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RELATIVE TILTING MODULES WITH RESPECT TO A SEMIDUALIZING MODULE

Maryam Salimi, Tehran

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Abstract. Let R be a commutative Noetherian ring, and let C be a semidualizing R-module. The notion of C-tilting R-modules is introduced as the relative setting of the notion of tilting R-modules with respect to C. Some properties of tilting and C-tilting modules and the relations between them are mentioned. It is shown that every finitely generated C-tilting R-module is C-projective. Finally, we investigate some kernel subcategories related to C-tilting modules.

Keywords: tilting module; semidualizing module; C-projective

MSC 2010: 13D05, 13D45

1. Introduction

Throughout this paper R is a commutative Noetherian ring and all modules are unital. Tilting modules are well-known and are useful in the representation theory of Artinian algebras, see for example [3] and [8]. Over the past few years, several mathematicians interested in the representation theory of finite dimensional algebras have developed a technique called "tilting". Given one algebra, one can take the endomorphism ring of a tilting module to get a different algebra which, although not Morita equivalent, has a similar module category. The conditions for an R-module T to be a tilting module are that T should have projective dimension at most one, $\mathrm{Ext}^1_R(T, T^{(k)})$ should be zero for every cardinal k, and there should be a short exact sequence $0 \to R \to T_0 \to T_1 \to 0$ of R-modules, where T_0 and T_1 are direct summands of direct sums of copies of T. In [3], the conditions are relaxed to allow T to have arbitrary finite projective dimension, as long as $\mathrm{Ext}^n_R(T, T^{(k)})$ vanishes for all positive n, and for every cardinal k, also any exact sequence of the form $0 \to R \to T_0 \to \ldots \to T_r \to 0$ in place of a short exact sequence.

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The notion of a "semidualizing module" is a central notion in relative homological algebra. This notion was first introduced by Foxby [6]. Then Vasconcelos [19] and Golod [7] rediscovered these modules using different terminology for different purposes. This notion has been investigated by many authors from different points of view; see for example [1], [4], [9] and [17].

Among various research areas on semidualizing modules, one sometimes focuses on extending the "absolute" classical notion of homological algebra to the "relative" setting with respect to a semidualizing module. For instance, this has been done for the classical and Gorenstein homological dimensions mainly through the works of Golod [7], Holm and Jørgensen [9] and White [20], and (co)homological theories have been extended to the relative setting with respect to a semidualizing module mainly through the works of Takahashi, White [17], Salimi, Tavasoli, Yassemi [14] and Salimi, Sather-Wagstaff, Tavasoli, Yassemi [13].

In this paper, we define relative tilting modules with respect to a semidualizing module and we adduce some examples of these modules. Also, we investigate some properties of C-tilting R-modules and we get relations between tilting and C-tilting R-modules, where C is a semidualizing R-module. Also, we show that every finitely generated C-tilting R-module is C-projective, where C is a semidualizing R-module. Finally, we investigate some kernel subcategories related to C-tilting modules.

2. Preliminaries

Throughout this paper, $\mathcal{M}(R)$ is the category of R-modules. Write $\mathcal{P}(R)$, $\mathcal{F}(R)$ and $\mathcal{I}(R)$ for the subcategories of projective, flat and injective R-modules, respectively. This section contains definitions and background information which will be used in the proof of our main results.

Definition 2.1. An *R-complex* is a sequence of R-module homomorphisms

$$Y = \dots \xrightarrow{\partial^Y_{n+1}} Y_n \xrightarrow{\partial^Y_n} Y_{n-1} \xrightarrow{\partial^Y_{n-1}} \dots$$

such that $\partial^Y_{n-1}\partial^Y_n = 0$ for each integer n. When Y is an R-complex, set $\mathrm{H}_n(Y) = \mathrm{Ker}(\partial^Y_n)/\mathrm{Im}(\partial^Y_{n+1})$ for each n. Given a subcategory $\mathcal{X}$ of $\mathcal{M}(R)$, an R-complex Y is $\mathrm{Hom}_R(\mathcal{X},-)$*-exact* if the complex $\mathrm{Hom}_R(X,Y)$ is exact for each X in $\mathcal{X}$. The term $\mathrm{Hom}_R(-,\mathcal{X})$*-exact* is defined similarly.

In this paper, resolutions are built from precovers, and coresolutions are built from preenvelopes, defined next. For more details about precovers and preenvelopes, the reader may consult [5], Chapters 5 and 6.

Definition 2.2. Let $\mathcal{X}$ be a subcategory of $\mathcal{M}(R)$ and let M be an R-module. An *$\mathcal{X}$-precover* of M is an R-module homomorphism $\varphi\colon X \to M$, where $X \in \mathcal{X}$, and such that the sequence

$$\mathrm{Hom}_R(X', \varphi)\colon \mathrm{Hom}_R(X', X) \to \mathrm{Hom}_R(X', M) \to 0$$

is exact for every $X' \in \mathcal{X}$. If every R-module admits $\mathcal{X}$-precover, then the class $\mathcal{X}$ is *precovering.* The terms *$\mathcal{X}$-preenvelope* and *preenveloping* are defined dually.

Assume that $\mathcal{X}$ is precovering. Then each R-module M has an *augmented proper $\mathcal{X}$-resolution*, that is, an R-complex

$$X^+ = \ldots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \xrightarrow{\tau} M \longrightarrow 0$$

such that $\mathrm{Hom}_R(Y, X^+)$ is exact for all $Y \in \mathcal{X}$. The truncated complex

$$X = \ldots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \longrightarrow 0$$

is a *proper $\mathcal{X}$-resolution* of M. The *$\mathcal{X}$-projective dimension* of M is

$$\mathcal{X}\text{-}\mathrm{pd}_R(M) = \inf\{\sup\{n\colon\ X_n \neq 0\}\colon\ X \text{ is a proper } \mathcal{X}\text{-resolution of } M\}.$$

Proper $\mathcal{X}$-coresolutions and $\mathcal{X}$- id are defined dually.

When $\mathcal{X}$ is the class of projective R-modules, we write $\mathrm{pd}_R(M)$ for the associated homological dimension and call it the *projective dimension* of M. Similarly, the *flat* and *injective dimensions* of M are, respectively, denoted by $\mathrm{fd}_R(M)$ and $\mathrm{id}_R(M)$.

Semidualizing modules, defined next, were first introduced by Foxby [6] and their investigation was furthered by Golod [7] and Vasconcelos [19]. For more details about semidualizing modules the reader may consult [15].

Definition 2.3. A finitely generated R-module C is *semidualizing* if the natural "homothety morphism" $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism and $\mathrm{Ext}_R^i(C, C) = 0$ for $i \geqslant 1$. An R-module D is *dualizing* if it is semidualizing and has finite injective dimension.

Let C be a semidualizing R-module. The classes of *C-projective*, *C-flat*, and *C-injective* modules, denoted respectively by $\mathcal{P}_C(R)$, $\mathcal{F}_C(R)$, and $\mathcal{I}_C(R)$ are defined as

$$\mathcal{P}_C(R) = \{M \cong P \otimes_R C\colon\ P \in \mathcal{P}(R)\},$$
$$\mathcal{F}_C(R) = \{M \cong F \otimes_R C\colon\ F \in \mathcal{F}(R)\},$$
$$\mathcal{I}_C(R) = \{M \cong \mathrm{Hom}_R(C, I)\colon\ I \in \mathcal{I}(R)\}.$$

Remark 2.4. Let C be a semidualizing R-module. In [10], Holm and White proved that the classes $\mathcal{P}_C(R)$ and $\mathcal{F}_C(R)$ are closed under coproducts and summands, and the class $\mathcal{I}_C(R)$ is closed under products and summands. Also, they proved that the classes $\mathcal{P}_C(R)$ and $\mathcal{F}_C(R)$ are precovering, and the class $\mathcal{I}_C(R)$ is preenveloping. Since R is Noetherian and C is finitely generated, it is straightforward to show that the class $\mathcal{F}_C(R)$ is closed under products, and $\mathcal{I}_C(R)$ is closed under coproducts.

Any semidualizing module defines two important classes of modules, namely the Auslander and Bass classes:

Definition 2.5. Let C be a semidualizing R-module. The *Auslander class* with respect to C is the class $\mathcal{A}_C(R)$ of R-modules M such that:

(i) $\mathrm{Tor}_i^R(C, M) = 0 = \mathrm{Ext}_R^i(C, C \otimes_R M)$ for all $i \geqslant 1$, and

(ii) the natural map $\gamma_M^C\colon M \to \mathrm{Hom}_R(C, C \otimes_R M)$ is an isomorphism.

The *Bass class* with respect to C is the class $\mathcal{B}_C(R)$ of R-modules M such that:

(i) $\mathrm{Ext}_R^i(C, M) = 0 = \mathrm{Tor}_i^R(C, \mathrm{Hom}_R(C, M))$ for all $i \geqslant 1$, and

(ii) the natural evaluation map $\xi_M^C\colon C \otimes_R \mathrm{Hom}_R(C, M) \to M$ is an isomorphism.

Remark 2.6. Let C be a semidualizing R-module. The class $\mathcal{A}_C(R)$ contains all R-modules of finite flat dimension and the class $\mathcal{B}_C(R)$ contains all R-modules of finite injective dimension. Also, Takahashi and White in [17], Corollary 2.9 showed that for an R-module M, if $\mathcal{P}_C\text{-}\mathrm{pd}_R(M) < \infty$ (or $\mathcal{I}_C\text{-}\mathrm{id}_R(M) < \infty$), then $M \in \mathcal{B}_C(R)$ (or $M \in \mathcal{A}_C(R)$).

The following functors are studied in [16] and [17].

Definition 2.7. Let C be a semidualizing R-module and let M and N be R-modules. Let L be a proper $\mathcal{P}_C$-resolution of M and let J be a proper $\mathcal{I}_C$-coresolution of N. For each i set

$$\mathrm{Ext}^i_{\mathcal{P}_C}(M, N) := \mathrm{H}_{-i}(\mathrm{Hom}_R(L, N)),$$
$$\mathrm{Ext}^i_{\mathcal{I}_C}(M, N) := \mathrm{H}_{-i}(\mathrm{Hom}_R(M, J)).$$

The following functors are studied in [13].

Definition 2.8. Let C be a semidualizing R-module and let M and N be R-modules. Let L be a proper $\mathcal{P}_C$-resolution of M. For each i set

$$\mathrm{Tor}_i^{\mathcal{P}_C}(M, N) := \mathrm{H}_i(L \otimes_R N).$$

Fact 2.9. *Let C be a semidualizing R-module, let $n \geqslant 0$ be an integer, and let M be an R-module. Then the following statements hold:*

(i) $\mathcal{F}_C$-$\mathrm{pd}_R(M) \leqslant n$ *if and only if* $\mathrm{Tor}_i^{\mathcal{P}_C}(M,-) = 0$ *for all* $i > n$, *see* [13], *Theorem* 4.5.

(ii) $\mathcal{P}_C$-$\mathrm{pd}_R(M) \leqslant n$ *if and only if* $\mathrm{Ext}^i_{\mathcal{P}_C}(M,-) = 0$ *for all* $i > n$, *see* [17], *Theorem* 3.2.

Lemma 2.10. *Let C be a semidualizing R-module and let $\mathbb{L} = (0 \to L' \to L \to L'' \to 0)$ be a complex of R-modules.*

(i) *If $\mathbb{L}$ is $\mathrm{Hom}_R(\mathcal{P}_C,-)$-exact (i.e. if $\mathrm{Hom}_R(C,\mathbb{L})$ is exact, e.g. if $L' \in \mathcal{B}_C(R)$ and $\mathbb{L}$ is exact), then there is a long exact sequence*

$$0 \to \mathrm{Ext}^0_{\mathcal{P}_C}(T,L') \to \mathrm{Ext}^0_{\mathcal{P}_C}(T,L) \to \mathrm{Ext}^0_{\mathcal{P}_C}(T,L'') \to \mathrm{Ext}^1_{\mathcal{P}_C}(T,L') \to \dots$$

that is natural in $\mathbb{L}$ and T.

(ii) *If $\mathbb{L}$ is $(\mathcal{P}_C \otimes_R -)$-exact (i.e. if $C \otimes_R \mathbb{L}$ is exact, e.g. if $L'' \in \mathcal{A}_C(R)$ and $\mathbb{L}$ is exact), then there is a long exact sequence*

$$\dots \to \mathrm{Tor}_1^{\mathcal{P}_C}(T,L'') \to \mathrm{Tor}_0^{\mathcal{P}_C}(T,L') \to \mathrm{Tor}_0^{\mathcal{P}_C}(T,L) \to \mathrm{Tor}_0^{\mathcal{P}_C}(T,L'') \to 0$$

that is natural in $\mathbb{L}$ and T.

Proof. (i) It comes from [5], Theorem 8.2.3.

(ii) By [13], Theorem 3.10, we have $\mathrm{Tor}_i^{\mathcal{P}_C}(M,N) \cong \mathrm{Tor}_i^R(\mathrm{Hom}_R(C,M), C \otimes_R N)$ for all R-modules M and N, and for all $i \geqslant 0$. From the long exact sequence in $\mathrm{Tor}_i^R(\mathrm{Hom}_R(C,T),-)$ associated to the exact sequence $0 \to C \otimes_R L' \to C \otimes_R L \to C \otimes_R L'' \to 0$, we get the assertion. $\square$

For a semidualizing R-module C and R-module M we have $\mathcal{F}_C$-$\mathrm{pd}_R(M) \leqslant \mathcal{P}_C$-$\mathrm{pd}_R(M)$, by [13], Proposition 5.2. Therefore Lemma 2.10 and Fact 2.9 imply the following.

Lemma 2.11. *Let C be a semidualizing R-module and let $0 \to L' \to L \to L'' \to 0$ be an exact sequence of R-modules. Assume that T is an R-module. Then the following statements hold.*

(i) *If $L' \in \mathcal{B}_C(R)$ and $\mathrm{Ext}^i_{\mathcal{P}_C}(T,L) = 0$ for all $i > 0$, then $\mathrm{Ext}^{i+1}_{\mathcal{P}_C}(T,L') \cong \mathrm{Ext}^i_{\mathcal{P}_C}(T,L'')$ for all $i > 0$. In particular, if $\mathcal{P}_C$-$\mathrm{pd}_R(T) \leqslant n$, then*

$$\mathrm{Ext}^n_{\mathcal{P}_C}(T,L'') = 0.$$

(ii) *If $L'' \in \mathcal{A}_C(R)$ and $\mathrm{Tor}_i^{\mathcal{P}_C}(T,L) = 0$ for all $i > 0$, then $\mathrm{Tor}_{i+1}^{\mathcal{P}_C}(T,L'') \cong \mathrm{Tor}_i^{\mathcal{P}_C}(T,L')$ for all $i > 0$. In particular, if $\mathcal{P}_C$-$\mathrm{pd}_R(T) \leqslant n$, then*

$$\mathrm{Tor}_n^{\mathcal{P}_C}(T,L') = 0.$$

Corollary 2.12. *Let C be a semidualizing R-module and let T be an R-module such that $\mathcal{P}_C$-$\mathrm{pd}_R(T) \leqslant n$. Then the following statements hold.*

(i) *Assume that $X_n \to X_{n-1} \to \ldots \to X_1 \to Y_0 \to 0$ is an exact sequence of R-modules such that $Y_0 \in \mathcal{B}_C(R)$ and $X_j \in \mathcal{B}_C(R)$ for all $1 \leqslant j \leqslant n$. If $\mathrm{Ext}^i_{\mathcal{P}_C}(T, X_j) = 0$ for all $i > 0$ and $1 \leqslant j \leqslant n$, then $\mathrm{Ext}^i_{\mathcal{P}_C}(T, Y_0) = 0$ for all $i > 0$.*

(ii) *Assume that $0 \to Y_0 \to X_1 \to X_1 \to \ldots \to X_n$ is an exact sequence of R-modules such that $Y_0 \in \mathcal{A}_C(R)$ and $X_j \in \mathcal{A}_C(R)$ for all $1 \leqslant j \leqslant n$. If $\mathrm{Tor}_i^{\mathcal{P}_C}(T, X_j) = 0$ for all $i > 0$ and $1 \leqslant j \leqslant n$, then $\mathrm{Tor}_i^{\mathcal{P}_C}(T, Y_0) = 0$ for all $i > 0$.*

P r o o f. We just prove item (i). The proof of item (ii) is dual.

(i) Consider the following short exact sequences of R-modules:

$$\begin{gathered} 0 \to Y_1 \to X_1 \to Y_0 \to 0 \\ 0 \to Y_2 \to X_2 \to Y_1 \to 0 \\ \vdots \\ 0 \to Y_n \to X_n \to Y_{n-1} \to 0. \end{gathered}$$

Note that $Y_j \in \mathcal{B}_C(R)$ for $0 \leqslant j \leqslant n$, by [15], Proposition 3.1.7. By Lemma 2.11, $\mathrm{Ext}^n_{\mathcal{P}_C}(T, Y_j) = 0$ for $0 \leqslant j \leqslant n-1$. Since Fact 2.9 implies that $\mathrm{Ext}^r_{\mathcal{P}_C}(T, -) = 0$ for $r > n$, using Lemma 2.10 we find that

$$0 = \mathrm{Ext}^{i+n-1}_{\mathcal{P}_C}(T, Y_{n-1}) \cong \mathrm{Ext}^{i+n-2}_{\mathcal{P}_C}(T, Y_{n-2}) \cong \ldots \cong \mathrm{Ext}^i_{\mathcal{P}_C}(T, Y_0)$$

for all $i > 0$. $\square$

The next lemma follows easily from Lemma 2.10.

Lemma 2.13. *Let C be a semidualizing R-module, let $r \geqslant 1$ be an integer and let T be an R-module. Then the following statements hold.*

(i) *Assume that $0 \to X \to V_0 \to \ldots \to V_r \to 0$ is an exact sequence of R-modules such that $X \in \mathcal{B}_C(R)$ and $V_j \in \mathcal{B}_C(R)$ for all $0 \leqslant j \leqslant r$. If $\mathrm{Ext}^i_{\mathcal{P}_C}(T, V_j) = 0$ for all $i > 0$ and $0 \leqslant j \leqslant r$, then $\mathrm{Ext}^i_{\mathcal{P}_C}(T, X) = 0$ for all $i \geqslant r+1$.*

(ii) *Assume that $0 \to V_r \to V_{r-1} \to \ldots \to V_0 \to X \to 0$ is an exact sequence of R-modules such that $X \in \mathcal{A}_C(R)$ and $V_j \in \mathcal{A}_C(R)$ for all $0 \leqslant j \leqslant r$. If $\mathrm{Tor}_i^{\mathcal{P}_C}(T, V_j) = 0$ for all $i > 0$ and $0 \leqslant j \leqslant r$, then $\mathrm{Tor}_i^{\mathcal{P}_C}(T, X) = 0$ for all $i \geqslant r+1$.*

Proposition 2.14. *Let C be a semidualizing R-module, let $r \geqslant 1$ be an integer and let T be an R-module. Then the following statements hold.*

(1) *Assume that there exists an exact sequence*

$$0 \to X \to V_0 \to \ldots \to V_r \to 0$$

of R-modules such that $X \in \mathcal{B}_C(R)$, $V_j \in \mathcal{B}_C(R)$ and $\mathrm{Ext}^i_{\mathcal{P}_C}(T, V_j) = 0$ for all $i > 0$ and $0 \leqslant j \leqslant r$. Then the following conditions are equivalent.

(i) $\mathrm{Ext}^i_{\mathcal{P}_C}(T, X) = 0$ *for all $i \geqslant 0$ and $i < r$.*

(ii) *The induced sequence*

$$0 \to \mathrm{Ext}^0_{\mathcal{P}_C}(T, V_0) \to \ldots \to \mathrm{Ext}^0_{\mathcal{P}_C}(T, V_{r-1}) \to \mathrm{Ext}^0_{\mathcal{P}_C}(T, V_r)$$

is exact.

When these equivalent conditions hold,

$$\mathrm{Ext}^r_{\mathcal{P}_C}(T, X) \cong \mathrm{Coker}(\mathrm{Ext}^0_{\mathcal{P}_C}(T, V_{r-1}) \to \mathrm{Ext}^0_{\mathcal{P}_C}(T, V_r)).$$

(2) *Assume that there exists an exact sequence*

$$0 \to V_r \to V_{r-1} \to \ldots \to V_0 \to X \to 0$$

of R-modules such that $X \in \mathcal{A}_C(R)$, $V_j \in \mathcal{A}_C(R)$ and $\mathrm{Tor}_i^{\mathcal{P}_C}(T, V_j) = 0$ for all $i > 0$ and $0 \leqslant j \leqslant r$. Then the following conditions are equivalent.

(i) $\mathrm{Tor}_i^{\mathcal{P}_C}(T, X) = 0$ *for all $i \geqslant 0$ and $i < r$.*

(ii) *The induced sequence*

$$\mathrm{Tor}_0^{\mathcal{P}_C}(T, V_r) \to \mathrm{Tor}_0^{\mathcal{P}_C}(T, V_{r-1}) \to \ldots \to \mathrm{Tor}_0^{\mathcal{P}_C}(T, V_0) \to 0$$

is exact.

When these equivalent conditions hold,

$$\mathrm{Tor}_r^{\mathcal{P}_C}(T, X) \cong \mathrm{Ker}(\mathrm{Tor}_0^{\mathcal{P}_C}(T, V_r) \to \mathrm{Tor}_0^{\mathcal{P}_C}(T, V_{r-1})).$$

Proof. We just prove item (1). The proof of item (2) is dual.

(1) It suffices to prove that item (ii) is equivalent to the fact that $\mathrm{Ext}^i_{\mathcal{P}_C}(T, X) = 0$ for all $0 \leqslant i \leqslant r-1$, by Lemma 2.13. Denote by X_i the kernel of the map $V_i \to V_{i+1}$ and put $X = X_0$. Note that $X_i \in \mathcal{B}_C(R)$ for all $0 \leqslant i \leqslant r$ by [15], Proposition 3.1.7. Then from each exact sequence $0 \to X_i \to V_i \to X_{i+1} \to 0$, we get the following exact sequence of R-modules

$$0 \to \mathrm{Ext}^0_{\mathcal{P}_C}(T, X_i) \to \mathrm{Ext}^0_{\mathcal{P}_C}(T, V_i) \to \mathrm{Ext}^0_{\mathcal{P}_C}(T, X_{i+1}) \to \mathrm{Ext}^1_{\mathcal{P}_C}(T, X_i) \to 0$$

for all $0 \leqslant i \leqslant r-2$. Therefore item (ii) is equivalent to the fact that $\mathrm{Ext}^0_{\mathcal{P}_C}(T, X) = 0$ and $\mathrm{Ext}^1_{\mathcal{P}_C}(T, X_j) = 0$ for all $0 \leqslant j \leqslant r-2$. On the other hand, the condition $\mathrm{Ext}^i_{\mathcal{P}_C}(T, V_j) = 0$ for all $i > 0$ and $0 \leqslant j \leqslant r$ implies that

$$\mathrm{Ext}^1_{\mathcal{P}_C}(T, X_{i-1}) \cong \mathrm{Ext}^2_{\mathcal{P}_C}(T, X_{i-2}) \cong \ldots \cong \mathrm{Ext}^i_{\mathcal{P}_C}(T, X_0)$$

by Lemma 2.11. Then we get the equivalence of (i) and (ii). Also, when these equivalent conditions hold, the exact sequence

$$0 \to \mathrm{Ext}^0_{\mathcal{P}_C}(T, X_{r-1}) \to \mathrm{Ext}^0_{\mathcal{P}_C}(T, V_{r-1}) \to \mathrm{Ext}^0_{\mathcal{P}_C}(T, X_r) \to \mathrm{Ext}^1_{\mathcal{P}_C}(T, X_{r-1}) \to 0$$

and the isomorphism $\mathrm{Ext}^r_{\mathcal{P}_C}(T, X) \cong \mathrm{Ext}^1_{\mathcal{P}_C}(T, X_{r-1})$ imply that $\mathrm{Ext}^r_{\mathcal{P}_C}(T, X) \cong \mathrm{Coker}(\mathrm{Ext}^0_{\mathcal{P}_C}(T, V_{r-1}) \to \mathrm{Ext}^0_{\mathcal{P}_C}(T, V_r))$. □

Convention 2.15. *Let M be an R-module. For every cardinal k let $M^{(k)}$ be the direct sum of k copies of M and let $\mathrm{Add}(M)$ be the collection of all direct summands of arbitrary direct sums of M.*

3. Relative tilting modules

Let C be a semidualizing R-module. In this section, we introduce the notion of C-tilting R-modules as the relative setting of the notion of tilting R-modules with respect to C and we adduce some examples of these modules. Also, this section contains some properties of tilting and C-tilting R-modules and the relations between them.

Definition 3.1. An R-module T is called *generalized tilting* when the following conditions are satisfied.

(1) $\mathrm{pd}_R(T) < \infty$.
(2) $\mathrm{Ext}^i_R(T, T^{(k)}) = 0$ for each i, $1 \leqslant i \leqslant \mathrm{pd}_R(T)$ and every cardinal k.
(3) There exists a long exact sequence $0 \to R \to T_0 \to T_1 \to \ldots \to T_r \to 0$, where $r \geqslant 0$ and $T_i \in \mathrm{Add}(T)$ for all $0 \leqslant i \leqslant r$.

Example 3.2. Every finitely generated free module is generalized tilting.

Definition 3.3. Let C be a semidualizing R-module. An R-module T is called *C-tilting* when the following conditions are satisfied.

(1) $\mathcal{P}_C$-$\mathrm{pd}_R(T) < \infty$.
(2) $\mathrm{Ext}^i_{\mathcal{P}_C}(T, T^{(k)}) = 0$ for each i, $1 \leqslant i \leqslant \mathcal{P}_C$-$\mathrm{pd}_R(T)$ and every cardinal k.
(3) There exists a long exact sequence $0 \to C \to T_0 \to T_1 \to \ldots \to T_r \to 0$, where $r \geqslant 0$ and $T_i \in \mathrm{Add}(T)$ for all $0 \leqslant i \leqslant r$.

Example 3.4. Let C be a semidualizing R-module and let $n \in \mathbb{N}$. Then C^n is C-tilting. Note that $C^n \in \mathcal{P}_C(R)$, therefore $\mathcal{P}_C\text{-pd}_R(C^n) = 0$ and for all cardinals k and all $i \geqslant 1$ we have $\operatorname{Ext}^i_{\mathcal{P}_C}(C^n, (C^n)^{(k)}) = 0$ by [17], Theorem 3.1.

Example 3.5. Let C be a dualizing R-module such that $\operatorname{id}_R(C) \leqslant n$.

(i) Consider the following minimal injective resolution of C:

$$0 \to C \to I_0 \to \ldots \to I_n \to 0.$$

Set $T = \bigoplus_{i \leqslant n} I_i$. Then T is a C-tilting R-module. Indeed, since T is an injective R-module, [12], Proposition 3.3 implies that $\mathcal{P}_C\text{-pd}_R(T) \leqslant n$. Note that T and $T^{(k)}$ belong to $\mathcal{B}_C(R)$ for all cardinal k. Therefore, by [17], Corollary 4.2, $\operatorname{Ext}^i_{\mathcal{P}_C}(T, T^{(k)}) \cong \operatorname{Ext}^i_R(T, T^{(k)}) = 0$ for all cardinal k and all i, $1 \leqslant i \leqslant \mathcal{P}_C\text{-pd}_R(T)$.

(ii) By [18], Proposition 2.4, $\mathcal{I}_C\text{-}\operatorname{id}_R(R) \leqslant n$. Therefore, by [17], Corollary 2.3, there exists the long exact sequence

$$0 \to R \to \operatorname{Hom}_R(C, I_0) \to \ldots \to \operatorname{Hom}_R(C, I_n) \to 0$$

of R-modules such that I_i is injective for all $i = 0, \ldots, n$. Set

$$T = \bigoplus_{i \leqslant n} \operatorname{Hom}_R(C, I_i).$$

Then T is a generalized tilting R-module. Indeed, since T is a C-injective R-module, [12], Proposition 3.3 implies that $\operatorname{pd}_R(T) \leqslant n$. Note that for all cardinal k and all i, $1 \leqslant i \leqslant \operatorname{pd}_R(T)$, we have

$$\begin{aligned} \operatorname{Ext}^i_R(T, T^{(k)}) &\cong \bigoplus_{j \leqslant n} \bigoplus_{r \leqslant n} \operatorname{Ext}^i_R(\operatorname{Hom}_R(C, I_j), (\operatorname{Hom}_R(C, I_r))^{(k)}) \\ &\cong \bigoplus_{j \leqslant n} \bigoplus_{r \leqslant n} \operatorname{Ext}^i_R(I_j, (I_r)^{(k)}) = 0, \end{aligned}$$

by [15], Lemma 3.1.13.

Remark 3.6. Let C be a semidualizing R-module. Then the following statements hold.

(i) Let T be a generalized tilting R-module. Then $\operatorname{pd}_R(T) < \infty$ and Remark 2.6 implies that $T \in \mathcal{A}_C(R)$.

(ii) Let T be a C-tilting R-module. Then $\mathcal{P}_C\text{-pd}_R(T) < \infty$ and Remark 2.6 implies that $T \in \mathcal{B}_C(R)$.

The functors $C \otimes_R -\colon \mathcal{A}_C(R) \to \mathcal{B}_C(R)$ and $\operatorname{Hom}_R(C,-)\colon \mathcal{B}_C(R) \to \mathcal{A}_C(R)$ establish an equivalence of categories between the Auslander class and the Bass class. This is usually called the Foxby equivalence between the two classes. Remark 3.6 leads to the following theorem.

Theorem 3.7. *Let C be a semidualizing R-module. Then the following statements hold.*

(i) *If T is a generalized tilting R-module, then $C \otimes_R T$ is a C-tilting R-module.*

(ii) *If T is a C-tilting R-module, then $\operatorname{Hom}_R(C,T)$ is a generalized tilting R-module.*

Proof. (i) Let T be a generalized tilting R-module. Then $\operatorname{pd}_R(T) < \infty$. By Remark 3.6, $T \in \mathcal{A}_C(R)$ and by [17], Theorem 2.11, $\mathcal{P}_C\text{-}\operatorname{pd}_R(C \otimes_R T) = \operatorname{pd}_R(\operatorname{Hom}_R(C, C \otimes_R T)) = \operatorname{pd}_R(T) < \infty$. By [17], Theorem 2.8, $C \otimes_R T \in \mathcal{B}_C(R)$, and by [15], Proposition 3.1.6, $(C \otimes_R T)^{(k)} \in \mathcal{B}_C(R)$ for every cardinal k. So, we have

$$\operatorname{Ext}^i_{\mathcal{P}_C}(C \otimes_R T, (C \otimes_R T)^{(k)}) \cong \operatorname{Ext}^i_R(C \otimes_R T, (C \otimes_R T)^{(k)}) \cong \operatorname{Ext}^i_R(T, T^{(k)}) = 0$$

or all i, $1 \leqslant i \leqslant \mathcal{P}_C\text{-}\operatorname{pd}_R(C \otimes_R T)$. In the above sequence, the first isomorphism follows from [17], Corollary 4.2, and the second isomorphism is induced by [15], Proposition 3.1.13. Since T is a generalized tilting R-module, there exists a long exact sequence $0 \to R \to T_0 \to \ldots \to T_r \to 0$, where $r \geqslant 0$, and $T_i \in \operatorname{Add}(T)$ for all i, $0 \leqslant i \leqslant r$. By [15], Proposition 3.1.6, $T_i \in \mathcal{A}_C(R)$ and therefore $\operatorname{Tor}^R_j(C, T_i) = 0$ for every $j \geqslant 1$ and $0 \leqslant i \leqslant r$. The fact that $R \in \mathcal{A}_C(R)$ and [15], Proposition 3.1.7 imply that there exists a long exact sequence $0 \to C \to C \otimes_R T_0 \to C \otimes_R T_1 \to \ldots \to C \otimes_R T_r \to 0$, where $T_i \in \operatorname{Add}(T)$ and therefore $C \otimes_R T_i \in \operatorname{Add}(C \otimes_R T)$ for all i, $0 \leqslant i \leqslant r$. So, we get the assertion.

(ii) Let T be a C-tilting R-module. Then $\mathcal{P}_C\text{-}\operatorname{pd}_R(T) < \infty$. By [17], Theorem 2.11, $\operatorname{pd}_R(\operatorname{Hom}_R(C,T)) = \mathcal{P}_C\text{-}\operatorname{pd}_R(T) < \infty$. Also, [17], Theorem 4.1 implies that

$$\operatorname{Ext}^i_R(\operatorname{Hom}_R(C,T), (\operatorname{Hom}_R(C,T))^{(k)}) \cong \operatorname{Ext}^i_{\mathcal{P}_C}(T, T^{(k)}) = 0$$

for every cardinal k since $T \in \mathcal{B}_C(R)$. Since T is C-tilting, there exists a long exact sequence $0 \to C \to T_0 \to T_1 \to \ldots \to T_r \to 0$, where $r \geqslant 0$ and $T_i \in \operatorname{Add}(T)$ for all i, $0 \leqslant i \leqslant r$. By [15], Proposition 3.1.6, $T_i \in \mathcal{B}_C(R)$ and therefore $\operatorname{Ext}^j_R(C, T_i) = 0$ for every $j \geqslant 1$ and all i, $0 \leqslant i \leqslant r$. The fact that $C \in \mathcal{B}_C(R)$ and [15], Proposition 3.1.7 imply the following long exact sequence:

$$0 \to \operatorname{Hom}_R(C,C) \cong R \to \operatorname{Hom}_R(C,T_0) \to \ldots \to \operatorname{Hom}_R(C,T_r) \to 0,$$

where $\operatorname{Hom}_R(C,T_i) \in \operatorname{Add}(\operatorname{Hom}_R(C,T))$. So, we get the assertion. $\square$

Note that Theorem 3.7 shows that the endomorphism algebras one gets from C-tilting modules are exactly the endomorphism algebras one gets from the generalized tilting modules, as follows.

First, if T is a generalized tilting module, then $C \otimes_R T$ is C-tilting, and we have

$$\begin{aligned}\operatorname{End}_R(C \otimes_R T) = \operatorname{Hom}_R(C \otimes_R T, C \otimes_R T) &\cong \operatorname{Hom}_R(T, \operatorname{Hom}_R(C, C \otimes_R T)) \\ &\cong \operatorname{Hom}_R(T,T) = \operatorname{End}_R(T).\end{aligned}$$

The second isomorphism here is from the condition $T \in \mathcal{A}_C(R)$. Note that the priori R-isomorphism $\operatorname{End}_R(C \otimes_R T) \cong \operatorname{End}_R(T)$ respects the ring structure (i.e. the composition product) in the endomorphism algebras.

On the other hand, if T is C-tilting, then $\operatorname{Hom}_R(C,T)$ is generalized tilting and a similar argument shows that

$$\begin{aligned}\operatorname{End}_R(\operatorname{Hom}_R(C,T)) &= \operatorname{Hom}_R(\operatorname{Hom}_R(C,T), \operatorname{Hom}_R(C,T)) \\ &\cong \operatorname{Hom}_R(T,T) = \operatorname{End}_R(T).\end{aligned}$$

Definition 3.8. Let C be a semidualizing R-module, and let T be a C-tilting R-module. We say that T is a *good C-tilting module* if the short exact sequence of condition (3) of Definition 3.3 has the form $0 \to C \to T_0 \to T_1 \to \ldots \to T_r \to 0$, where the T_i's are direct summands of finite direct sums of copies of T. If $C = R$, then we simply say T is a *good tilting module*, instead of saying T is a good R-tilting module.

Remark 3.9. Let C be a semidualizing R-module. Using the proof of Theorem 3.7, one can easily check the following statements.

(i) If T is a good tilting R-module, then $C \otimes_R T$ is a good C-tilting R-module.

(ii) If T is a good C-tilting R-module, then $\operatorname{Hom}_R(C,T)$ is a good tilting R-module.

Let T be a good tilting R-module and let $S = \operatorname{End}_R(T)$. In [2], Proposition 1.4, it is proved that $\operatorname{End}_S(T) \cong R$. In the following, we investigate this property for a good C-tilting R-module.

Proposition 3.10. *Let C be a semidualizing R-module and let T be a good C-tilting R-module. Assume that $S = \operatorname{End}_R(T)$. Then the S-module $\operatorname{Hom}_R(C,T)$ has a projective resolution $0 \to Q_r \to \ldots \to Q_0 \to \operatorname{Hom}_R(C,T) \to 0$, where Q_i's are direct summands of a finite direct sums of copies of S, and $\operatorname{Ext}^i_S(\operatorname{Hom}_R(C,T),T) = 0$ for all $i > 0$ and $\operatorname{End}_S(T) \cong R$.*

P r o o f. Let T be a good C-tilting R-module. So, there exists the exact sequence

$$(\dagger) \qquad 0 \to C \to T_0 \to T_1 \to \ldots \to T_r \to 0,$$

where T_i's are direct summands of finite direct sums of copies of T. Therefore $T_i \in \mathcal{B}_C(R)$ for all $0 \leqslant i \leqslant r$ by [15], Proposition 3.1.6 and Remark 3.6. By [17], Corollary 4.2, $\mathrm{Ext}^j_R(T_i, T) \cong \mathrm{Ext}^j_{\mathcal{P}_C}(T_i, T) = 0$ for all $j \geqslant 1$, and $0 \leqslant i \leqslant r$. Denote by K_i the kernel of the map $T_i \to T_{i+1}$ for $1 \leqslant i \leqslant r-1$. Applying the functor $\mathrm{Hom}_R(-, T)$ to (†), we get by dimension shifting that $\mathrm{Ext}^j_R(K_i, T) = 0$ for all $j \geqslant 1$ and $0 \leqslant i \leqslant r-1$. Therefore we get the following exact sequence of S-modules:

$$(\dagger\dagger) \qquad 0 \to \mathrm{Hom}_R(T_r, T) \to \ldots \to \mathrm{Hom}_R(T_0, T) \to \mathrm{Hom}_R(C, T) \to 0.$$

Note that the S-module $\mathrm{Hom}_R(T_i, T)$ is a direct summand of a finite direct sums of copies of S. Hence the canonical map $\delta_{T_i}\colon T_i \to \mathrm{Hom}_S(\mathrm{Hom}_R(T_i, T), T)$ is an isomorphism, and $\mathrm{Ext}^1_S(\mathrm{Hom}_R(T_i, T), T) = 0$ for $0 \leqslant i \leqslant r$. For an R-module M, we denote the S-module $\mathrm{Hom}_R(M, T)$ by M^* and the R-module $\mathrm{Hom}_S(\mathrm{Hom}_R(M, T), T)$ by M^{**}. Applying the functor $\mathrm{Hom}_S(-, T)$ to (††), we get the following commutative diagrams with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r] \arrow[d, "\delta_C"] & T_0 \arrow[r] \arrow[d, "\delta_{T_0}"] & K_1 \arrow[rr] \arrow[d, "\delta_{K_1}"] & & 0 \\
0 \arrow[r] & C^{**} \arrow[r] & T_0^{**} \arrow[r] & K_1^{**} \arrow[r] & \mathrm{Ext}^1_S(\mathrm{Hom}_R(C,T),T) \arrow[r] & 0
\end{tikzcd}$$

$$\vdots$$

$$\begin{tikzcd}
0 \arrow[r] & K_{r-1} \arrow[r] \arrow[d, "\delta_{K_{r-1}}"] & C \otimes_R T_{r-1} \arrow[r] \arrow[d, "\delta_{T_{r-1}}"] & C \otimes_R T_r \arrow[rr] \arrow[d, "\delta_{T_r}"] & & 0 \\
0 \arrow[r] & K_{r-1}^{**} \arrow[r] & T_{r-1}^{**} \arrow[r] & T_r^{**} \arrow[r] & \mathrm{Ext}^1_S(K_{r-1}^*, T) \arrow[r] & 0.
\end{tikzcd}$$

Since δ_{T_i}'s are isomorphisms, we get

$$\mathrm{Ext}^1_S(\mathrm{Hom}_R(C, T), T) = 0, \quad 0 = \mathrm{Ext}^1_S(K_i^*, T) \cong \mathrm{Ext}^{i+1}_S(\mathrm{Hom}_R(C, T), T)$$

for every $0 \leqslant i \leqslant r-1$. Therefore $\mathrm{Ext}^i_S(\mathrm{Hom}_R(C, T), T) = 0$ for all $i > 0$ since $\mathrm{pd}_S(\mathrm{Hom}_R(C, T)) \leqslant r$. Also, $C \cong C^{**} = \mathrm{Hom}_S(\mathrm{Hom}_R(C, T), T)$. On the other hand,

$$\begin{aligned} R &\cong \mathrm{Hom}_R(C, C) \cong \mathrm{Hom}_R(C, \mathrm{Hom}_S(\mathrm{Hom}_R(C, T), T)) \\ &\cong \mathrm{Hom}_S(C \otimes_R \mathrm{Hom}_R(C, T), T) \cong \mathrm{Hom}_S(T, T) = \mathrm{End}_S(T). \end{aligned}$$

So, we get the assertion. □

Fact 3.11. *Let C be a dualizing R-module with $\mathrm{id}_R(C) \leqslant 1$ and let*

$$P_0 = \{\mathfrak{p} \in \mathrm{Spec}(R)\colon \mathrm{ht}(\mathfrak{p}) = 0\},$$
$$P_1 = \{\mathfrak{p} \in \mathrm{Spec}(R)\colon \mathrm{ht}(\mathfrak{p}) = 1\}.$$

By [18], *Proposition* 3.1, *the minimal injective coresolution of* C *has the form*

$$0 \to C \to G \xrightarrow{\pi} \bigoplus_{\mathfrak{p}\in P_1} \mathrm{E}(R/\mathfrak{p}) \to 0,$$

where $G = \bigoplus_{\mathfrak{p}\in P_0} \mathrm{E}(R/\mathfrak{p})$.

Proposition 3.12. *Let* C *be a dualizing* R*-module with* $\mathrm{id}_R(C) \leqslant 1$ *and let* P_0, P_1 *and* π *be as in Fact* 3.11. *Consider a subset* $P \subseteq P_1$ *and put* $C_P = \pi^{-1}\Big(\bigoplus_{\mathfrak{p}\in P} \mathrm{E}(R/\mathfrak{p})\Big)$ *and* $T_P = C_P \bigoplus \Big(\bigoplus_{\mathfrak{p}\in P} \mathrm{E}(R/\mathfrak{p})\Big)$. *Then* T_P *is a* C*-tilting* R*-module.*

P r o o f. Note that there is a commutative diagram with exact rows as follows.

$$\begin{tikzcd}
(*) & 0 \arrow[r] & C \arrow[r] \arrow[d, "\cong"] & C_P \arrow[r] \arrow[d, hook] & \bigoplus_{\mathfrak{p}\in P} \mathrm{E}(R/\mathfrak{p}) \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] & C \arrow[r] & G \arrow[r, "\pi"] & \bigoplus_{\mathfrak{p}\in P_1} \mathrm{E}(R/\mathfrak{p}) \arrow[r] & 0.
\end{tikzcd}$$

The exact row $(*)$ implies that $\mathrm{id}_R(C_P) < \infty$. Therefore, $\mathrm{id}_R(T_P) < \infty$ and so $\mathcal{P}_C\text{-}\mathrm{pd}_R(T_P) \leqslant 1$, by [12], Proposition 3.3. By the Snake lemma $G/C_P \cong \bigoplus_{\mathfrak{p}\in P_1 - P} \mathrm{E}(R/\mathfrak{p})$, therefore for each cardinal k we have the following exact sequence of R-modules:

$$(**) \qquad 0 \longrightarrow C_P^{(k)} \longrightarrow G^{(k)} \longrightarrow \bigoplus_{\mathfrak{p}\in P_1 - P} \mathrm{E}(R/\mathfrak{p})^{(k)} \longrightarrow 0.$$

By [5], Theorem 3.3.8, we have $\mathrm{Hom}_R(\mathrm{E}(R/\mathfrak{p}), (G/C_P)^{(k)}) = 0$ for every $\mathfrak{p} \in P$ and so the exact sequence $(**)$ implies that $\mathrm{Ext}^1_R(\mathrm{E}(R/\mathfrak{p}), C_P^{(k)}) = 0$ for every $\mathfrak{p} \in P$. Therefore, the exact sequence $(*)$ implies that $\mathrm{Ext}^1_R(C_P, C_P^{(k)}) = 0$ for every cardinal k, because $\mathrm{id}_R(C_P) < \infty$ implies that $C_P^{(k)} \in \mathcal{B}_C(R)$. So, we have

$$\begin{aligned}
\mathrm{Ext}^1_{\mathcal{P}_C}(T_P, (T_P)^{(k)}) &\cong \mathrm{Ext}^1_R(T_P, (T_P)^{(k)}) \\
&\cong \mathrm{Ext}^1_R\bigg(C_P \oplus \Big(\bigoplus_{\mathfrak{p}\in P} \mathrm{E}(R/\mathfrak{p})\Big), \Big(C_P \oplus \Big(\bigoplus_{\mathfrak{p}\in P} \mathrm{E}(R/\mathfrak{p})\Big)\Big)^{(k)}\bigg) \\
&\cong \mathrm{Ext}^1_R(C_P, (C_P)^{(k)}) \oplus \mathrm{Ext}^1_R\bigg(C_P, \Big(\bigoplus_{\mathfrak{p}\in P} \mathrm{E}(R/\mathfrak{p})\Big)^{(k)}\bigg) \\
&\quad \oplus \mathrm{Ext}^1_R\bigg(\bigoplus_{\mathfrak{p}\in P} \mathrm{E}(R/\mathfrak{p}), \Big(\bigoplus_{\mathfrak{p}\in P} \mathrm{E}(R/\mathfrak{p})\Big)^{(k)}\bigg) \\
&\quad \oplus \mathrm{Ext}^1_R\bigg(\bigoplus_{\mathfrak{p}\in P} \mathrm{E}(R/\mathfrak{p}), (C_P)^{(k)}\bigg) = 0
\end{aligned}$$

for every cardinal k. In the above sequence, the first isomorphism follows from [17], Corollory 4.2. Note that $C_P \in \operatorname{Add}(T_P)$ and $\bigoplus_{\mathfrak{p}\in P} \mathrm{E}(R/\mathfrak{p}) \in \operatorname{Add}(T_P)$. So, the exact sequence $(*)$ is the desired sequence. $\square$

Remark 3.13. In the notation of Proposition 3.12,

$$\begin{aligned}\operatorname{Hom}_R(C, T_P) &\cong \operatorname{Hom}_R\Bigl(C, C_P \oplus \Bigl(\bigoplus_{\mathfrak{p}\in P} \mathrm{E}(R/\mathfrak{p})\Bigr)\Bigr)\\ &\cong \operatorname{Hom}_R(C, C_P) \oplus \operatorname{Hom}_R\Bigl(C, \bigoplus_{\mathfrak{p}\in P} \mathrm{E}(R/\mathfrak{p})\Bigr)\\ &\cong \operatorname{Hom}_R(C, C_P) \oplus \Bigl(\bigoplus_{\mathfrak{p}\in P} \operatorname{Hom}_R(C, \mathrm{E}(R/\mathfrak{p}))\Bigr)\end{aligned}$$

is generalized tilting by Theorem 3.7.

In the sequel, we show that every finitely generated C-tilting R-module is C-projective, where C is a semidualizing R-module. First we prove the following lemma.

Lemma 3.14. *Let $M \neq 0$ be a finitely generated R-module and let $n = \operatorname{pd}_R(M) < \infty$. Then $\operatorname{Ext}^n_R(M, M) \neq 0$.*

Proof. It suffices to prove the statement for a local case since $n = \operatorname{pd}_R(M) < \infty$ implies that there is a maximal ideal $\mathfrak{m}$ such that $\operatorname{pd}_{R_\mathfrak{m}}(M_\mathfrak{m}) = n$ and $0 \neq \operatorname{Ext}^n_{R_\mathfrak{m}}(M_\mathfrak{m}, M_\mathfrak{m}) \cong (\operatorname{Ext}^n_R(M, M))_\mathfrak{m}$, as desired. Now assume that $(R, \mathfrak{m})$ is a local ring. If $n = 0$, then $0 \neq M \cong R^m$ for some $m \in \mathbb{N}$, so $\operatorname{Hom}_R(M, M) \cong R^{m^2} \neq 0$. Therefore, suppose that $1 \leqslant n = \operatorname{pd}_R(M) < \infty$. Consider the projective resolution of M

$$0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to M \to 0,$$

where P_i is a finitely generated projective R-module for all i, $0 \leqslant i \leqslant n$. Let N be the $(n-1)$th syzygy of M. Then $\operatorname{pd}_R(N) = 1$. Assume that $\operatorname{Ext}^n_R(M, M) = 0$. Then $\operatorname{Ext}^1_R(N, M) \cong \operatorname{Ext}^n_R(M, M) = 0$. Since $M \neq 0$ is a finitely generated R-module, M contains a maximal submodule K and we have the following short exact sequence of R-modules:

$$(***) \qquad 0 \to K \to M \to R/\mathfrak{m} \to 0.$$

Applying $\operatorname{Hom}_R(N, -)$ on sequence $(***)$, we get that $\operatorname{Ext}^1_R(N, R/\mathfrak{m}) = 0$. So N is a projective R-module, which is contradiction. $\square$

Proposition 3.15. *Let C be a semidualizing R-module and let $T \neq 0$ be a finitely generated R-module. Then the following statements hold.*

(i) *If $n = \mathcal{P}_C\text{-}\operatorname{pd}_R(T) < \infty$, then $\operatorname{Ext}^n_{\mathcal{P}_C}(T, T) \neq 0$.*

(ii) *If T is a C-tilting R-module, then T is C-projective.*

P r o o f. (i) It is clear that $\operatorname{Hom}_R(C,T)$ is a finitely generated R-module and by [15], Corollary 2.1.17, $\operatorname{Hom}_R(C,T)\neq 0$. Also, $1\leqslant n=\operatorname{pd}_R(\operatorname{Hom}_R(C,T))<\infty$ by [17], Theorem 2.11. By Lemma 3.14, we have

$$\operatorname{Ext}^n_R(\operatorname{Hom}_R(C,T),\operatorname{Hom}_R(C,T))\neq 0.$$

On the other hand, $\operatorname{Ext}^n_{\mathcal{P}_C}(T,T)\cong\operatorname{Ext}^n_R(\operatorname{Hom}_R(C,T),\operatorname{Hom}_R(C,T))$ by [17], Theorem 4.1. So, we get the assertion.

(ii) Using (i) and the definition of C-tilting R-module, we get the assertion. $\square$

Example 3.16. Note that Proposition 3.12 shows that the conclusion of Proposition 3.15 (ii) fails when T is not finitely generated.

4. Kernels of some appropriate subcategories related to relative tilting modules

An R-module T is called *classical tilting*, in sense of [11], if the following statements hold.

(i) T has a projective resolution $0\to P_n\to P_{n-1}\to\ldots\to P_0\to 0$, where P_i is finitely generated.

(ii) $\operatorname{Ext}^i_R(T,T)=0$ for all $1\leqslant i\leqslant n$.

(iii) There exists an exact sequence $0\to R\to T_0\to\ldots\to T_n\to 0$ such that T_i is direct summand of finite direct sum copies of T.

In [11], the author proved that if T is a classical tilting R-module, and $S=\operatorname{End}_R(T)$, then T is a classical tilting S-module. Also, for any $r\geqslant 0$ there is a tilting category equivalence:

$$\bigcap_{i>0,i\neq r}\operatorname{Ker}(\operatorname{Ext}^i_R(T,-))\ \underset{\operatorname{Tor}^S_r(T,-)}{\overset{\operatorname{Ext}^r_R(T,-)}{\rightleftarrows}}\ \bigcap_{i>0,i\neq r}\operatorname{Ker}(\operatorname{Tor}^S_i(T,-)).$$

Now, assume that C is a semidualizing R-module and T is a finitely generated good C-tilting R-module. Then $\operatorname{Hom}_R(C,T)$ is a classical tilting R-module by Remark 3.9. Also, $\operatorname{End}_R(\operatorname{Hom}_R(C,T))\cong\operatorname{End}_R(T)$. Therefore, it is clear that we have the following tilting category equivalence:

$$\bigcap_{i>0,i\neq r}\operatorname{Ker}(\operatorname{Ext}^i_R(\operatorname{Hom}_R(C,T),-))\ \underset{\operatorname{Tor}^S_r(\operatorname{Hom}_R(C,T),-)}{\overset{\operatorname{Ext}^r_R(\operatorname{Hom}_R(C,T),-)}{\rightleftarrows}}\ \bigcap_{i>0,i\neq r}\operatorname{Ker}(\operatorname{Tor}^S_i(\operatorname{Hom}_R(C,T),-)).$$

However, for a C-tilting R-module T, one can ask the following question.

Question: Are there other appropriate functors for transfering properties between R and $\operatorname{End}_R(T)$?

In the sequel, we investigate kernels of some appropriate subcategories related to C-tilting R-modules.

Lemma 4.1. *Let C be a semidualizing R-module, let T be an R-module and let T' be a finitely generated R-module. Assume that $S = \operatorname{End}_R(T)$. Then the following conditions are equivalent for a fixed integer $i \geqslant 1$.*

(i) $\operatorname{Ext}^i_{\mathcal{P}_C}(T', T) = 0$.

(ii) $\operatorname{Tor}_i^{\mathcal{P}_C}(T', \operatorname{Hom}_S(T, I)) = 0$ *for all injective S-modules I.*

(iii) $\operatorname{Ext}^i_{\mathcal{P}_C}(T', T \otimes_S P) = 0$ *for all projective S-modules P.*

P r o o f. Let $\ldots \to C \otimes_R P_n \to \ldots \to C \otimes_R P_0 \to T' \to 0$ be an augmented proper $\mathcal{P}_C$-resolution of T' such that P_i's are finitely generated projective R-modules. For $i \geqslant 1$, $\operatorname{Ext}^i_{\mathcal{P}_C}(T', T) = 0$ if and only if the sequence $(*)$: $\operatorname{Hom}_R(C \otimes_R P_{i-1}, T) \to \operatorname{Hom}_R(C \otimes_R P_i, T) \to \operatorname{Hom}_R(C \otimes_R P_{i+1}, T)$ is exact. And the latter is equivalent to the fact that for any injective S-module I, $\operatorname{Hom}_S(\operatorname{Hom}_R(C \otimes_R P_{i+1}, T), I) \to \operatorname{Hom}_S(\operatorname{Hom}_R(C \otimes_R P_i, T), I) \to \operatorname{Hom}_S(\operatorname{Hom}_R(C \otimes_R P_{i-1}, T), I)$ is exact. By [5], Theorem 3.2.11 $\operatorname{Hom}_S(\operatorname{Hom}_R(C \otimes_R P_j, T), I) \cong (C \otimes_R P_j) \otimes_R \operatorname{Hom}_S(T, I)$. Then (i) $\Leftrightarrow$ (ii). Furthermore, the exactness of $(*)$ is equivalent to the fact that for any projective S-module P, $\operatorname{Hom}_R(C \otimes_R P_{i-1}, T) \otimes_S P \to \operatorname{Hom}_R(C \otimes_R P_i, T) \otimes_S P \to \operatorname{Hom}_R(C \otimes_R P_{i+1}, T) \otimes_S P$ is exact. By [5], Therorem 3.2.14, $\operatorname{Hom}_R(C \otimes_R P_j, T) \otimes_S P \cong \operatorname{Hom}_R(C \otimes_R P_j, T \otimes_S P)$ for all $j \geqslant 0$. Then (i) $\Leftrightarrow$ (iii). $\square$

Proposition 4.2. *Let C be a semidualizing R-module, let $T \in \mathcal{B}_C(R)$ and let $S = \operatorname{End}_R(T)$. Then the following statements hold.*

(i) *If I is an injective S-module, then $\operatorname{Hom}_S(T, I) \in \mathcal{A}_C(R)$.*

(ii) *If Y is an S-module such that $\operatorname{id}_S Y < \infty$ and $\operatorname{Ext}^i_S(T, Y) = 0$ for all $i > 0$, then $\operatorname{Hom}_S(T, Y) \in \mathcal{A}_C(R)$.*

(iii) *If P is a projective S-module, then $T \otimes_S P \in \mathcal{B}_C(R)$.*

(iv) *If Y is an S-module such that $\operatorname{pd}_S Y < \infty$ and $\operatorname{Tor}_i^S(T, Y) = 0$ for all $i > 0$, then $T \otimes_S Y \in \mathcal{B}_C(R)$.*

P r o o f. We just prove items (i) and (ii). The proof of items (iii) and (iv) is similar.

(i) For all $i > 0$,

$$\operatorname{Tor}_i^R(C, \operatorname{Hom}_S(T, I)) \cong \operatorname{Hom}_S(\operatorname{Ext}^i_R(C, T), I) = 0.$$

In the above sequence, the isomorphism follows from [5], Theorem 3.2.13, and the equality follows from $T \in \mathcal{B}_C(R)$. For all $i > 0$,

$$\begin{aligned}\operatorname{Ext}^i_R(C, C \otimes_R \operatorname{Hom}_S(T, I)) &\cong \operatorname{Ext}^i_R(C, \operatorname{Hom}_S(\operatorname{Hom}_R(C, T), I)) \\ &\cong \operatorname{Hom}_S(\operatorname{Tor}^R_i(C, \operatorname{Hom}_R(C, T)), I) = 0.\end{aligned}$$

In the above sequence, the first isomorphism follows from [5], Theorem 3.2.11, the second isomorphism follows from [5], Theorem 3.2.1, and the equality holds since $T \in \mathcal{B}_C(R)$. Also, it is routine to show that the following diagram commutes:

$$\begin{tikzcd} \operatorname{Hom}_S(T, I) \arrow[r, "\gamma^C_{\operatorname{Hom}_S(T,I)}"] \arrow[d, "\operatorname{Hom}_S(\xi^C_T, I)"'] & \operatorname{Hom}_R(C, C \otimes_R \operatorname{Hom}_S(T, I)) \arrow[d, "f"', "\cong"] \\ \operatorname{Hom}_S(C \otimes_R \operatorname{Hom}_R(C, T), I) \arrow[r, "g", "\cong"'] & \operatorname{Hom}_R(C, \operatorname{Hom}_S(\operatorname{Hom}_R(C, T), I)). \end{tikzcd}$$

In the above diagram, f is an isomorphism by [5], Theorem 3.2.11, g is an isomorphism by [5], Theorem 2.1.10 and $\operatorname{Hom}_S(\xi^C_T, I)$ is an isomorphism since $T \in \mathcal{B}_C(R)$. Hence $\gamma^C_{\operatorname{Hom}_S(T,I)}$ is an isomorphism.

(ii) It follows from (i) and [15], Corollary 3.1.8. □

Proposition 4.3. *Let C be a semidualizing R-module, let T be a finitely generated C-tilting R-module and let $S = \operatorname{End}_R(T)$. Assume that Y is an S-module such that $\operatorname{id}_S Y < \infty$ and assume that $\operatorname{Ext}^i_S(T, Y) = 0$ for all $i > 0$. Then $\operatorname{Tor}^R_i(T, \operatorname{Hom}_S(T, Y)) = 0$ for all $i > 0$ and $T \otimes_R \operatorname{Hom}_S(T, Y) \cong Y$.*

Proof. Let $\mathcal{P}_C$-$\operatorname{pd}_R T = n$. Take an augmented injective coresolution of Y,

$$0 \to Y \to I^0 \to \ldots \to I^{n-1} \to I^n \to \ldots$$

Applying $\operatorname{Hom}_S(T, -)$ to the above sequence, we get the exact sequence of R-modules

$$0 \to \operatorname{Hom}_S(T, Y) \to \operatorname{Hom}_S(T, I^0) \to \ldots \to \operatorname{Hom}_S(T, I^n) \to \operatorname{Hom}_S(T, I^{n+1}) \to \ldots,$$

since $\operatorname{Ext}^i_S(T, Y) = 0$ for all $i > 0$. By Lemma 4.1, $\operatorname{Tor}^{\mathcal{P}_C}_i(T, \operatorname{Hom}_S(T, I^j)) = 0$ for all $i > 0$, $j \geqslant 0$, since $\operatorname{Ext}^i_{\mathcal{P}_C}(T, T) = 0$. On the other hand, for $j \geqslant 0$

$$T \otimes_R \operatorname{Hom}_S(T, I^j) \cong \operatorname{Hom}_S(\operatorname{Hom}_R(T, T), I^j) \cong I^j,$$

by [5], Theorem 3.2.11. Also, $\operatorname{Hom}_S(T, Y) \in \mathcal{A}_C(R)$ and $\operatorname{Hom}_S(T, I^j) \in \mathcal{A}_C(R)$ for $j \geqslant 0$, by Proposition 4.2. By [13], Proposition 4.3, $\operatorname{Tor}^{\mathcal{P}_C}_i(T, \operatorname{Hom}_S(T, Y)) \cong$

$\mathrm{Tor}_i^R(T, \mathrm{Hom}_S(T,Y))$ and $\mathrm{Tor}_i^{\mathcal{P}_C}(T, \mathrm{Hom}_S(T,I^j)) \cong \mathrm{Tor}_i^R(T, \mathrm{Hom}_S(T,I^j))$ for all i and j. Also, by Corollary 2.12, $\mathrm{Tor}_i^{\mathcal{P}_C}(T, \mathrm{Hom}_S(T,Y)) = 0$ for all $i > 0$. Therefore, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & T \otimes_R \mathrm{Hom}_S(T,Y) \arrow[r] \arrow[d] & T \otimes_R \mathrm{Hom}_S(T,I^0) \arrow[r] \arrow[d, "\cong"] & T \otimes_R \mathrm{Hom}_S(T,I^1) \arrow[d, "\cong"] \\
0 \arrow[r] & Y \arrow[r] & I^0 \arrow[r] & I^1.
\end{tikzcd}$$

Hence $T \otimes_R \mathrm{Hom}_S(T,Y) \cong Y$. $\square$

Proposition 4.4. *Let C be a semidualizing R-module, let T be a finitely generated C-tilting R-module and let $S = \mathrm{End}_R(T)$. Assume that Y is an S-module such that $\mathrm{pd}_S Y < \infty$ and assume that $\mathrm{Tor}_i^S(T,Y) = 0$ for all $i > 0$. Then $\mathrm{Ext}_R^i(T, T \otimes_S Y) = 0$ for all $i > 0$ and $\mathrm{Hom}_R(T, T \otimes_S Y) \cong Y$.*

Proof. Let $\mathcal{P}_C$-$\mathrm{pd}_R T = n$. Take an augmented projective resolution of Y as follows:

$$\ldots \to P_n \to P_{n-1} \to \ldots \to P_0 \to Y \to 0.$$

Applying $T \otimes_S -$ to the above sequence, we get the exact sequence of R-modules

$$\ldots \to T \otimes_S P_n \to T \otimes_S P_{n-1} \to \ldots \to T \otimes_S Y \to 0,$$

since $\mathrm{Tor}_i^S(T,Y) = 0$ for all $i > 0$. By Lemma 4.1, $\mathrm{Ext}_{\mathcal{P}_C}^i(T, T \otimes_S P_j) = 0$ for all $i > 0$, $j \geqslant 0$, since $\mathrm{Ext}_{\mathcal{P}_C}^i(T,T) = 0$. On the other hand, for $j \geqslant 0$

$$\mathrm{Hom}_R(T, T \otimes_S P_j) \cong \mathrm{Hom}_R(T,T) \otimes_S P_j \cong S \otimes_S P_j \cong P_j,$$

by [5], Theorem 3.2.14. Also, $T \otimes_S Y \in \mathcal{B}_C(R)$ and $T \otimes_S P_j \in \mathcal{B}_C(R)$ for $j \geqslant 0$ by Proposition 4.2. By Corollary 2.12, $\mathrm{Ext}_{\mathcal{P}_C}^i(T, T \otimes_S Y) = 0$ for all $i > 0$. Also, $\mathrm{Ext}_{\mathcal{P}_C}^i(T, T \otimes_S P_j) \cong \mathrm{Ext}_R^i(T, T \otimes_S P_j)$ and $\mathrm{Ext}_{\mathcal{P}_C}^i(T, T \otimes_S Y) \cong \mathrm{Ext}_R^i(T, T \otimes_S Y)$ for all i and j by [17], Corollary 4.2. Therefore, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
\mathrm{Hom}_R(T, T \otimes_S P_1) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_R(T, T \otimes_S P_0) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_R(T, T \otimes_S Y) \arrow[r] \arrow[d] & 0 \\
P_1 \arrow[r] & P_0 \arrow[r] & Y \arrow[r] & 0.
\end{tikzcd}$$

Hence $\mathrm{Hom}_R(T, T \otimes_S Y) \cong Y$. $\square$

Theorem 4.5. *Let C be a semidualizing R-module, let T be a finitely generated C-tilting R-module and let $S = \mathrm{End}_R(T)$. Assume that $r \geqslant 0$ is an integer and assume that Y is an S-module such that $\mathrm{id}_S Y < \infty$ and $\mathrm{Ext}^i_S(T,Y) = 0$ for all $i \geqslant 0$, $i \neq r$. Then $\mathrm{Tor}^R_i(T, \mathrm{Ext}^r_S(T,Y)) = 0$ for all $i \geqslant 0$, $i \neq r$ and $\mathrm{Tor}^R_r(T, \mathrm{Ext}^r_S(T,Y)) \cong Y$.*

Proof. By Proposition 4.3, we may assume that $r \geqslant 1$. Take an augmented injective coresolution of Y,

$$0 \to Y \to I^0 \to \ldots \to I^{r-1} \to I^r \to \ldots.$$

Denote by J^i the kernel of the map $I^i \to I^{i+1}$ for all $i \geqslant 0$. Note that $\mathrm{Ext}^i_S(T, J^r) = 0$ since $\mathrm{Ext}^i_S(T, J^r) \cong \mathrm{Ext}^{i+r}_S(T,Y) = 0$ for all $i > 0$. Also, $\mathrm{id}_S J^r < \infty$. Hence Proposition 4.2 implies that $\mathrm{Hom}_S(T, J^r) \in \mathcal{A}_C(R)$. So, $\mathrm{Tor}^R_i(T, \mathrm{Hom}_S(T, J^r)) = 0$ for all $i > 0$ and $T \otimes_R \mathrm{Hom}_S(T, J^r) \cong J^r$, by Proposition 4.3. Applying $\mathrm{Hom}_S(T, -)$ to the injective coresolution, we get the exact sequence of R-modules

$$0 \to \mathrm{Hom}_S(T, I^0) \to \ldots \to \mathrm{Hom}_S(T, I^{r-1}) \to \mathrm{Hom}_S(T, J^r) \to X \to 0,$$

where $\mathrm{Ext}^1_S(T, J^{r-1}) \cong \ldots \cong \mathrm{Ext}^r_S(T,Y) \cong X$. Note that $X \in \mathcal{A}_C(R)$, by Proposition 4.2 and [15], Corollary 3.1.8. By Proposition 2.14 and [13], Proposition 4.3, we have $\mathrm{Tor}^{\mathcal{P}_C}_i(T,X) \cong \mathrm{Tor}^R_i(T,X) = 0$ for all $i \geqslant 0$, $i \neq r$ and

$$\begin{aligned}\mathrm{Tor}^{\mathcal{P}_C}_r(T,X) &\cong \mathrm{Tor}^R_r(T,X) \cong \mathrm{Ker}(T \otimes_R \mathrm{Hom}_S(T, I^0) \to T \otimes_R \mathrm{Hom}_S(T, I^1)) \\ &\cong \mathrm{Ker}(I^0 \to I^1) \cong Y.\end{aligned}$$

$\square$

Theorem 4.6. *Let C be a semidualizing R-module, let T be a finitely generated C-tilting R-module and let $S = \mathrm{End}_R(T)$. Assume that $r \geqslant 0$ is an integer and assume that Y is an S-module such that $\mathrm{pd}_S Y < \infty$ and $\mathrm{Tor}^i_S(T,Y) = 0$ for all $i \geqslant 0$, $i \neq r$. Then $\mathrm{Ext}^i_R(T, T \otimes_S Y) = 0$ for all $i \geqslant 0$, $i \neq r$, and $\mathrm{Ext}^r_R(T, \mathrm{Tor}^S_r(T,Y)) \cong Y$.*

Proof. The proof is dual of the proof of Theorem 4.5. $\square$

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Author's address: Maryam Salimi, Department of Mathematics, East Tehran Branch, Islamic Azad University, Tehran, Iran, e-mail: `maryamsalimi@ipm.ir`.