mod by [19], Lemma 4.2. By Lemma 3.3, $\operatorname{Hom}_\Delta(U, U) \cong \Gamma$ and $\operatorname{Ext}^{\geqslant 1}_\Delta(U, U) = 0$. As ${}_R\Gamma^* \in C^{\perp}$, we get $\operatorname{Ext}^{\geqslant 1}_\Gamma(U, U) \cong \operatorname{Ext}^{\geqslant 1}_R(C, U) = 0$ by Lemma 3.5 and (1). Hence ${}_\Gamma U$ is a generalized tilting module by Lemma 2.3.

(2) $\Rightarrow$ (1): Since ${}_\Gamma U$ is a generalized tilting module, $\operatorname{Ext}^{\geqslant 1}_\Gamma(U, U) = 0$. So $\operatorname{Ext}^{\geqslant 1}_R(C, U) \cong \operatorname{Ext}^{\geqslant 1}_\Gamma(U, U) = 0$ by Lemma 3.5. Thus ${}_RU \in C^{\perp}$ and ${}_RU \in {}_C\mathcal{X}$. On the other hand, ${}_\Gamma\Gamma^*$ is injective by [19], Lemma 4.2. So ${}_\Gamma\Gamma^* \in \operatorname{Res}({}_\Gamma U)$ by [3], Theorem 3.2.13 and Lemma 3.1. Also ${}_RU \in \operatorname{Res}({}_RC)$, and we get ${}_R\Gamma^* \in \operatorname{Res}_RC$ by Lemma 3.4. Hence ${}_R\Gamma^* \in {}_C\mathcal{X}$.

By the assumption and Lemmas 3.4 and 3.5, we can get ${}_\Gamma M \in {}_U\mathcal{X} \Leftrightarrow {}_RM \in {}_C\mathcal{X}$. □

Remark 3.7. By [18], Corollary 2.16, if the projective dimensions of the generalized tilting module ${}_RC$ is finite, then ${}_C\mathcal{X} = C^{\perp}$. So we have that $C^{\perp} \subseteq \operatorname{Res}({}_RC)$. Hence we can easily show [19], Theorem 4.6 and [14], Corollary 2.5 in case ${}_R\Gamma^*$ has C-grade 0 by Theorem 3.6.

To end this section, we consider the split extension rings.

If $\Gamma = R \ltimes Q$ is a split extension of R by Q, i.e., $\Gamma = R \oplus Q$, of course Γ and R are ring extensions of each other via the ring homomorphisms $R \to \Gamma$, $r \mapsto (r, 0)$ and $\Gamma \to R$, $(r, q) \mapsto r$. And Theorem 3.6 can be improved as follows.

Theorem 3.8. *Let $\Gamma = R \ltimes Q$ be a split extension of R by Q and let ${}_RC$ be a left R-module with $\operatorname{End}_RC \cong S$. Assume that $C_S \in \operatorname{gen}^*(S)$, then ${}_\Gamma U$ is a generalized tilting module such that ${}_R(Q \otimes_R C) \in \operatorname{Res}({}_RC)$ and $\operatorname{Tor}^R_{\geqslant 1}(Q, C) = 0$ if and only if ${}_RC$ is a generalized tilting module with ${}_RQ^* \in {}_C\mathcal{X}$ and ${}_R(Q \otimes_R C) \in {}_C\mathcal{X}$.*

P r o o f. ($\Rightarrow$): Since Γ is a split extension of R, there exists a ring homomorphism $\Gamma \to R$. So R is also a ring extension of Γ and $R \otimes_\Gamma U \cong R \otimes_\Gamma (\Gamma \otimes_R C) \cong C$. By Theorem 3.6, we only need to show ${}_\Gamma R^* = {}_\Gamma\operatorname{Hom}_\Gamma(R, \operatorname{Hom}_R(\Gamma, D)) \cong {}_\Gamma D \in {}_U\mathcal{X}$ and ${}_\Gamma(R \otimes_\Gamma U) \in {}_U\mathcal{X}$. In fact, since ${}_\Gamma\Gamma^*$ is injective, ${}_\Gamma\Gamma^* \in {}_U\mathcal{X}$. But $\Gamma \cong R \oplus Q$ and ${}_\Gamma\Gamma^* \cong D \oplus \operatorname{Hom}_R(Q, D)$, so ${}_\Gamma D \in {}_U\mathcal{X}$ by Remark 3.2. On the other hand, since ${}_\Gamma U$ is a generalized tilting module, ${}_\Gamma U \in {}_U\mathcal{X}$. But ${}_\Gamma U \cong (R \oplus Q) \otimes_R C \cong {}_\Gamma C \oplus Q \otimes_R C$. So ${}_\Gamma(R \otimes_\Gamma U) \cong {}_\Gamma C \in {}_U\mathcal{X}$. Hence ${}_RC$ is a generalized tilting module and ${}_RQ^* \in {}_C\mathcal{X}$ and ${}_R(Q \otimes_R C) \in {}_C\mathcal{X}$ by Theorem 3.6.

($\Leftarrow$): By the assumption, ${}_RU \cong (R \oplus Q) \otimes_R C \cong {}_RC \oplus Q \otimes_R C$. Since ${}_RC$ is a generalized tilting module, ${}_RC \in {}_C\mathcal{X}$. Also ${}_R(Q \otimes_R C) \in {}_C\mathcal{X}$, so we get ${}_RU \in {}_C\mathcal{X}$. On the other hand, ${}_R\Gamma \cong {}_RR \oplus {}_RQ$. Since ${}_RR^* \cong {}_RD$ is injective, ${}_RR^* \in {}_C\mathcal{X}$. Also

${}_RQ^* \in {}_C\mathcal{X}$, which yields ${}_R\Gamma^* \in {}_C\mathcal{X}$. By Theorem 3.6, ${}_\Gamma U$ is a generalized tilting module. Clearly ${}_R(Q \otimes_R C) \in \mathrm{Res}({}_RC)$ and $\mathrm{Tor}^R_{\geqslant 1}(Q, C) = 0$. $\square$

4. Applications to semidualizing bimodules

Definition 4.1 ([9], Definition 2.1). An (R, S)-bimodule ${}_RC_S$ is called *semidualizing* if

(1) ${}_RC \in \mathrm{gen}^*(R)$;
(2) $C_S \in \mathrm{gen}^*(S)$;
(3) the natural homothety map ${}_RR_R \to \mathrm{Hom}_S(C, C)$ is an isomorphism;
(4) the natural homothety map ${}_SS_S \to \mathrm{Hom}_R(C, C)$ is an isomorphism;
(5) $\mathrm{Ext}^{\geqslant 1}_R(C, C) = 0 = \mathrm{Ext}^{\geqslant 1}_S(C, C)$.

Over any rings R and S, Holm and White [9], Definition 4.1 also defined the Auslander class $\mathcal{A}_C(S)$ and the Bass class $\mathcal{B}_C(R)$ induced by a semidualizing bimodule ${}_RC_S$.

Definition 4.2. The Bass class $\mathcal{B}_C(R)$ with respect to ${}_SC_R$ consists of all right R-modules N satisfying

(1) $\mathrm{Ext}^i_R(C, N) = 0$ for all $i \geqslant 1$;
(2) $\mathrm{Tor}^S_i(\mathrm{Hom}_R(C, N), C) = 0$ for all $i \geqslant 1$;
(3) the natural evaluation homomorphism $\nu_N\colon \mathrm{Hom}_R(C, N) \otimes_S C \to N$ is an isomorphism.

The Auslander class $\mathcal{A}_C(S)$ with respect to ${}_SC_R$ consists of all left S-modules M satisfying

(1) $\mathrm{Tor}^S_i(C, M) = 0$ for all $i \geqslant 1$;
(2) $\mathrm{Ext}^i_R(C, C \otimes_S M) = 0$ for all $i \geqslant 1$;
(3) the natural evaluation homomorphism $\mu_N\colon M \to \mathrm{Hom}_R(C, C \otimes_S M)$ is an isomorphism.

Following from [9], Definition 4.1 and Lemma 3.1, we know that $\mathcal{B}_C(R) = C^\perp \cap \mathrm{Res}({}_RC) = {}_C\mathcal{X}$. Hence we can get a semidualizing (Γ, Δ)-bimodule $\Gamma \otimes_R C$ over the ring extension by Theorem 3.6.

Corollary 4.3. *Let $\xi\colon R \to \Gamma$ be a ring homomorphism and ${}_RC_S$ a semidualizing bimodule. The following conditions are equivalent:*

(1) ${}_R\Gamma^* \in \mathcal{B}_C(R)$ *and* ${}_RU \in \mathcal{B}_C(R)$;
(2) ${}_\Gamma U_\Delta$ *is a semidualizing bimodule and* ${}_RU \in \mathrm{Res}_RC$ *and* $\mathrm{Tor}^R_{\geqslant 1}(\Gamma, C) = 0$.

When they are satisfied, for any Γ*-module* M *and* Δ*-module* N *we have*

$$_{\Gamma}M \in \mathcal{B}_U(\Gamma) \Leftrightarrow {}_R M \in \mathcal{B}_C(R) \quad \textit{and} \quad N_\Delta \in \mathcal{A}_U(\Delta) \Leftrightarrow N_S \in \mathcal{A}_C(S).$$

When both R and Γ are commutative noetherian rings, Christensen studied the semidualizing complexes and Auslander categories under base change in [2], Theorem 5.1. He proved that when C is a semidualizing complex over R, the complex $C \otimes_R \Gamma$ is semidualizing over Γ if and only if $\Gamma \in {}_C\mathcal{A}(R)$, where ${}_C\mathcal{A}(R)$ is the C-Auslander class.

By Corollary 4.3, we can easily prove the module version of [2], Theorem 5.1 generally.

Corollary 4.4. *Let* R *and* Γ *be two commutative rings,* $\xi\colon R \to \Gamma$ *a ring homomorphism and* C *a semidualizing module over* R*. The following conditions are equivalent:*

(1) ${}_R\Gamma^* \in \mathcal{B}_C(R)$;

(2) $U = \Gamma \otimes_R C$ *is a semidualizing* Γ*-module and* $\operatorname{Tor}^R_{\geqslant 1}(C, \Gamma) = 0$.

P r o o f. (1) $\Rightarrow$ (2): By [9], Definition 4.1 and [3], Theorems 3.2.1 and 3.2.13, we can easily show ${}_R\Gamma^* \in \mathcal{B}_C(R)$ if and only if ${}_R\Gamma \in \mathcal{A}_C(R)$. On the other hand, $U = \Gamma \otimes_R C \cong C \otimes_R \Gamma$, as R is commutative. Thus $U \in \mathcal{B}_C(R)$ by [9], Proposition 4.1. Hence ${}_\Gamma U_\Delta$ is a semidualizing bimodule by Corollary 4.3. Also we have $\Delta = \operatorname{Hom}_\Gamma(U, U) = \operatorname{Hom}_\Gamma(\Gamma \otimes_R C, \Gamma \otimes_R C) \cong \operatorname{Hom}_R(C, \Gamma \otimes_R C) \cong \Gamma$, where the first isomorphism follows from the Hom-tensor adjointness and the second from the fact that ${}_R\Gamma \in \mathcal{A}_C(R)$. Hence $U = \Gamma \otimes_R C$ is a semidualizing Γ-module.

(2) $\Rightarrow$ (1): Since $\operatorname{Tor}^R_{\geqslant 1}(C, \Gamma) = 0$ and U is semidualizing, $\operatorname{Ext}^{\geqslant 1}_R(C, U) \cong \operatorname{Ext}^{\geqslant 1}_\Gamma(U, U) = 0$ by Lemma 3.5. Also $\Gamma \cong \operatorname{Hom}_\Gamma(U, U) \cong \operatorname{Hom}_R(C, C \otimes_R \Gamma)$ and $\operatorname{Tor}^R_{\geqslant 1}(C, \Gamma) = 0$, which yields ${}_R\Gamma \in \mathcal{A}_C(R)$ and $\Gamma^* \in \mathcal{B}_C(R)$. □

Particularly, if Γ is a flat R-module, then clearly $\Gamma \in \mathcal{A}_C(R)$ and $\operatorname{Tor}^R_{\geqslant 1}(C, \Gamma) = 0$. So $\Gamma \otimes_R C$ is a semidualizing Γ-bimodule in case C is a semidualizing module over R by Corollary 4.4, which is also proved by Holm and White in [9], Proposition 3.2.

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Author's address: Zhen Zhang, Department of Mathematics, Qilu Normal University, Jinan 250200, P. R. China, e-mail: `157642043@qq.com`.