THE CONSTRUCTION OF 3-LIE 2-ALGEBRAS

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Abstract. We construct a 3-Lie 2-algebra from a 3-Leibniz algebra and a Rota-Baxter 3-Lie algebra. Moreover, we give some examples of 3-Leibniz algebras.

Keywords: 3-Leibniz algebras; Rota-Baxter 3-Lie algebras; 3-Lie 2-algebras

MSC 2010: 17B99, 55U15

1. Introduction

Higher categorical structures play an important role in both string theory and physics. Some higher categorical structures are obtained by categorifying existing mathematical concepts. One of the simplest higher structures is a categorical vector space, that is, a 2-vector space. A categorical Lie algebra introduced by Baez and Crans in [1], which is called a Lie 2-algebra, is a 2-vector space equipped with a skew-symmetric bilinear functor whose Jacobi identity is replaced by the Jacobiator satisfying some coherence laws of its own. Lie 2-algebra theories are widely developed in [2], [6]–[8], [12]–[17]. Recently, the notion of 3-Lie 2-algebras has been introduced in [18], which is a categorical 3-Lie algebra. It is shown that the category of 3-Lie 2-algebras is equivalent to the category of 2-term 3-Lie_∞-algebras, see [18], so a 3-Lie 2-algebra is defined by a 3-Lie_∞-algebra.

In this paper, we construct a 3-Lie 2-algebra from a 3-Leibniz algebra and a Rota-Baxter 3-Lie algebra.

The paper is organized as follows. In Section 2, we construct a 3-Lie 2-algebra from a 3-Leibniz algebra, and give some examples of 3-Lie 2-algebras from associative

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trialgebras, dendriform algebras, tridendriform algebras and dialgebras. In Section 3, we construct a 3-Lie 2-algebra from a Rota-Baxter 3-Lie algebra.

2. From 3-Leibniz algebras to 3-Lie 2-algebras

The class of n-Leibniz algebras introduced by Casas, Loday and Pirashvili in [5] can be regarded as a natural generalization of n-Lie algebras, which are of a considerable importance in Nambu mechanics. In particular, for $n = 3$ one recovers a 3-Leibniz algebra, see [4]. In this section, we construct a 3-Lie 2-algebra from a 3-Leibniz algebra.

Definition 2.1 ([4])**.** A 3-Leibniz algebra is a vector space $\mathcal{L}$ with a trilinear map $[\cdot,\cdot,\cdot]\colon \mathcal{L}\times\mathcal{L}\times\mathcal{L}\to\mathcal{L}$ such that for any $x_1,x_2,x_3,x_4,x_5\in\mathcal{L}$ the following equality is satisfied:

$$[x_1,x_2,[x_3,x_4,x_5]] = [[x_1,x_2,x_3],x_4,x_5] + [x_3,[x_1,x_2,x_4],x_5] + [x_3,x_4,[x_1,x_2,x_5]]. \tag{2.1}$$

The left center of 3-Leibniz algebras is given by

$$Z(\mathcal{L}) = \{x\in\mathcal{L};\ [x,y,z]=0,\ \forall y,z\in\mathcal{L}\}. \tag{2.2}$$

The left ideal I of 3-Leibniz algebras is given by $[I,\mathcal{L},\mathcal{L}]\subseteq I$. It is obvious that $Z(\mathcal{L})$ is a left ideal of the 3-Leibniz algebra $\mathcal{L}$.

A 3-Lie 2-algebra can be regarded as a categorification of a 3-Lie algebra, which is equivalent to a 2-term 3-Lie$_\infty$-algebra. An explicit description of 3-Lie 2-algebra is given in [18].

Definition 2.2 ([18])**.** A 2-term 3-Lie$_\infty$-algebra consists of the following data:

- ▷ a complex of vector spaces $L_1 \xrightarrow{d} L_0$;
- ▷ completely skew-symmetric trilinear maps $l_3\colon L_i\times L_j\times L_k\to L_{i+j+k}$, where $0\leqslant i+j+k\leqslant 1$;
- ▷ a multilinear map $l_5\colon (L_0\wedge L_0)\otimes(L_0\wedge L_0\wedge L_0)\to L_1$, such that for any $x,y,x_i\in L_0$ and $a,b,c\in L_1$, the following equalities are satisfied:
  - (a) $dl_3(x,y,a) = l_3(x,y,da)$,
  - (b) $l_3(a,b,c)=0$, $l_3(a,b,x)=0$,
  - (c) $l_3(da,b,x) = l_3(a,db,x)$,
  - (d) $dl_5(x_1,x_2,x_3,x_4,x_5) = -l_3(x_1,x_2,l_3(x_3,x_4,x_5)) + l_3(x_3,l_3(x_1,x_2,x_4),x_5) + l_3(l_3(x_1,x_2,x_3),x_4,x_5) + l_3(x_3,x_4,l_3(x_1,x_2,x_5))$,

(e) $l_5(da, x_2, x_3, x_4, x_5) = -l_3(a, x_2, l_3(x_3, x_4, x_5)) + l_3(x_3, l_3(a, x_2, x_4), x_5)$
$+ l_3(l_3(a, x_2, x_3), x_4, x_5) + l_3(x_3, x_4, l_3(a, x_2, x_5)),$

(f) $l_5(x_1, x_2, da, x_4, x_5) = -l_3(x_1, x_2, l_3(a, x_4, x_5)) + l_3(a, l_3(x_1, x_2, x_4), x_5)$
$+ l_3(l_3(x_1, x_2, a), x_4, x_5) + l_3(a, x_4, l_3(x_1, x_2, x_5)),$

(g)
$$\begin{aligned} &l_3(l_5(x_1, x_2, x_3, x_4, x_5), x_6, x_7) + l_3(x_5, l_5(x_1, x_2, x_3, x_4, x_6), x_7)\\ &\quad + l_3(x_1, x_2, l_5(x_3, x_4, x_5, x_6, x_7)) + l_3(x_5, x_6, l_5(x_1, x_2, x_3, x_4, x_7))\\ &\quad + l_5(x_1, x_2, l_3(x_3, x_4, x_5), x_6, x_7) + l_5(x_1, x_2, l_3(x_3, x_4, x_6), x_7)\\ &\quad + l_5(x_1, x_2, x_5, x_6, l_3(x_3, x_4, x_7))\\ &= l_3(x_3, x_4, l_5(x_1, x_2, x_5, x_6, x_7)) + l_5(l_3(x_1, x_2, x_3), x_4, x_5, x_6, x_7)\\ &\quad + l_5(x_3, l_3(x_1, x_2, x_4), x_5, x_6, x_7) + l_5(x_3, x_4, l_3(x_1, x_2, x_5), x_6, x_7)\\ &\quad + l_5(x_3, x_4, x_5, l_3(x_1, x_2, x_6), x_7) + l_5(x_1, x_2, x_3, x_4, l_3(x_5, x_6, x_7))\\ &\quad + l_5(x_3, x_4, x_5, x_6, l_3(x_1, x_2, x_7)). \end{aligned}$$

A 2-term 3-Lie$_\infty$-algebra is denoted by $\mathbb{L} = (L_1, L_0, d, l_3, l_5)$, or simply $\mathbb{L}$.

If $d = 0$ ($l_5 = 0$), then the 2-term 3-Lie$_\infty$-algebra is called skeletal (strict).

Let $(\mathcal{L}, [\cdot,\cdot,\cdot])$ be a 3-Leibniz algebra. Define a completely skew-symmetric bracket $[\![\cdot,\cdot,\cdot]\!]$ on $\mathcal{L}$ by

$$[\![x_1, x_2, x_3]\!] = \frac{1}{6}\sum_{\sigma}(-1)^{\tau(\sigma)}[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}], \tag{2.3}$$

where σ runs over the symmetric group S_3 and the number $\tau(\sigma)$ is equal to 0 or 1 depending on the parity of the permutation σ. The corresponding operator J with respect to $[\![\cdot,\cdot]\!]$ is given by

$$\begin{aligned} J_{x_1,x_2,x_3,x_4,x_5} = {} & [\![[\![x_1, x_2, x_3]\!], x_4, x_5]\!] + [\![x_3, [\![x_1, x_2, x_4]\!], x_5]\!]\\ & + [\![x_3, x_4, [\![x_1, x_2, x_5]\!]]\!] - [\![x_1, x_2, [\![x_3, x_4, x_5]\!]]\!]. \end{aligned} \tag{2.4}$$

Lemma 2.3. *Let $(\mathcal{L}, [\cdot,\cdot,\cdot])$ be a 3-Leibniz algebra. For any $x_i \in \mathcal{L}$, we have*

$$\begin{aligned} &[\![J_{x_1,x_2,x_3,x_4,x_5}, x_6, x_7]\!] + [\![x_5, J_{x_1,x_2,x_3,x_4,x_6}, x_7]\!] + [\![x_1, x_2, J_{x_3,x_4,x_5,x_6,x_7}]\!]\\ &\quad + [\![x_5, x_6, J_{x_1,x_2,x_3,x_4,x_7}]\!] - [\![x_3, x_4, J_{x_1,x_2,x_5,x_6,x_7}]\!]\\ &\quad + J_{x_1,x_2,[\![x_3,x_4,x_5]\!],x_6,x_7} + J_{x_1,x_2,x_5,[\![x_3,x_4,x_6]\!],x_7} + J_{x_1,x_2,x_5,x_6,[\![x_3,x_4,x_7]\!]}\\ &\quad - J_{[\![x_1,x_2,x_3]\!],x_4,x_5,x_6,x_7} - J_{x_3,[\![x_1,x_2,x_4]\!],x_5,x_6,x_7} - J_{x_3,x_4,[\![x_1,x_2,x_5]\!],x_6,x_7}\\ &\quad - J_{x_3,x_4,x_5,[\![x_1,x_2,x_6]\!],x_7} - J_{x_1,x_2,x_3,x_4,[\![x_5,x_6,x_7]\!]} - J_{x_3,x_4,x_5,x_6,[\![x_1,x_2,x_7]\!]} = 0. \end{aligned} \tag{2.5}$$

P r o o f. Since the bracket $\llbracket\cdot,\cdot,\cdot\rrbracket$ defined by equation (2.3) is completely skew-symmetric, we have

$$\begin{aligned}
&\llbracket J_{x_1,x_2,x_3,x_4,x_5}, x_6, x_7\rrbracket + \llbracket x_5, J_{x_1,x_2,x_3,x_4,x_6}, x_7\rrbracket + \llbracket x_1, x_2, J_{x_3,x_4,x_5,x_6,x_7}\rrbracket\\
&\quad + \llbracket x_5, x_6, J_{x_1,x_2,x_3,x_4,x_7}\rrbracket - \llbracket x_3, x_4, J_{x_1,x_2,x_5,x_6,x_7}\rrbracket\\
&\quad + J_{x_1,x_2,\llbracket x_3,x_4,x_5\rrbracket,x_6,x_7} + J_{x_1,x_2,x_5,\llbracket x_3,x_4,x_6\rrbracket,x_7} + J_{x_1,x_2,x_5,x_6,\llbracket x_3,x_4,x_7\rrbracket}\\
&\quad - J_{\llbracket x_1,x_2,x_3\rrbracket,x_4,x_5,x_6,x_7} - J_{x_3,\llbracket x_1,x_2,x_4\rrbracket,x_5,x_6,x_7} - J_{x_3,x_4,\llbracket x_1,x_2,x_5\rrbracket,x_6,x_7}\\
&\quad - J_{x_3,x_4,x_5,\llbracket x_1,x_2,x_6\rrbracket,x_7} - J_{x_1,x_2,x_3,x_4,\llbracket x_5,x_6,x_7\rrbracket} - J_{x_3,x_4,x_5,x_6,\llbracket x_1,x_2,x_7\rrbracket}\\
&= \llbracket J_{x_1,x_2,x_3,x_4,x_5}, x_6, x_7\rrbracket + \llbracket x_5, J_{x_1,x_2,x_3,x_4,x_6}, x_7\rrbracket + \llbracket x_1, x_2, J_{x_3,x_4,x_5,x_6,x_7}\rrbracket\\
&\quad + \llbracket x_5, x_6, J_{x_1,x_2,x_3,x_4,x_7}\rrbracket - \llbracket x_3, x_4, J_{x_1,x_2,x_5,x_6,x_7}\rrbracket\\
&\quad + \llbracket\llbracket x_1, x_2, \llbracket x_3, x_4, x_5\rrbracket\rrbracket, x_6, x_7\rrbracket + \llbracket\llbracket x_3, x_4, x_5\rrbracket, \llbracket x_1, x_2, x_6\rrbracket, x_7\rrbracket\\
&\quad + \llbracket\llbracket x_3, x_4, x_5\rrbracket, x_6, \llbracket x_1, x_2, x_7\rrbracket\rrbracket + \llbracket x_1, \llbracket\llbracket x_3, x_4, x_5\rrbracket, x_6, x_7\rrbracket, x_2\rrbracket\\
&\quad + \llbracket\llbracket x_1, x_2, x_5\rrbracket, \llbracket x_3, x_4, x_6\rrbracket, x_7\rrbracket + \llbracket x_5, \llbracket x_1, x_2, \llbracket x_3, x_4, x_6\rrbracket\rrbracket, x_7\rrbracket\\
&\quad + \llbracket x_5, \llbracket x_3, x_4, x_6\rrbracket, \llbracket x_1, x_2, x_7\rrbracket\rrbracket + \llbracket x_1, \llbracket x_5, \llbracket x_3, x_4, x_6\rrbracket, x_7\rrbracket, x_2\rrbracket\\
&\quad + \llbracket\llbracket x_1, x_2, x_5\rrbracket, x_6, \llbracket x_3, x_4, x_7\rrbracket\rrbracket + \llbracket x_5, \llbracket x_1, x_2, x_6\rrbracket, \llbracket x_3, x_4, x_7\rrbracket\rrbracket\\
&\quad + \llbracket x_5, x_6, \llbracket x_1, x_2, \llbracket x_3, x_4, x_7\rrbracket\rrbracket\rrbracket + \llbracket x_1, \llbracket x_5, x_6, \llbracket x_3, x_4, x_7\rrbracket\rrbracket, x_2\rrbracket\\
&\quad - \llbracket\llbracket\llbracket x_1, x_2, x_3\rrbracket, x_4, x_5\rrbracket, x_6, x_7\rrbracket - \llbracket x_5, \llbracket\llbracket x_1, x_2, x_3\rrbracket, x_4, x_6\rrbracket, x_7\rrbracket\\
&\quad - \llbracket x_5, x_6, \llbracket\llbracket x_1, x_2, x_3\rrbracket, x_4, x_7\rrbracket\rrbracket - \llbracket\llbracket x_1, x_2, x_3\rrbracket, \llbracket x_5, x_6, x_7\rrbracket, x_4\rrbracket\\
&\quad - \llbracket\llbracket x_3, \llbracket x_1, x_2, x_4\rrbracket, x_5\rrbracket, x_6, x_7\rrbracket - \llbracket x_5, \llbracket x_3, \llbracket x_1, x_2, x_4\rrbracket, x_6\rrbracket, x_7\rrbracket\\
&\quad - \llbracket x_5, x_6, \llbracket x_3, \llbracket x_1, x_2, x_4\rrbracket, x_7\rrbracket\rrbracket - \llbracket x_3, \llbracket x_5, x_6, x_7\rrbracket, \llbracket x_1, x_2, x_4\rrbracket\rrbracket\\
&\quad - \llbracket\llbracket x_3, x_4, \llbracket x_1, x_2, x_5\rrbracket\rrbracket, x_6, x_7\rrbracket - \llbracket\llbracket x_1, x_2, x_5\rrbracket, \llbracket x_3, x_4, x_6\rrbracket, x_7\rrbracket\\
&\quad - \llbracket\llbracket x_1, x_2, x_5\rrbracket, x_6, \llbracket x_3, x_4, x_7\rrbracket\rrbracket - \llbracket x_3, \llbracket\llbracket x_1, x_2, x_5\rrbracket, x_6, x_7\rrbracket, x_4\rrbracket\\
&\quad - \llbracket\llbracket x_3, x_4, x_5\rrbracket, \llbracket x_1, x_2, x_6\rrbracket, x_7\rrbracket - \llbracket x_5, \llbracket x_3, x_4, \llbracket x_1, x_2, x_6\rrbracket\rrbracket, x_7\rrbracket\\
&\quad - \llbracket x_5, \llbracket x_1, x_2, x_6\rrbracket, \llbracket x_3, x_4, x_7\rrbracket\rrbracket - \llbracket x_3, \llbracket x_5, \llbracket x_1, x_2, x_6\rrbracket, x_7\rrbracket, x_4\rrbracket\\
&\quad - \llbracket\llbracket x_1, x_2, x_3\rrbracket, x_4, \llbracket x_5, x_6, x_7\rrbracket\rrbracket - \llbracket x_3, \llbracket x_1, x_2, x_4\rrbracket, \llbracket x_5, x_6, x_7\rrbracket\rrbracket\\
&\quad - \llbracket x_3, x_4, \llbracket x_1, x_2, \llbracket x_5, x_6, x_7\rrbracket\rrbracket\rrbracket - \llbracket x_1, \llbracket x_3, x_4, \llbracket x_5, x_6, x_7\rrbracket\rrbracket, x_2\rrbracket\\
&\quad - \llbracket\llbracket x_3, x_4, x_5\rrbracket, x_6, \llbracket x_1, x_2, x_7\rrbracket\rrbracket - \llbracket x_5, \llbracket x_3, x_4, x_6\rrbracket, \llbracket x_1, x_2, x_7\rrbracket\rrbracket\\
&\quad - \llbracket x_5, x_6, \llbracket x_3, x_4, \llbracket x_1, x_2, x_7\rrbracket\rrbracket\rrbracket - \llbracket x_3, \llbracket x_5, x_6, \llbracket x_1, x_2, x_7\rrbracket\rrbracket, x_4\rrbracket\\
&= 0.
\end{aligned}$$

The proof is finished. □

Now, we construct a 3-Lie 2-algebra from a 3-Leibniz algebra $(\mathcal{L}, [\cdot,\cdot,\cdot])$ and its center $Z(\mathcal{L})$. We can consider the graded vector space $\mathbb{L} = Z(\mathcal{L}) \oplus \mathcal{L}$, where $Z(\mathcal{L})$ is of degree 1, $\mathcal{L}$ is of degree 0. Define a degree -1 differential $d = i\colon Z(\mathcal{L}) \to \mathcal{L}$, which is the inclusion. Define a degree 0 completely skew-symmetric trilinear map l_3 and a degree 1 multilinear map l_5 on $\mathbb{L}$ by

$$\begin{cases} l_3(x_1,x_2,x_3) = [\![x_1,x_2,x_3]\!] = \dfrac{1}{6}\sum_{\sigma}(-1)^{\tau(\sigma)}[x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}] \quad \forall\, x_i \in \mathcal{L}, \\ l_3(a,x_1,x_2) = -l_3(a,x_2,x_1) = -l_3(x_1,a,x_2) = l_3(x_2,a,x_1) \\ \qquad = l_3(x_1,x_2,a) = -l_3(x_2,x_1,a) = [\![a,x_1,x_2]\!] \\ \qquad = \dfrac{1}{6}([x_1,x_2,a] + [x_2,a,x_1] - [x_1,a,x_2] - [x_2,x_1,a]) \\ \qquad\qquad \forall\, x_i \in \mathcal{L},\ a \in Z(\mathcal{L}), \\ l_3(a,b,c) = l_3(a,b,x) = 0 \quad \forall\, a,b \in Z(\mathcal{L}),\ x \in \mathcal{L}, \\ l_5(x_1,x_2,x_3,x_4,x_5) = J_{x_1,x_2,x_3,x_4,x_5} \quad \forall\, x_i \in \mathcal{L}. \end{cases}$$

We obtain the following theorem:

Theorem 2.4. *With the above notations,* $(\mathbb{L} = Z(\mathcal{L}) \oplus \mathcal{L}, d, l_3, l_5)$ *is a 3-Lie 2-algebra.*

P r o o f. It is clear that conditions (a)–(f) in Definition 2.2 hold from the definition of d, l_3 and l_5. By Lemma 2.3, condition (g) in Definition 2.2 holds. Then $(\mathbb{L} = Z(\mathcal{L}) \oplus \mathcal{L}, d, l_3, l_5)$ is a 3-Lie 2-algebra. $\square$

The above theorem gives a method to construct 3-Lie 2-algebras. By using this theorem, we can obtain the following examples from associative trialgebras, dendriform algebras, tridendriform algebras and dialgebras, respectively.

An associative trialgebra, see [10], is a vector space A equipped with three binary associative operations:

$$\dashv\colon\ A \otimes A \to A,$$
$$\perp\colon\ A \otimes A \to A,$$
$$\vdash\colon\ A \otimes A \to A$$

(called left, middle, and right, respectively), satisfying the following relations:

(a) $(x \dashv y) \dashv z = x \dashv (y \vdash z) = x \dashv (y \perp z)$,
(b) $(x \vdash y) \dashv z = x \vdash (y \dashv z)$,
(c) $(x \dashv y) \vdash z = x \vdash (y \vdash z) = (x \perp y) \vdash z$,
(d) $(x \perp y) \vdash z = x \perp (y \dashv z)$,
(e) $(x \dashv y) \perp z = x \perp (y \vdash z)$,
(f) $(x \vdash y) \perp z = x \vdash (y \perp z)$.

Example 2.5. An associative trialgebra is a 3-Leibniz algebra with respect to the bracket

$$[x,y,z] = z \dashv (x \perp y - y \perp x) - (x \perp y - y \perp x) \vdash z.$$

Then a 3-Lie 2-algebra can be obtained by Theorem 2.4.

A dendriform algebra, see [11], is a vector space D equipped with two bilinear operations

$$\begin{aligned} \succ\colon\ & D \otimes D \to D, \\ \prec\colon\ & D \otimes D \to D \end{aligned}$$

such that the following equalities hold:

(a) $(x \prec y) \prec z = x \prec (y \prec z + y \succ z)$,
(b) $x \succ (y \succ z) = (x \prec y + x \succ y) \succ z$,
(c) $(x \succ y) \prec z = x \succ (y \prec z)$.

Example 2.6. A dendriform algebra $(D, \prec, \succ)$ is a 3-Leibniz algebra associated to the operator

$$\begin{aligned} \{x,y,z\} = {} & z \prec (x \prec y + x \succ y) + z \succ (x \prec y + x \succ y) \\ & - (x \prec y + x \succ y) \prec z - (x \prec y + x \succ y) \succ z. \end{aligned}$$

Then a 3-Lie 2-algebra can be obtained by Theorem 2.4.

A tridendriform algebra, see [10], is a quadruple $(T, \prec, \succ, \cdot)$ consisting of a vector space T and three bilinear maps

$$\begin{aligned} \prec\colon\ & T \otimes T \to T, \\ \succ\colon\ & T \otimes T \to T, \\ \cdot\colon\ & T \otimes T \to T \end{aligned}$$

such that the following equalities hold:

(a) $(x \prec y) \prec z = x \prec (y \prec z + y \succ z + y \cdot z)$,
(b) $(x \succ y) \prec z = x \succ (y \prec z)$,
(c) $x \succ (y \succ z) = (x \prec y + x \succ y + x \cdot y) \succ z$,
(d) $(x \succ y) \cdot z = x \succ (y \cdot z)$,
(e) $(x \prec y) \cdot z = x \cdot (y \succ z)$,
(f) $(x \cdot y) \prec z = x \cdot (y \prec z)$,
(g) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Example 2.7. A tridendriform algebra $(T, \prec, \succ, \cdot)$ is a 3-Leibniz algebra associated to the operator

$$\begin{aligned}\{x,y,z\} = z \prec (x \prec y + x \succ y + x \cdot y) + z \succ (x \prec y + x \succ y + x \cdot y) \\ + z \cdot (x \prec y + x \succ y + x \cdot y) - (x \prec y + x \succ y + x \cdot y) \prec z \\ - (x \prec y + x \succ y + x \cdot y) \succ z - (x \prec y + x \succ y + x \cdot y) \cdot z.\end{aligned}$$

Then a 3-Lie 2-algebra can be obtained by Theorem 2.4.

A dialgebra, see [9], is a vector space V equipped with two bilinear operations

$$\begin{aligned}\dashv\colon\ V \otimes V \to V, \\ \vdash\colon\ V \otimes V \to V\end{aligned}$$

such that the following equalities hold:

(a) $x \vdash (y \dashv z) = (x \vdash y) \dashv z$,

(b) $x \dashv (y \dashv z) = (x \dashv y) \dashv z = x \dashv (y \vdash z)$,

(c) $x \vdash (y \vdash z) = (x \vdash y) \vdash z = (x \dashv y) \vdash z$.

Example 2.8. A dialgebra algebra $(V, \dashv, \vdash)$ is a 3-Leibniz algebra associated to the operator

$$\{x,y,z\} = z \dashv (x \vdash y) - (x \vdash y) \vdash z.$$

Then a 3-Lie 2-algebra can be obtained by Theorem 2.4.

3. From Rota-Baxter 3-Lie algebras to 3-Lie 2-algebras

In this section, we construct a 3-Lie 2-algebra from a Rota-Baxter 3-Lie algebra.

Definition 3.1 ([3])**.** A Rota-Baxter 3-Lie algebra is a 3-Lie algebra $(\mathfrak{R}, [\cdot,\cdot,\cdot])$ with a linear map $R\colon L \to L$ such that

$$\begin{aligned}[R(x_1), R(x_2), R(x_3)] = R\big([R(x_1), R(x_2), x_3] + [R(x_1), x_2, R(x_3)] \\ + [x_1, R(x_2), R(x_3)] + \lambda[R(x_1), x_2, x_3] \\ + \lambda[x_1, R(x_2), x_3] + \lambda[x_1, x_2, R(x_3)] + \lambda^2[x_1, x_2, x_3]\big),\end{aligned} \tag{3.1}$$

where R is called a Rota-Baxter operator of weight λ. A Rota-Baxter 3-Lie algebra is denoted by $(\mathfrak{R}, [\cdot,\cdot,\cdot], R)$.

Theorem 3.2. *Let $(\mathfrak{R}, [\cdot,\cdot,\cdot], R)$ be a Rota-Baxter 3-Lie algebra with a Rota-Baxter operator of weight 0. Let $V_1 = \mathfrak{R}$, $V_0 = \mathfrak{R}$. On the complex of vector spaces $V_1 = \mathfrak{R} \xrightarrow{R} \mathfrak{R} = V_0$, define a trilinear map l_3 by*

$$\begin{cases} l_3(x,y,z) = [R(x), R(y), R(z)] \quad \forall\, x,y,z \in V_0, \\ l_3(x,y,a) = [R(x), R(y), R(a)] \quad \forall\, x,y \in V_0,\ a \in V_1, \\ l_3(a,b,c) = 0 \quad \forall\, a,b,c \in V_1, \\ l_3(a,b,x) = 0 \quad \forall\, x \in V_0,\ a,b \in V_1. \end{cases}$$

If the Rota-Baxter operator R satisfies $R^2 = R$, then $(V_1 = \mathfrak{R} \xrightarrow{R} \mathfrak{R} = V_0, l_3)$ is a strict 3-Lie 2-algebra.

Proof. Since $R^2 = R$, we have

$$R([R(x_1), R(x_2), R(x_3)]) = [R(x_1), R(x_2), R(x_3)].$$

For any $x, y \in V_0$, $a, b \in V_1$, we have

$$\begin{aligned} Rl_3(x,y,a) &= R([R(x), R(y), R(a)]) = [R(x), R(y), R(a)] \\ &= [R(x), R(y), R^2(a)] = l_3(x,y,R(a)), \end{aligned}$$

and

$$\begin{aligned} l_3(R(a),b,x) &= [R^2(a), R(b), R(x)] = [R(a), R(b), R(x)] \\ &= [R(a), R^2(b), R(x)] = l_3(a, R(b), x), \end{aligned}$$

which implies that equalities (a) and (c) hold in Definition 2.2.

For any $x_i \in V_0$, we have

$$\begin{aligned} & l_3(x_3, l_3(x_1,x_2,x_4), x_5) + l_3(l_3(x_1,x_2,x_3), x_4, x_5) \\ &\quad + l_3(x_3, x_4, l_3(x_1,x_2,x_5)) - l_3(x_1, x_2, l_3(x_3,x_4,x_5)) \\ &= [R(x_3), [R(x_1), R(x_2), R(x_4)], R(x_5)] + [[R(x_1), R(x_2), R(x_3)], R(x_4), R(x_5)] \\ &\quad + [R(x_3), R(x_4), [R(x_1), R(x_2), R(x_5)]] - [R(x_1), R(x_2), [R(x_3), R(x_4), R(x_5)]] \\ &= 0. \end{aligned}$$

Similarly, we obtain the other equalities in Definition 2.2. □

Proposition 3.3. *Let $(\mathfrak{R}, [\cdot,\cdot,\cdot], R)$ be a Rota-Baxter 3-Lie algebra with a Rota-Baxter operator of weight 0. Then $(\mathfrak{R}, [\cdot,\cdot,\cdot]')$ is a 3-Lie algebra, where the trilinear map $[\cdot,\cdot,\cdot]' \colon \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ is defined by*

$$[x,y,z]' = [R(x), R(y), z] + [R(x), y, R(z)] + [x, R(y), R(z)].$$

P r o o f. It is straightforward to calculate by equation (3.1) and Definition 2.1. □

Definition 3.4 ([18]). A representation of a 3-Lie algebra L on the vector space V is a bilinear map $\varrho\colon \wedge^2 L \to \mathfrak{gl}(V)$ such that

$$[\varrho(\mathfrak{X}, \varrho(\mathfrak{Y})] = \varrho([\mathfrak{X}, \mathfrak{Y}]_F) \quad \forall\, \mathfrak{X} = x_1 \wedge x_2,\ \mathfrak{Y} = y_1 \wedge y_2,$$
$$\varrho(x, [y_1, y_2, y_3]) = \varrho(y_2, y_3)\varrho(x, y_1) - \varrho(y_1, y_3)\varrho(x, y_2) + \varrho(y_1, y_2)\varrho(x, y_3) \quad \forall\, x, y_i \in L,$$

where $[\mathfrak{X}, \mathfrak{Y}]_F = [x_1, x_2, y_1] \wedge y_2 + y_1 \wedge [x_1, x_2, y_2]$. We denote a representation by $(V; \varrho)$.

On a Rota-Baxter 3-Lie algebra $(\mathfrak{R}, [\cdot,\cdot,\cdot], R)$ with a Rota-Baxter operator of weight 0, we define a left multiplication $l\colon \wedge^2\mathfrak{R} \to \mathfrak{gl}(\mathfrak{R})$ by

$$l_{x,y}(z) = [R(x), R(y), z] \quad \forall\, x, y, z \in \mathfrak{R}.$$

It is clear that l is a representation of the 3-Lie algebra $(\mathfrak{R}, [\cdot,\cdot,\cdot]')$ on $\mathfrak{R}$.

Definition 3.5. Let $(\mathfrak{R}, [\cdot,\cdot,\cdot], R)$ be a Rota-Baxter 3-Lie algebra with a Rota-Baxter operator of weight 0. A symplectic structure on $(\mathfrak{R}, [\cdot,\cdot,\cdot], R)$ is a non-degenerate skew-symmetric bilinear form $\omega\colon \wedge^2\mathfrak{R} \to \mathbb{R}$, such that for all $x, y, z, t \in \mathfrak{R}$ the following equality holds:

$$\omega([x, y, z], t) - \omega([x, y, t], z) + \omega([x, z, t], y) - \omega([y, z, t], x) = 0. \tag{3.2}$$

The algebra $(\mathfrak{R}, [\cdot,\cdot,\cdot], R, \omega)$ is called a symplectic Rota-Baxter 3-Lie algebra.

Theorem 3.6. *Let $(\mathfrak{R}, [\cdot,\cdot,\cdot], R)$ be a Rota-Baxter 3-Lie algebra with a Rota-Baxter operator of weight 0 and ω a symplectic structure on $\mathfrak{R}$ satisfying*

$$\omega([R(x), R(y), z], t) = \omega([x, y, t], z). \tag{3.3}$$

On the complex of vector spaces $\mathfrak{R}^* \xrightarrow{d=(\omega^*)^{-1}} \mathfrak{R}$, define a completely skew-symmetric trilinear map l_3 by

$$\begin{cases} l_3(x, y, z) = [x, y, z] \quad \forall\, x, y, z \in \mathfrak{R}, \\ l_3(x, y, f) = l^*_{x,y} f \quad \forall\, x, y \in \mathfrak{R},\ f \in \mathfrak{R}^*, \\ l_3(x, f, g) = 0 \quad \forall\, x \in \mathfrak{R},\ f, g \in \mathfrak{R}^*, \\ l_3(f, g, h) = 0 \quad \forall\, f, g, h \in \mathfrak{R}^*, \end{cases}$$

where l^* is the dual representation of l and $\omega^*\colon \mathfrak{R} \to \mathfrak{R}^*$ is given by $\omega^*(x)(y) = \omega(x, y)$. Then $(\mathfrak{R}^* \xrightarrow{d=(\omega^*)^{-1}} \mathfrak{R}, l_3)$ is a strict 3-Lie 2-algebra.

P r o o f. We only need to prove equalities (a) and (e) in Definition 2.2, and the other equalities hold similarly. For any $x, y, z, t \in \mathfrak{R}$, set $f = \omega^*(z)$ and $g = \omega^*(t)$, by equation (3.3), we have

$$\begin{aligned}\langle dl_3(x, y, f), g\rangle &= -\langle l_3(x, y, f), (\omega^*)^{-1} g\rangle \\ &= -\langle l^*_{x,y}\omega^*(z), t\rangle \\ &= \langle \omega^*(z), [R(x), R(y), t]\rangle \\ &= \omega(z, [R(x), R(y), t]) \\ &= \omega(t, [x, y, z])\end{aligned}$$

and

$$\begin{aligned}\langle l_3(x, y, df), g\rangle &= \langle l_3(x, y, z), \omega^*(t)\rangle \\ &= \langle [x, y, z], \omega^*(t)\rangle \\ &= \omega(t, [x, y, z]).\end{aligned}$$

Since ω is non-degenerate, we obtain $dl_3(x, y, f) = l_3(x, y, df)$.

Furthermore, for any $x_i \in \mathfrak{R}$, set $f = \omega^*(x_1)$, we have

$$\begin{aligned}&\langle -l_3(f, x_2, l_3(x_3, x_4, x_5)), x_6\rangle + \langle l_3(x_3, l_3(f, x_2, x_4), x_5), x_6\rangle \\ &\qquad + \langle l_3(l_3(f, x_2, x_3), x_4, x_5), x_6\rangle + \langle l_3(x_3, x_4, l_3(f, x_2, x_5)), x_6\rangle \\ &\quad = \langle -l^*_{x_2,[x_3,x_4,x_5]}\omega^*(x_1), x_6\rangle - \langle l^*_{x_3,x_5} l^*_{x_2,x_4}\omega^*(x_1), x_6\rangle \\ &\qquad + \langle l^*_{x_4,x_5} l^*_{x_2,x_3}\omega^*(x_1), x_6\rangle + \langle l^*_{x_3,x_4} l^*_{x_2,x_5}\omega^*(x_1), x_6\rangle \\ &\quad = \omega(x_1, [R(x_2), R[x_3, x_4, x_5], x_6]) \\ &\qquad - \omega(x_1, [R(x_2), R(x_4), [R(x_3), R(x_5), x_6]]) \\ &\qquad + \omega(x_1, [R(x_2), R(x_3), [R(x_4), R(x_5), x_6]]) \\ &\qquad + \omega(x_1, [R(x_2), R(x_5), [R(x_3), R(x_4), x_6]]) \\ &\quad = \omega(x_6, [x_2, [x_3, x_4, x_5], x_1]) + \omega(x_6, [x_3, x_5, [x_2, x_4, x_1]]) \\ &\qquad - \omega(x_6, [x_4, x_5, [x_2, x_3, x_1]]) - \omega(x_6, [x_3, x_4, [x_2, x_5, x_1]]) \\ &\quad = 0,\end{aligned}$$

which implies that equality (e) holds in Definition 2.2. The proof is finished. □

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