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INCLUSION RELATIONS BETWEEN HARMONIC BERGMAN-BESOV AND WEIGHTED BLOCH SPACES ON THE UNIT BALL

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Abstract. We consider harmonic Bergman-Besov spaces b^p_α and weighted Bloch spaces b^∞_α on the unit ball of $\mathbb{R}^n$ for the full ranges of parameters $0 < p < \infty$, $\alpha \in \mathbb{R}$, and determine the precise inclusion relations among them. To verify these relations we use Carleson measures and suitable radial differential operators. For harmonic Bergman spaces various characterizations of Carleson measures are known. For weighted Bloch spaces we provide a characterization when $\alpha > 0$.

Keywords: harmonic Bergman-Besov space; weighted harmonic Bloch space; Carleson measure; Berezin transform

MSC 2010: 31B05, 42B35

1. Introduction

Let $n \geqslant 2$ be an integer and $\mathbb{B} = \mathbb{B}_n$ be the open unit ball in $\mathbb{R}^n$. We denote the normalized Lebesgue volume measure on $\mathbb{B}$ by $\mathrm{d}\nu$ so that $\nu(\mathbb{B}) = 1$. For $\alpha \in \mathbb{R}$, we define the weighted measures $\mathrm{d}\nu_\alpha$ on $\mathbb{B}$ by

$$\mathrm{d}\nu_\alpha(x) = c_\alpha(1-|x|^2)^\alpha\,\mathrm{d}\nu(x).$$

These measures are finite when $\alpha > -1$ and in this case we choose c_α so that $\nu_\alpha(\mathbb{B}) = 1$. When $\alpha \leqslant -1$, we set $c_\alpha = 1$. For a measure μ on $\mathbb{B}$, we denote the Lebesgue classes with respect to μ by $L^p(\mu)$, $0 < p < \infty$. For short, we also write $L^p_\alpha = L^p(\mathrm{d}\nu_\alpha)$ and $L^p = L^p_0$.

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For a multi-index $m = (m_1, \ldots, m_n)$, where $m_1, \ldots, m_n$ are non-negative integers, and for smooth f we write

$$\partial^m f = \frac{\partial^{|m|} f}{\partial x_1^{m_1} \ldots \partial x_n^{m_n}},$$

where $|m| = m_1 + \ldots + m_n$.

Let $h(\mathbb{B})$ be the space of all complex-valued harmonic functions on $\mathbb{B}$. When $\alpha > -1$ and $0 < p < \infty$, the harmonic weighted Bergman space b^p_α is defined by $b^p_\alpha = h(\mathbb{B}) \cap L^p_\alpha$. This family of spaces can be extended to all $\alpha \in \mathbb{R}$. We call the extended family Bergman-Besov spaces, though they are sometimes called Bergman-Sobolev spaces or just Besov spaces.

For $\alpha \in \mathbb{R}$ and $0 < p < \infty$, let N be a non-negative integer such that

$$\alpha + pN > -1. \tag{1.1}$$

The harmonic Bergman-Besov space b^p_α consists of all $f \in h(\mathbb{B})$ such that

$$(1 - |x|^2)^N \partial^m f \in L^p_\alpha,$$

for every multi-index m with $|m| = N$. That is, $f \in h(\mathbb{B})$ belongs to b^p_α if and only if for every multi-index m with $|m| = N$, we have

$$\int_{\mathbb{B}} |\partial^m f(x)|^p (1 - |x|^2)^{\alpha + pN} \, \mathrm{d}\nu(x) < \infty.$$

The space b^p_α does not depend on the choice of N as long as (1.1) is satisfied. When $\alpha > -1$, one can choose $N = 0$ and the resulting space is a harmonic weighted Bergman space. When $\alpha = -1$ and $p = 2$, the space b^2_{-1} is the harmonic Hardy space. When $\alpha = -n$, the measure $\mathrm{d}\nu_{-n}$ is Möbius invariant and the spaces b^p_{-n} are called harmonic Besov spaces by many authors. If, in addition, $p = 2$, the space b^2_{-n} is the harmonic Dirichlet space.

In the definition of b^p_α, instead of partial derivatives one can use radial derivatives or more effectively certain radial differential operators D^t_s defined in terms of reproducing kernels of harmonic Bergman spaces. These matters are studied in detail in [12] for $1 \leqslant p < \infty$ and in [8] for $0 < p < 1$ and will be reviewed in Section 2.

For each $\alpha \in \mathbb{R}$ and $0 < p < \infty$ the spaces b^p_α are nontrivial, since they clearly contain harmonic polynomials. As will be verified later, all of them are different, though some of them are included in some others. One of the aims of this paper is to determine exactly when a Bergman-Besov space b^p_α is included in another b^q_β. The result is divided into two cases depending on whether $q < p$ or $q \geqslant p$.

Theorem 1.1. *Let $0 < q < p < \infty$ and $\alpha, \beta \in \mathbb{R}$. Then*

$$b^p_\alpha \subset b^q_\beta \quad \textit{if and only if} \quad \frac{\alpha+1}{p} < \frac{\beta+1}{q}.$$

Theorem 1.2. *Let $0 < p \leqslant q < \infty$ and $\alpha, \beta \in \mathbb{R}$. Then*

$$b^p_\alpha \subset b^q_\beta \quad \textit{if and only if} \quad \frac{\alpha+n}{p} \leqslant \frac{\beta+n}{q}.$$

We emphasize that both theorems are two-directional and together they completely determine all possible inclusion relations between Bergman-Besov spaces. If we assign to each b^p_α the point (p, α) in the right half of $p\alpha$-plane, then the inclusion relations are shown in Figure 1. Moreover, the inclusions in the above theorems are strict (see Corollary 4.3 below).

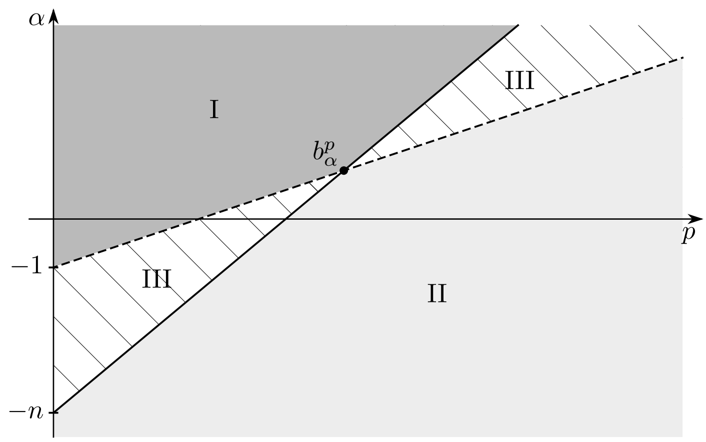

Figure 1. If (q, β) is in region I, then $b^p_\alpha \subset b^q_\beta$. If (q, β) is in region II, then $b^p_\alpha \supset b^q_\beta$. If (q, β) is in region III, then neither b^p_α nor b^q_β contains the other.

Theorems 1.1 and 1.2 complete the inclusion relations stated in [12], Section 13.2, where one-directional partial results are shown. For the *holomorphic* analogues of these theorems see [21].

Roughly speaking, the "$p = \infty$" case of Bergman-Besov spaces b^p_α is the family of Bloch spaces b^∞_α. Let $\alpha \in \mathbb{R}$. Pick a non-negative integer N such that

$$\alpha + N > 0. \tag{1.2}$$

The weighted harmonic Bloch space b^∞_α consists of all $f \in h(\mathbb{B})$ such that

$$\sup_{x \in \mathbb{B}} (1 - |x|^2)^{\alpha+N} |\partial^m f(x)| < \infty,$$

for every multi-index m with $|m| = N$. We mention two special cases. When $\alpha = 0$, taking $N = 1$ shows

$$b_0^\infty = \Big\{ f \in h(\mathbb{B})\colon \sup_{x\in\mathbb{B}} (1-|x|^2)|\nabla f(x)| < \infty \Big\}.$$

This is the most studied member of the family. When $\alpha > 0$, one can choose $N = 0$ and

$$b_\alpha^\infty = \Big\{ f \in h(\mathbb{B})\colon \sup_{x\in\mathbb{B}} (1-|x|^2)^\alpha |f(x)| < \infty \Big\}, \quad \alpha > 0.$$

As before, the spaces b_α^∞ do not depend on the choice of N as long as (1.2) is satisfied and partial derivatives can be replaced with radial derivatives or the operators D_s^t. These are studied in detail in [9].

The second aim of this paper is to determine the exact inclusion relation between a Bergman-Besov space b_α^p and a weighted Bloch space b_β^∞.

Theorem 1.3. *Let $0 < p < \infty$ and $\alpha, \beta \in \mathbb{R}$. Then*
(a) $b_\beta^\infty \subset b_\alpha^p$ *if and only if* $\beta < (\alpha+1)/p$,
(b) $b_\alpha^p \subset b_\beta^\infty$ *if and only if* $\beta \geqslant (\alpha+n)/p$.

The *holomorphic* analogue of the above theorem is proved in [20] for $\alpha = 0$, $\beta > -1$ and in [21] for $\alpha, \beta \in \mathbb{R}$.

To prove the above theorems we will use Carleson measures. Various characterizations of Carleson measures for harmonic Bergman spaces are known. We will recall them in Section 3 and add a new characterization. On the other hand, as far as we know, characterizations of Carleson measures for harmonic Bloch spaces do not exist. In Subsection 3.2 we will provide a characterization for b_α^∞, $\alpha > 0$.

2. Preliminaries

In this section we collect some known facts that will be used later. For two positive expressions X and Y we write $X \lesssim Y$ if there exists a positive constant C, whose exact value is inessential, such that $X \leqslant CY$. If both $X \lesssim Y$ and $Y \lesssim X$, we write $X \sim Y$.

The Pochhammer symbol $(a)_b$ is defined by

$$(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)},$$

when a and $a+b$ are off the pole set $-\mathbb{N}$ of the gamma function. By Stirling formula

$$\frac{(a)_c}{(b)_c} \sim c^{a-b}, \quad c \to \infty. \tag{2.1}$$

A harmonic function f on $\mathbb{B}$ has a homogeneous expansion, that is, there exist homogeneous harmonic polynomials f_k of degree k such that $f(x) = \sum_{k=0}^{\infty} f_k(x)$. The series uniformly and absolutely converges on compact subsets of $\mathbb{B}$.

2.1. Pseudohyperbolic metric. The canonical Möbius transformation on $\mathbb{B}$ that exchanges a and 0 is

$$\varphi_a(x) = \frac{(1-|a|^2)(a-x) + |a-x|^2 a}{[x,a]^2}.$$

Here the bracket $[x,a]$ is defined by

$$[x,a] = \sqrt{1 - 2x \cdot a + |x|^2|a|^2},$$

where $x \cdot a$ denotes the inner product of x and a in $\mathbb{R}^n$. The pseudohyperbolic distance between $x, y \in \mathbb{B}$ is

$$\varrho(x,y) = |\varphi_x(y)| = \frac{|x-y|}{[x,y]}.$$

For a proof of the following lemma see [2], Lemma 2.2.

Lemma 2.1. *Let $a, x, y \in \mathbb{B}$. Then*

$$\frac{1-\varrho(x,y)}{1+\varrho(x,y)} \leqslant \frac{[x,a]}{[y,a]} \leqslant \frac{1+\varrho(x,y)}{1-\varrho(x,y)}.$$

The following two lemmas show that if $x, y \in \mathbb{B}$ are close in the pseudohyperbolic metric, then certain quantities are comparable. Both of them easily follow from Lemma 2.1 (note that $[x,x] = 1 - |x|^2$).

Lemma 2.2. *Let $0 < \delta < 1$. Then*

$$[x,y] \sim 1 - |x|^2 \sim 1 - |y|^2$$

for all $x, y \in \mathbb{B}$ with $\varrho(x,y) < \delta$.

Lemma 2.3. *Let $0 < \delta < 1$. Then*

$$[x,a] \sim [y,a]$$

for all $a, x, y \in \mathbb{B}$ with $\varrho(x,y) < \delta$.

For $0 < \delta < 1$ and $x \in \mathbb{B}$ we denote the pseudohyperbolic ball with center x and radius δ by $E_\delta(x)$. The pseudohyperbolic ball $E_\delta(x)$ is also a Euclidean ball with center c and radius r, where

$$c = \frac{(1-\delta^2)x}{1-\delta^2|x|^2} \quad \text{and} \quad r = \frac{(1-|x|^2)\delta}{1-\delta^2|x|^2}.$$

It follows that for fixed $0 < \delta < 1$, we have $\nu(E_\delta(x)) \sim (1-|x|^2)^n$. More generally, for $\alpha \in \mathbb{R}$, by Lemma 2.2

$$\nu_\alpha(E_\delta(x)) = c_\alpha \int_{E_\delta(x)} (1-|y|^2)^\alpha \,\mathrm{d}\nu(y) \sim (1-|x|^2)^\alpha \nu(E_\delta(x)) \sim (1-|x|^2)^{\alpha+n}. \tag{2.2}$$

Let (a_k) be a sequence of points in $\mathbb{B}$ and $0 < \delta < 1$. We say that (a_k) is δ-separated if $\varrho(a_j, a_k) \geqslant \delta$ for all $j \neq k$. For a proof of the following lemma see, for example, [16].

Lemma 2.4. *Let $0 < \delta < 1$. There exists a sequence of points (a_k) in $\mathbb{B}$ satisfying the following properties:*

(i) *(a_k) is δ-separated,*

(ii) $\bigcup\limits_{k=1}^{\infty} E_\delta(a_k) = \mathbb{B}$,

(iii) *there exists a positive integer N such that every $x \in \mathbb{B}$ belongs to at most N of the balls $E_\delta(a_k)$.*

2.2. Reproducing kernels and the operators D_s^t. When $\alpha > -1$, the point evaluation functional $f \mapsto f(x)$ is bounded on the Hilbert space b^2_α, so by the Riesz representation theorem there exists $R_\alpha(x,y)$ such that

$$f(x) = \int_{\mathbb{B}} f(y)\overline{R_\alpha(x,y)} \,\mathrm{d}\nu_\alpha(y) \quad \forall f \in b^2_\alpha,\ \forall\, x \in \mathbb{B},\ \alpha > -1.$$

It is well-known that R_α is real-valued and $R_\alpha(x,y) = R_\alpha(y,x)$. The homogeneous expansion of $R_\alpha(x,y)$ can be expressed in terms of zonal harmonics (see [7], [17])

$$R_\alpha(x,y) = \sum_{k=0}^{\infty} \frac{(1+\frac{1}{2}n+\alpha)_k}{(\frac{1}{2}n)_k} Z_k(x,y) =: \sum_{k=0}^{\infty} \gamma_k(\alpha) Z_k(x,y), \quad \alpha > -1.$$

For definition and details about $Z_k(x,y)$, see [1], Chapter 5. By (2.1)

$$\gamma_k(\alpha) \sim k^{\alpha+1}. \tag{2.3}$$

The reproducing kernels $R_\alpha(x,y)$ can be extended to all $\alpha \in \mathbb{R}$ (see [11], [12]), where the crucial point is not the precise form of the kernel but preserving the property (2.3).

Definition 2.5. Let $\alpha \in \mathbb{R}$. Define

$$\gamma_k(\alpha) := \begin{cases} \dfrac{(1+\frac{1}{2}n+\alpha)_k}{(\frac{1}{2}n)_k} & \text{if } \alpha > -(1+\frac{1}{2}n), \\ \dfrac{((1)_k)^2}{(1-(\frac{1}{2}n+\alpha))_k(\frac{1}{2}n)_k} & \text{if } \alpha \leqslant -(1+\frac{1}{2}n), \end{cases}$$

and $R_\alpha(x,y) := \sum\limits_{k=0}^{\infty} \gamma_k(\alpha) Z_k(x,y)$.

By (2.1), the property (2.3) holds for all $\alpha \in \mathbb{R}$. Using the coefficients in the extended kernels we define the radial differential operators D_s^t.

Definition 2.6. Let $f \in h(\mathbb{B})$ and $f = \sum\limits_{k=0}^{\infty} f_k$ be its homogeneous expansion. For $s,t \in \mathbb{R}$ we define

$$D_s^t f = \sum_{k=0}^{\infty} \frac{\gamma_k(s+t)}{\gamma_k(s)} f_k.$$

$D_s^t f$ is also in $h(\mathbb{B})$ and the map $D_s^t\colon h(\mathbb{B}) \to h(\mathbb{B})$ is continuous in the topology of uniform convergence on compact subsets (see [12]). By (2.3), $\gamma_k(s+t)/\gamma_k(s) \sim k^t$ and, roughly speaking, D_s^t multiplies the kth homogeneous part of f by k^t. For $t > 0$ the operator D_s^t acts as a differential operator and for $t < 0$ as an integral operator. The parameter s plays a minor role and is used to have the precise relation $D_s^t R_s(x,y) = R_{s+t}(x,y)$. Compared to partial or radial derivatives, an important property of D_s^t is that it is invertible, the inverse of D_s^t being D_{s+t}^{-t}.

As mentioned in the introduction, the spaces b_α^p and b_α^∞ can equivalently be defined by using the operators D_s^t. Given $0 < p < \infty$ and $\alpha \in \mathbb{R}$, pick $s,t \in \mathbb{R}$ such that $\alpha + pt > -1$. The harmonic Bergman-Besov space b_α^p consists of all $f \in h(\mathbb{B})$ such that (see [12] for $1 \leqslant p < \infty$, and [8] for $0 < p < 1$)

$$\|f\|_{b_\alpha^p}^p = \|(1-|x|^2)^t D_s^t f\|_{L_\alpha^p}^p = c_\alpha \int_{\mathbb{B}} |D_s^t f(x)|^p (1-|x|^2)^{\alpha+pt}\, \mathrm{d}\nu(x) < \infty.$$

Strictly speaking, the norm (quasinorm for $0 < p < 1$) depends on s and t but this is not mentioned as it is known that every choice of the pair (s,t) leads to an equivalent norm.

Given $\alpha \in \mathbb{R}$, pick $s,t \in \mathbb{R}$ such that $\alpha + t > 0$. The harmonic Bloch space b_α^∞ consists of all $f \in h(\mathbb{B})$ such that (see [9])

$$\|f\|_{b_\alpha^\infty} = \sup_{x \in \mathbb{B}} (1-|x|^2)^{\alpha+t} |D_s^t f(x)| < \infty.$$

The most important property of the operator D_s^t that we will use later is that it allows us to pass from one Bergman-Besov (or Bloch) space to another. More precisely, we have the following isomorphisms.

Lemma 2.7. *Let $0 < p < \infty$ and $\alpha, s, t \in \mathbb{R}$.*

(a) *The map $D_s^t \colon b_\alpha^p \to b_{\alpha+pt}^p$ is an isomorphism.*

(b) *The map $D_s^t \colon b_\alpha^\infty \to b_{\alpha+t}^\infty$ is an isomorphism.*

For a proof of part (a) of the above lemma see [12], Corollary 9.2 for $1 \leqslant p < \infty$ and [8] for $0 < p < 1$. For part (b) see [9], Proposition 4.6.

2.3. Estimates of reproducing kernels. In the case $\alpha > -1$, the reproducing kernels $R_\alpha(x, y)$ are well-studied by various authors. We recall some of their properties below. For extension of these properties to $\alpha \in \mathbb{R}$ we refer to [12].

For a proof of the following pointwise estimate see [6], [13], [19] for $\alpha > -1$ and [12] for $\alpha \in \mathbb{R}$.

Lemma 2.8. *Let $\alpha \in \mathbb{R}$. For all $x, y \in \mathbb{B}$,*

$$|R_\alpha(x,y)| \lesssim \begin{cases} \dfrac{1}{[x,y]^{\alpha+n}} & \text{if } \alpha > -n, \\ 1 + \log \dfrac{1}{[x,y]} & \text{if } \alpha = -n, \\ 1 & \text{if } \alpha < -n. \end{cases}$$

On the diagonal $x = y$, the above estimate holds in two directions. For a proof see [17] for $\alpha > -1$ and [9] for $\alpha \in \mathbb{R}$.

Lemma 2.9. *Let $\alpha \in \mathbb{R}$. For all $x \in \mathbb{B}$,*

$$R_\alpha(x,x) \sim \begin{cases} \dfrac{1}{(1-|x|^2)^{\alpha+n}} & \text{if } \alpha > -n, \\ 1 + \log \dfrac{1}{1-|x|^2} & \text{if } \alpha = -n, \\ 1 & \text{if } \alpha < -n. \end{cases}$$

The next lemma shows that the first part of the above estimate continues to hold when x and y are close enough in the pseudohyperbolic metric. It can be proved along the same lines as [17], Proposition 5.

Lemma 2.10. *Let $\alpha > -n$. There exists $0 < \delta < 1$ such that for every $x \in \mathbb{B}$ and $y \in E_\delta(x)$,*

$$R_\alpha(x,y) \sim \frac{1}{(1-|x|^2)^{\alpha+n}}.$$

The next lemma gives an estimate of weighted L^p norms of reproducing kernels. For $\alpha > -1$ and $c > 0$, it is proved in [17], Proposition 8. For a full proof see [12], Theorem 1.5.

Lemma 2.11. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$ and $\beta > -1$. Set $c = p(\alpha + n) - (\beta + n)$. Then*

$$\int_{\mathbb{B}} |R_\alpha(x,y)|^p (1-|y|^2)^\beta \, \mathrm{d}\nu(y) \sim \begin{cases} \dfrac{1}{(1-|x|^2)^c} & \text{if } c > 0, \\ 1 + \log \dfrac{1}{1-|x|^2} & \text{if } c = 0, \\ 1 & \text{if } c < 0. \end{cases}$$

By Lemma 2.8, when $\alpha > -n$, the kernel $R_\alpha(x,y)$ is dominated by $1/[x,y]^{\alpha+n}$. The next lemma estimates the weighted integrals of these dominating terms. For a proof see, for example, [14], Proposition 2.2, or [19], Lemma 4.4.

Lemma 2.12. *Let $\beta > -1$ and $s \in \mathbb{R}$. Then*

$$\int_{\mathbb{B}} \frac{(1-|y|^2)^\beta}{[x,y]^{\beta+n+s}} \, \mathrm{d}\nu(y) \sim \begin{cases} \dfrac{1}{(1-|x|^2)^s} & \text{if } s > 0, \\ 1 + \log \dfrac{1}{1-|x|^2} & \text{if } s = 0, \\ 1 & \text{if } s < 0. \end{cases}$$

Integral operators involving $R_\alpha(x,y)$ or the above dominating terms are widely used in the study of Bergman spaces. For $s, \beta \in \mathbb{R}$ define the operator

$$E_{s,\beta} f(x) := (1-|x|^2)^s \int_{\mathbb{B}} f(y) \frac{(1-|y|^2)^\beta}{[x,y]^{\beta+n+s}} \, \mathrm{d}\nu(y).$$

For a proof of the following lemma see, for example, [12], Theorem 1.6.

Lemma 2.13. *Let $1 \leqslant p < \infty$ and $\alpha, \beta, s \in \mathbb{R}$. The operator $E_{s,\beta} \colon L^p_\alpha \to L^p_\alpha$ is bounded if and only if $-ps < \alpha + 1 < p(\beta + 1)$.*

Finally, we mention the following well-known growth estimate for elements of the Bergman space b^p_α, $\alpha > -1$. See [12], Theorem 13.1 for generalization to $\alpha \in \mathbb{R}$.

Lemma 2.14. *Let $0 < p < \infty$ and $\alpha > -1$. Then for all $f \in b^p_\alpha$ and for all $x \in \mathbb{B}$,*

$$|f(x)| \lesssim \frac{\|f\|_{b^p_\alpha}}{(1-|x|^2)^{(\alpha+n)/p}}.$$

3. Carleson measures

Let $X \subset h(\mathbb{B})$ be a Banach or more generally a quasi-Banach space of harmonic functions and let $0 < q < \infty$. A positive Borel measure μ on $\mathbb{B}$ is called a q-Carleson measure for X if the inclusion $i\colon X \to L^q(\mu)$ is bounded, that is, if

$$\left(\int_{\mathbb{B}} |f(x)|^q \, \mathrm{d}\mu(x)\right)^{1/q} \lesssim \|f\|_X \quad \forall f \in X.$$

Carleson measures for various function spaces are extensively studied; a frequent theme is characterizing Carleson measures in terms of averaging functions and Berezin transforms.

For $0 < \delta < 1$ the averaging function $\widehat{\mu}_\delta$ is defined by

$$\widehat{\mu}_\delta(x) = \frac{\mu(E_\delta(x))}{\nu(E_\delta(x))}, \quad x \in \mathbb{B}.$$

More generally, for $\alpha \in \mathbb{R}$ we define

$$\widehat{\mu}_{\alpha,\delta}(x) := \frac{\mu(E_\delta(x))}{\nu_\alpha(E_\delta(x))}, \quad x \in \mathbb{B}.$$

By (2.2), $\widehat{\mu}_{\alpha,\delta}(x) \sim \mu(E_\delta(x))/(1-|x|^2)^{\alpha+n}$. The following lemma shows that weighted L^p behaviour of $\widehat{\mu}_{\alpha,\delta}$ is independent of δ. For a proof see [2], Proposition 3.6, for $\alpha = 0$. The proof also works for all $\alpha \in \mathbb{R}$.

Lemma 3.1. *Let $0 < p < \infty$, $\alpha, \beta \in \mathbb{R}$ and $0 < \delta, \varepsilon < 1$. Then $\widehat{\mu}_{\alpha,\delta} \in L^p_\beta$ if and only if $\widehat{\mu}_{\alpha,\varepsilon} \in L^p_\beta$.*

The Berezin transform of a positive measure μ on $\mathbb{B}$ is

$$\widetilde{\mu}(x) = \int_{\mathbb{B}} \frac{|R(x,y)|^2}{\|R(x,\cdot)\|^2_{L^2}} \, \mathrm{d}\mu(y),$$

where $R(x,y) = R_0(x,y)$. More generally, instead of $R(x,y)$ one can use weighted kernels $R_\alpha(x,y)$. We will restrict ourselves to the case $\alpha > -1$.

For $\alpha > -1$ and $t > 1$, the (α,t)-Berezin transform of μ is defined by

$$\widetilde{\mu}_{\alpha,t}(x) := \int_{\mathbb{B}} \frac{|R_\alpha(x,y)|^t}{\|R_\alpha(x,\cdot)\|^t_{L^t_\alpha}} \, \mathrm{d}\mu(y).$$

Since $(\alpha+n)t - (\alpha+n) > 0$, by Lemma 2.11

$$\widetilde{\mu}_{\alpha,t}(x) \sim (1-|x|^2)^{(\alpha+n)t-(\alpha+n)} \int_{\mathbb{B}} |R_\alpha(x,y)|^t \, \mathrm{d}\mu(y). \tag{3.1}$$

Applying also Lemma 2.8 we obtain the estimate

$$\widetilde{\mu}_{\alpha,t}(x) \lesssim (1-|x|^2)^{(\alpha+n)t-(\alpha+n)} \int_{\mathbb{B}} \frac{\mathrm{d}\mu(y)}{[x,y]^{(\alpha+n)t}}. \tag{3.2}$$

Using the dominating term on the right-hand side, for $\alpha > -1$ and $s > 0$, we define (α, s)-Berezin-type transform $\bar{\mu}_{\alpha,s}$ by

$$\bar{\mu}_{\alpha,s}(x) := (1-|x|^2)^s \int_{\mathbb{B}} \frac{\mathrm{d}\mu(y)}{[x,y]^{\alpha+n+s}}.$$

The following proposition shows that L^p_α behaviour of $\widetilde{\mu}_{\alpha,t}$, $\bar{\mu}_{\alpha,s}$ and $\widehat{\mu}_{\alpha,\delta}$ are the same when $p > 1$.

Proposition 3.2. *Let $1 < p < \infty$ and $\alpha > -1$. The following are equivalent:*
(a) $\widehat{\mu}_{\alpha,\delta} \in L^p_\alpha$ *for some (every)* $0 < \delta < 1$.
(b) $\bar{\mu}_{\alpha,s} \in L^p_\alpha$ *for some (every)* $s > 0$.
(c) $\widetilde{\mu}_{\alpha,t} \in L^p_\alpha$ *for some (every)* $t > 1$.

Proof. (a) $\Rightarrow$ (b): Suppose $\widehat{\mu}_{\alpha,\delta} \in L^p_\alpha$ for some $0 < \delta < 1$. By Lemma 2.3 and (2.2)

$$\int_{\mathbb{B}} \frac{\mathrm{d}\mu(y)}{[x,y]^{\alpha+n+s}} \sim \int_{\mathbb{B}} \frac{1}{(1-|y|^2)^{\alpha+n}} \int_{E_\delta(y)} \frac{\mathrm{d}\nu_\alpha(z)}{[x,z]^{\alpha+n+s}}\, \mathrm{d}\mu(y).$$

Applying Fubini's theorem (note that $z \in E_\delta(y)$ if and only if $y \in E_\delta(z)$) and Lemma 2.2 we obtain

$$\begin{aligned}\int_{\mathbb{B}} \frac{\mathrm{d}\mu(y)}{[x,y]^{\alpha+n+s}} &\sim \int_{\mathbb{B}} \frac{1}{[x,z]^{\alpha+n+s}} \int_{E_\delta(z)} \frac{\mathrm{d}\mu(y)}{(1-|y|^2)^{\alpha+n}}\, \mathrm{d}\nu_\alpha(z) \\ &\sim \int_{\mathbb{B}} \frac{\widehat{\mu}_{\alpha,\delta}(z)}{[x,z]^{\alpha+n+s}}\, \mathrm{d}\nu_\alpha(z).\end{aligned} \tag{3.3}$$

Since $\widehat{\mu}_{\alpha,\delta} \in L^p_\alpha$, by Lemma 2.13, $\bar{\mu}_{\alpha,s} \in L^p_\alpha$ for every $s > 0$.

(b) $\Rightarrow$ (c): This part follows from (3.2).

(c) $\Rightarrow$ (a): Suppose $\widetilde{\mu}_{\alpha,t} \in L^p_\alpha$ for some $t > 1$. Pick $0 < \delta_0 < 1$ as promised in Lemma 2.10. By (3.1) and Lemma 2.10

$$\begin{aligned}\widetilde{\mu}_{\alpha,t}(x) &\gtrsim (1-|x|^2)^{(\alpha+n)t-(\alpha+n)} \int_{E_{\delta_0}(x)} |R_\alpha(x,y)|^t\, \mathrm{d}\mu(y) \\ &\sim \frac{\mu(E_{\delta_0}(x))}{(1-|x|^2)^{\alpha+n}} = \widehat{\mu}_{\alpha,\delta_0}(x).\end{aligned} \tag{3.4}$$

Hence $\widehat{\mu}_{\alpha,\delta_0} \in L^p_\alpha$. By Lemma 3.1, $\widehat{\mu}_{\alpha,\delta} \in L^p_\alpha$ for every $0 < \delta < 1$. $\square$

The next proposition is about a similar result concerning pointwise bounds.

Proposition 3.3. *Suppose $\gamma \geqslant 0$ and $\alpha > -1$. The following are equivalent:*

(a) $\widehat{\mu}_{\alpha,\delta}(x) \lesssim (1-|x|^2)^{\gamma}$ *for some (every)* $0 < \delta < 1$.

(b) $\bar{\mu}_{\alpha,s}(x) \lesssim (1-|x|^2)^{\gamma}$ *for some (every)* $s > \gamma$.

(c) $\widetilde{\mu}_{\alpha,t}(x) \lesssim (1-|x|^2)^{\gamma}$ *for some (every)* $t > (\alpha+n+\gamma)/(\alpha+n)$.

Proof. The proof is similar to the proof of the previous proposition. To see that (a) implies (b) suppose that (a) holds for some $0 < \delta < 1$. By (3.3)

$$\begin{aligned}(1-|x|^2)^s \int_{\mathbb{B}} \frac{\mathrm{d}\mu(y)}{[x,y]^{\alpha+n+s}} &\sim (1-|x|^2)^s \int_{\mathbb{B}} \frac{\widehat{\mu}_{\alpha,\delta}(y)}{[x,y]^{\alpha+n+s}}\,\mathrm{d}\nu_{\alpha}(y) \\ &\lesssim (1-|x|^2)^s \int_{\mathbb{B}} \frac{(1-|y|^2)^{\alpha+\gamma}}{[x,y]^{\alpha+n+s}}\,\mathrm{d}\nu(y).\end{aligned}$$

Part (b) follows from Lemma 2.12. That (b) implies (c) is immediate from (3.2). To see that (c) implies (a), pick δ_0 as in Lemma 2.10. Relation (3.4) shows that (a) holds with $\delta = \delta_0$. That it holds for every $0 < \delta < 1$ is a consequence of Lemma 3.2 of [5]. □

3.1. q-Carleson measures for harmonic Bergman spaces.

Characterizations of q-Carleson measures for harmonic Bergman spaces b^p_{α}, $\alpha > -1$ in terms of $\widehat{\mu}_{\alpha,\delta}$ and $\widetilde{\mu}_{\alpha,t}$ are established by various authors in more general settings. In this subsection we will recall these results and add a new characterization in terms of $\bar{\mu}_{\alpha,s}$. The characterizations are divided into two cases depending on whether $q < p$ or $q \geqslant p$. In the case $q < p$ note that the conjugate exponent of p/q is $p/(p-q)$.

Theorem 3.4. *Let $0 < q < p < \infty$, $\alpha > -1$ and $\mu \geqslant 0$. The following are equivalent:*

(a) μ *is a q-Carleson measure for* b^p_{α}.

(b) $\widehat{\mu}_{\alpha,\delta} \in L^{p/(p-q)}_{\alpha}$ *for some (every)* $0 < \delta < 1$.

(c) $\widetilde{\mu}_{\alpha,t} \in L^{p/(p-q)}_{\alpha}$ *for some (every)* $t > 1$.

(d) $\bar{\mu}_{\alpha,s} \in L^{p/(p-q)}_{\alpha}$ *for some (every)* $s > 0$.

Proof. That (a) and (b) are equivalent is proved in [15] and [16] for the unweighted holomorphic Bergman space on the unit disc $\mathbb{D}$. As mentioned in the remarks of [15], the method works also for weighted harmonic Bergman spaces on the unit ball of $\mathbb{R}^n$. The equivalence of (a), (b) and (c) is proved in [4], Theorem 3.4, not just for the ball but for bounded smooth domains. The equivalence of (b), (c) and (d) follows from Proposition 3.2. □

We now consider the case $q \geqslant p$.

Theorem 3.5. *Let $0 < p \leqslant q < \infty$, $\alpha > -1$ and $\mu \geqslant 0$. The following are equivalent:*

(a) *μ is a q-Carleson measure for b^p_α.*

(b) *$\widehat{\mu}_{\alpha,\delta} \lesssim (1-|x|^2)^{(\alpha+n)(q/p-1)}$ for some (every) $0 < \delta < 1$.*

(c) *$\widetilde{\mu}_{\alpha,t} \lesssim (1-|x|^2)^{(\alpha+n)(q/p-1)}$ for some (every) $t > q/p$.*

(d) *$\bar{\mu}_{\alpha,s} \lesssim (1-|x|^2)^{(\alpha+n)(q/p-1)}$ for some (every) $s > (\alpha+n)(q/p-1)$.*

Note that (b) is equivalent to

$$\mu(E_\delta(x)) \lesssim (1-|x|^2)^{(\alpha+n)q/p} \quad \text{for some (every) } 0 < \delta < 1$$

and (d) is equivalent to

$$(1-|x|^2)^c \int_{\mathbb{B}} \frac{\mathrm{d}\mu(y)}{[x,y]^{(\alpha+n)q/p+c}} \lesssim 1 \quad \text{for some (every) } c > 0.$$

Proof. Equivalence of (a) and (b) is proved in [18]. Equivalence of (a), (b) and (c) is proved in [4], Theorem 3.1, for bounded smooth domains. That (b), (c) and (d) are equivalent follows from Proposition 3.3. $\square$

3.2. q-Carleson measures for harmonic Bloch spaces b^∞_α, $\alpha > 0$. In this subsection we will characterize q-Carleson measures for b^∞_α, $\alpha > 0$, in terms of a weighted integral of μ. It is also possible to give characterizations in terms of $\widehat{\mu}_{\alpha,\delta}$, $\widetilde{\mu}_{\alpha,t}$ or $\bar{\mu}_{\alpha,s}$ similar to the previous two theorems, but the following result will be sufficient for our purposes.

Theorem 3.6. *Let $0 < q < \infty$, $\alpha > 0$ and $\mu \geqslant 0$. The following are equivalent:*

(a) *μ is a q-Carleson measure for b^∞_α.*

(b) *$\int_{\mathbb{B}} \mathrm{d}\mu(x)/(1-|x|^2)^{\alpha q} < \infty$.*

For the *holomorphic* counterpart of the above theorem see [10], Theorem 1.2. The proof in [10] uses the so-called holomorphic Ryll-Wojtaszczyk polynomials which do not seem to have harmonic analogues. Our proof below is based on an idea of Luecking, see [16], and employs Khinchine's inequality.

Define the Rademacher functions r_k on $\mathbb{R}$ by

$$r_1(t) = \begin{cases} 1 & \text{if } 0 \leqslant t-[t] < \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} \leqslant t-[t] < 1 \end{cases}$$

$$r_k(t) = r_1(2^{k-1}t), \quad k = 2, 3, \ldots$$

Let $(c_k) \in l^2$ be a sequence of complex numbers and $f(t) = \sum\limits_{k=1}^{\infty} c_k r_k(t)$. Khinchine's inequality states that for any $0 < q < \infty$, the $L^q[0,1]$ norm of f is comparable to the l^2 norm of (c_k).

Lemma 3.7 (Khinchine's inequality)**.** *Let $0 < q < \infty$ and $(c_k) \in l^2$. The series $\sum\limits_{k=1}^{\infty} c_k r_k(t)$ converges almost everywhere and if $f(t) = \sum\limits_{k=1}^{\infty} c_k r_k(t)$, then*

$$\Big(\int_0^1 |f(t)|^q \,\mathrm{d}t\Big)^{1/q} \sim \Big(\sum_{k=1}^{\infty} |c_k|^2\Big)^{1/2}.$$

A proof of Khinchine's inequality can be found in [22], Section V.8. In the proof of Theorem 3.6 we will use some special functions in b^∞_α that are defined in the next lemma. For the $\alpha = 0$ version of this lemma, see [3].

Lemma 3.8. *Let $0 < \delta < 1$ and (a_k) be a δ-separated sequence. Let $\alpha > 0$ and $s > \alpha - 1$. Then for $(\lambda_k) \in l^\infty$*

$$f(x) = \sum_{k=1}^{\infty} \lambda_k (1 - |a_k|^2)^{s+n-\alpha} R_s(x, a_k) \tag{3.5}$$

is in b^∞_α and $\|f\|_{b^\infty_\alpha} \lesssim \|\lambda_k\|_{l^\infty}$.

Proof. The first estimate in Lemma 2.8 shows

$$|f(x)| \lesssim \|\lambda_k\|_{l^\infty} \sum_{k=1}^{\infty} \frac{(1 - |a_k|^2)^{s+n-\alpha}}{[x, a_k]^{s+n}}.$$

By Lemma 2.2, 2.3 and (2.2)

$$\frac{(1 - |a_k|^2)^{s+n-\alpha}}{[x, a_k]^{s+n}} \sim \int_{E_{\delta/2}(a_k)} \frac{(1 - |y|^2)^{s-\alpha}}{[x, y]^{s+n}} \,\mathrm{d}\nu(y).$$

Using this and the fact that the balls $E_{\delta/2}(a_k)$ are disjoint we obtain

$$|f(x)| \lesssim \|\lambda_k\|_{l^\infty} \sum_{k=1}^{\infty} \int_{E_{\delta/2}(a_k)} \frac{(1 - |y|^2)^{s-\alpha}}{[x, y]^{s+n}} \,\mathrm{d}\nu(y) \leqslant \|\lambda_k\|_{l^\infty} \int_{\mathbb{B}} \frac{(1 - |y|^2)^{s-\alpha}}{[x, y]^{s+n}} \,\mathrm{d}\nu(y).$$

The required result follows from Lemma 2.12. $\square$

We are now ready to prove Theorem 3.6.

P r o o f of Theorem 3.6. (a) $\Rightarrow$ (b): Pick $s > \alpha - 1$. By Lemma 2.10 there exists $0 < \delta < 1$ such that

$$R_s(x,y) \sim \frac{1}{(1-|y|^2)^{s+n}} \quad \text{when } x \in E_\delta(y). \tag{3.6}$$

Let (a_k) be such that the properties listed in Lemma 2.4 hold with δ as above.

Let $(\lambda_k) \in l^\infty$ and f be defined as in (3.5). By assumption (a) and Lemma 3.8

$$\int_{\mathbb{B}} \left| \sum_{k=1}^{\infty} \lambda_k (1-|a_k|^2)^{s+n-\alpha} R_s(x,a_k) \right|^q \mathrm{d}\mu(x) \lesssim \|f\|^q_{b^\infty_\alpha} \lesssim \|\lambda_k\|^q_{l^\infty}.$$

For $t \in [0,1]$ replace λ_k with $\lambda_k r_k(t)$ (which does not affect the l^∞ norm) and integrate as t ranges from 0 to 1 to obtain

$$\int_0^1 \int_{\mathbb{B}} \left| \sum_{k=1}^{\infty} \lambda_k r_k(t) (1-|a_k|^2)^{s+n-\alpha} R_s(x,a_k) \right|^q \mathrm{d}\mu(x)\,\mathrm{d}t \lesssim \|\lambda_k\|^q_{l^\infty}.$$

Applying Fubini's theorem shows

$$\int_{\mathbb{B}} \int_0^1 \left| \sum_{k=1}^{\infty} \lambda_k r_k(t) (1-|a_k|^2)^{s+n-\alpha} R_s(x,a_k) \right|^q \mathrm{d}t\,\mathrm{d}\mu(x) \lesssim \|\lambda_k\|^q_{l^\infty}.$$

To use Khinchine's inequality we first check that $(\lambda_k(1-|a_k|^2)^{s+n-\alpha} R_s(x,a_k))$ is in l^2. For fixed $x \in \mathbb{B}$, we have $|R_s(x,a_k)| \lesssim 1$ and by Lemma 2.2, (2.2) and Lemma 2.4 (i),

$$\begin{aligned}\sum_{k=1}^{\infty} (1-|a_k|^2)^{2(s+n-\alpha)} &\sim \sum_{k=1}^{\infty} \int_{E_{\delta/2}(a_k)} (1-|y|^2)^{2(s-\alpha)+n}\,\mathrm{d}\nu(y) \\ &\leqslant \int_{\mathbb{B}} (1-|y|^2)^{2(s-\alpha)+n}\,\mathrm{d}\nu(y),\end{aligned}$$

which is finite since $2(s-\alpha)+n > 0$. We now apply Khinchine's inequality and deduce

$$\int_{\mathbb{B}} \left(\sum_{k=1}^{\infty} |\lambda_k|^2 (1-|a_k|^2)^{2(s+n-\alpha)} |R_s(x,a_k)|^2 \right)^{q/2} \mathrm{d}\mu(x) \lesssim \|\lambda_k\|^q_{l^\infty}.$$

By part (iii) of Lemma 2.4 for any function $g \geqslant 0$ on $\mathbb{B}$, we have

$$\sum_{j=1}^{\infty} \int_{E_\delta(a_j)} g\,\mathrm{d}\mu \leqslant N \int_{\mathbb{B}} g\,\mathrm{d}\mu.$$

Thus

$$\sum_{j=1}^{\infty} \int_{E_\delta(a_j)} \left(\sum_{k=1}^{\infty} |\lambda_k|^2 (1-|a_k|^2)^{2(s+n-\alpha)} |R_s(x,a_k)|^2 \right)^{q/2} \mathrm{d}\mu(x) \lesssim \|\lambda_k\|_{l^\infty}^q.$$

In the second sum above, taking only the term corresponding to $k=j$ gives

$$\sum_{j=1}^{\infty} \int_{E_\delta(a_j)} |\lambda_j|^q (1-|a_j|^2)^{q(s+n-\alpha)} |R_s(x,a_j)|^q \,\mathrm{d}\mu(x) \lesssim \|\lambda_k\|_{l^\infty}^q.$$

Setting $\lambda_j = 1$ for $j = 1, 2, \ldots$ and using (3.6) we conclude that

$$\sum_{j=1}^{\infty} \frac{\mu(E_\delta(a_j))}{(1-|a_j|^2)^{\alpha q}} \lesssim 1.$$

Finally, by part (ii) of Lemma 2.4 and Lemma 2.2

$$\int_{\mathbb{B}} \frac{\mathrm{d}\mu(x)}{(1-|x|^2)^{\alpha q}} \leqslant \sum_{j=1}^{\infty} \int_{E_\delta(a_j)} \frac{\mathrm{d}\mu(x)}{(1-|x|^2)^{\alpha q}} \lesssim \sum_{j=1}^{\infty} \frac{\mu(E_\delta(a_j))}{(1-|a_j|^2)^{\alpha q}} \lesssim 1.$$

(b) $\Rightarrow$ (a): This part immediately follows from the fact that when $\alpha > 0$ and $f \in b_\alpha^\infty$, we have $(1-|x|^2)^\alpha |f(x)| \leqslant \|f\|_{b_\alpha^\infty}$ for all $x \in \mathbb{B}$. □

4. Inclusion relations

We begin with the inclusion relations between two Bergman-Besov spaces. For future reference we record the following immediate consequence of Lemma 2.7.

Lemma 4.1. *Let $0 < p, q < \infty$ and $\alpha, \beta, t \in \mathbb{R}$. Then*

$$b_\alpha^p \subset b_\beta^q \Leftrightarrow b_{\alpha+pt}^p \subset b_{\beta+qt}^q.$$

P r o o f. Pick any $s \in \mathbb{R}$, apply (the invertible operator) D_s^t to the lefthand side and use Lemma 2.7. □

P r o o f of Theorem 1.1. We first assume $\alpha, \beta > -1$. Since pointwise evaluation is bounded on b_α^p and b_β^q, by the closed graph theorem

$$\begin{aligned} b_\alpha^p \subset b_\beta^q &\Leftrightarrow \|f\|_{b_\beta^q} \lesssim \|f\|_{b_\alpha^p} \quad \forall f \in b_\alpha^p, \\ &\Leftrightarrow \left(\int_{\mathbb{B}} |f(x)|^q (1-|x|^2)^\beta \,\mathrm{d}\nu(x) \right)^{1/q} \lesssim \|f\|_{b_\alpha^p} \quad \forall f \in b_\alpha^p. \end{aligned}$$

That is, $b^p_\alpha \subset b^q_\beta$ if and only if

$$\mathrm{d}\mu(x) = (1-|x|^2)^\beta \,\mathrm{d}\nu(x)$$

is a q-Carleson measure for b^p_α. Using part (b) of Theorem 3.4 and (2.2) we deduce that

$$\begin{aligned} b^p_\alpha \subset b^q_\beta &\Leftrightarrow (1-|x|^2)^{\beta-\alpha} \in L^{p/(p-q)}_\alpha \\ &\Leftrightarrow \int_{\mathbb{B}} (1-|x|^2)^{(\beta-\alpha)p/(p-q)+\alpha} \,\mathrm{d}\nu(x) < \infty. \end{aligned}$$

The last integral is finite if and only if

$$(\beta-\alpha)\frac{p}{p-q} + \alpha > -1,$$

which is equivalent to $(\alpha+1)/p < (\beta+1)/q$. This finishes the proof for $\alpha, \beta > -1$.

We generalize above to all $\alpha, \beta \in \mathbb{R}$ by using Lemma 4.1. Let $\alpha, \beta \in \mathbb{R}$. Pick arbitrary $s \in \mathbb{R}$ and $t \in \mathbb{R}$ such that

$$\alpha + pt > -1, \quad \beta + qt > -1.$$

By Lemma 4.1, we have $b^p_\alpha \subset b^q_\beta \Leftrightarrow b^p_{\alpha+pt} \subset b^q_{\beta+qt}$. The first part of the proof shows that the inclusion on the right holds if and only if $(\alpha+pt+1)/p < (\beta+qt+1)/q$. Canceling t we get the desired result. $\square$

Proof of Theorem 1.2. The idea of the proof is similar to the previous one, the main difference is that we refer to Theorem 3.5 instead of Theorem 3.4.

First assume $\alpha, \beta > -1$. As in the previous proof, $b^p_\alpha \subset b^q_\beta$ if and only if

$$\mathrm{d}\mu(x) = (1-|x|^2)^\beta \,\mathrm{d}\nu(x)$$

is a q-Carleson measure for b^p_α. By part (b) of Theorem 3.5 and (2.2) this is true if and only if

$$(1-|x|^2)^{\beta-\alpha} \lesssim (1-|x|^2)^{(\alpha+n)(q/p-1)}.$$

This estimate holds if and only if $\beta - \alpha \geqslant (\alpha+n)(q/p-1)$, which is equivalent to $(\alpha+n)/p \leqslant (\beta+n)/q$.

Generalization to $\alpha, \beta \in \mathbb{R}$ follows from Lemma 4.1 as in the previous proof. $\square$

Combining Theorems 1.1 and 1.2 leads to the following corollary.

Corollary 4.2. *All Bergman-Besov spaces are different.*

P r o o f. Suppose first that $p = q$. In this case $b^p_\alpha \subset b^p_\beta$ if and only if $\alpha < \beta$ and the inclusion is strict. This is well-known for $\alpha, \beta > -1$ and can easily be generalized to $\alpha, \beta \in \mathbb{R}$ by using Lemma 4.1 and arguing as in the proof of Theorem 1.1. So $b^p_\alpha = b^p_\beta$ if and only if $\alpha = \beta$.

We now consider the case $p \neq q$. Without loss of generality assume $q < p$. Suppose $b^p_\alpha = b^q_\beta$. Since $b^p_\alpha \subset b^q_\beta$, by Theorem 1.1, we must have $(\alpha+1)/p < (\beta+1)/q$. Adding this to the inequality $(n-1)/p < (n-1)/q$ we obtain

$$\frac{\alpha + n}{p} < \frac{\beta + n}{q}.$$

On the other hand, by Theorem 1.2, the inclusion $b^q_\beta \subset b^p_\alpha$ implies

$$\frac{\beta + n}{q} \leqslant \frac{\alpha + n}{p}.$$

This contradiction shows $b^p_\alpha \neq b^q_\beta$. □

Corollary 4.3. *The inclusions in Theorems* 1.1 *and* 1.2 *are strict unless* $p = q$ *and* $\alpha = \beta$.

P r o o f. By the previous corollary $b^p_\alpha \neq b^q_\beta$ unless $p = q$ and $\alpha = \beta$. □

We now turn to the inclusion relations between Bergman-Besov and Bloch spaces.

P r o o f of Theorem 1.3. We prove part (a) first with the assumptions $\alpha > -1$ and $\beta > 0$. As before, the boundedness of pointwise evaluation functional on b^p_α and b^∞_β and the closed graph theorem imply

$$b^\infty_\beta \subset b^p_\alpha \ \Leftrightarrow\ \|f\|_{b^p_\alpha} \lesssim \|f\|_{b^\infty_\beta} \quad \forall f \in b^\infty_\beta.$$

In other words, $b^\infty_\beta \subset b^p_\alpha$ if and only if

$$\mathrm{d}\mu(x) = (1 - |x|^2)^\alpha \,\mathrm{d}\nu(x)$$

is a p-Carleson measure for b^∞_β. By Theorem 3.6 this holds if and only if

$$\int_{\mathbb{B}} \frac{(1 - |x|^2)^\alpha}{(1 - |x|^2)^{\beta p}} \,\mathrm{d}\nu(x) < \infty.$$

Since the above integral is finite if and only if $\alpha - \beta p > -1$, equivalently $\beta < (\alpha+1)/p$, the proof is complete in the case $\alpha > -1$ and $\beta > 0$.

We generalize the above result to all $\alpha, \beta \in \mathbb{R}$ by using Lemma 2.7. Let $\alpha, \beta \in \mathbb{R}$. Pick arbitrary $s \in \mathbb{R}$ and $t \in \mathbb{R}$ such that

$$\alpha + pt > -1, \quad \beta + t > 0. \tag{4.1}$$

Since $b^\infty_\beta \subset b^p_\alpha$ if and only if $D^t_s b^\infty_\beta \subset D^t_s b^p_\alpha$, by Lemma 2.7

$$b^\infty_\beta \subset b^p_\alpha \Leftrightarrow b^\infty_{\beta+t} \subset b^p_{\alpha+pt}. \tag{4.2}$$

By the first part of the proof, the righthand side holds if and only if $\beta + t < (\alpha + pt + 1)/p$, which is same as $\beta < (\alpha + 1)/p$.

We now prove part (b). We, again, first consider the case $\alpha > -1$ and $\beta > 0$. To see the "if" part suppose $\beta \geqslant (\alpha + n)/p$. If $f \in b^p_\alpha$, then by Lemma 2.14

$$(1 - |x|^2)^{(\alpha+n)/p} |f(x)| \lesssim \|f\|_{b^p_\alpha}.$$

Using first $\beta > 0$ and then $\beta \geqslant (\alpha + n)/p$, we obtain

$$\|f\|_{b^\infty_\beta} = \sup_{x\in\mathbb{B}} (1 - |x|^2)^\beta |f(x)| \leqslant \sup_{x\in\mathbb{B}} (1 - |x|^2)^{(\alpha+n)/p} |f(x)| \lesssim \|f\|_{b^p_\alpha}.$$

Hence $b^p_\alpha \subset b^\infty_\beta$.

To see the "only if" part suppose $\beta < (\alpha + n)/p$. If $b^p_\alpha \subset b^\infty_\beta$, then as before we must have

$$\|f\|_{b^\infty_\beta} \lesssim \|f\|_{b^p_\alpha} \quad \forall f \in b^p_\alpha. \tag{4.3}$$

Choose γ such that $\beta < \gamma < (\alpha + n)/p$. For $a \in \mathbb{B}$, define $g_a\colon \mathbb{B} \to \mathbb{R}$,

$$g_a(x) = R_{\gamma-n}(x, a).$$

Lemma 2.11 implies that $g_a \in b^p_\alpha$ for each $a \in \mathbb{B}$ and

$$\|g_a\|_{b^p_\alpha} = \left(c_\alpha \int_{\mathbb{B}} |R_{\gamma-n}(x, a)|^p (1 - |x|^2)^\alpha \, \mathrm{d}\nu(x) \right)^{1/p} \sim 1.$$

On the other hand, by Lemma 2.9

$$\|g_a\|_{b^\infty_\beta} = \sup_{x\in\mathbb{B}} (1 - |x|^2)^\beta |R_{\gamma-n}(x, a)| \geqslant (1 - |a|^2)^\beta |R_{\gamma-n}(a, a)| \sim \frac{1}{(1 - |a|^2)^{\gamma-\beta}}.$$

The contradiction with (4.3) as $|a| \to 1^-$ shows that $b^p_\alpha \subset b^\infty_\beta$ only if $\beta \geqslant (\alpha + n)/p$. This completes the proof of part (b) when $\alpha > -1$ and $\beta > 0$.

To generalize to all $\alpha, \beta \in \mathbb{R}$ we argue as in part (a). Choose t satisfying (4.1). Similar to (4.2) we have $b^p_\alpha \subset b^\infty_\beta$ if and only if $b^p_{\alpha+pt} \subset b^\infty_{\beta+t}$ and last inclusion holds if and only if $\beta \geqslant (\alpha + n)/p$. $\square$

It is known that weighted Bloch spaces with different parameters are different, that is $b^\infty_\alpha = b^\infty_\beta$ if and only if $\alpha = \beta$; see, for example, [9], Remark 4.9. Theorem 1.3 immediately implies that a Bergman-Besov space and a weighted Bloch space cannot be the same. Together with Corollary 4.2 we deduce the following.

Corollary 4.4. *All Bergman-Besov spaces and all weighted Bloch spaces are different.*

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