

FINITE  $p$ -GROUPS WITH EXACTLY TWO NONLINEAR  
NON-FAITHFUL IRREDUCIBLE CHARACTERS

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*Abstract.* Let  $G$  be a finite group with exactly two nonlinear non-faithful irreducible characters. We discuss the properties of  $G$  and classify finite  $p$ -groups with exactly two nonlinear non-faithful irreducible characters.

*Keywords:*  $p$ -group; nonlinear irreducible character; non-faithful character

*MSC 2010:* 20C15

1. INTRODUCTION

Iranmanesh and Saeidi [4] studied finite groups with exactly one nonlinear non-faithful irreducible character. And Saeidi [6] classified solvable groups with a unique nonlinear non-faithful irreducible character. We consider the following case in this note.

**Hypothesis (\*):**

*A finite group has exactly two nonlinear non-faithful irreducible characters.*

Let  $G$  be a finite group with exactly two nonlinear non-faithful irreducible characters  $\chi_1, \chi_2$ . Let  $K_1 = \ker \chi_1$ ,  $K_2 = \ker \chi_2$ , and write  $L = K_1 \cap K_2$ .

In this note, we will show some properties of groups which satisfy Hypothesis (\*). Our main conclusion is the classification of finite  $p$ -groups with exactly two nonlinear non-faithful irreducible characters. In fact, we have the following result.

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**Theorem 1.1.** *A  $p$ -group  $G$  has exactly two nonlinear non-faithful irreducible characters if and only if one of the following assertions holds.*

- (1)  $G$  is a 2-group with nilpotence class 2,  $G/\mathbf{Z}(G)$  is elementary abelian,  $\mathbf{Z}(G) \cong C_2 \times C_2$  and  $|G'| = 2$ .
- (2)  $G$  is a group of order 32 of nilpotence class 3 and  $\mathbf{Z}(G) \cong C_2$  or  $\mathbf{Z}(G) \cong C_4$ .
- (3)  $G$  is a group of order 81 of nilpotence class 3.

All groups considered in this note are finite. The notation and terminology are standard, and one can refer to [5] and [3].

## 2. PRELIMINARIES

We first discuss the intersection of the kernels of irreducible nonlinear characters of a finite group. There is a modular form of the intersection of the kernels of irreducible Brauer characters in [8]. Moreover, in this section, we give some properties of groups which satisfy Hypothesis (\*), and state facts which are important to prove the main theorem of this note.

**Proposition 2.1.** *Let  $\text{Irr}(G)$  and  $\text{Lin}(G)$  be the sets of irreducible characters and linear characters of a group  $G$ , respectively. If  $\text{Irr}(G) \supsetneq \text{Lin}(G)$ , then*

- (i)  $\bigcap_{\varphi \in \text{Irr}(G) - \text{Lin}(G)} \ker \varphi = 1$ . In particular, if  $G$  has a unique nonlinear irreducible character  $\chi$ , then  $\chi$  is faithful; and if  $G$  has exactly two nonlinear irreducible characters  $\chi, \varphi$ , then  $\ker \chi \cap \ker \varphi = 1$ .
- (ii) if  $G$  satisfies Hypothesis (\*) and  $L > 1$ , then  $\mathbf{Z}(G)$  is cyclic, where  $\mathbf{Z}(G)$  is the center of  $G$ ;
- (iii) if  $G$  satisfies Hypothesis (\*), then every normal subgroup of  $G$  not containing  $G'$  is among  $K_1, K_2$  and  $L$ ;
- (iv) if  $\mathbf{Z}(G) \neq 1$  and  $G$  satisfies Hypothesis (\*) and  $L > 1$ , then  $L$  is a minimal normal subgroup of  $G$  of a prime order. Moreover, if  $G' \cap L = 1$ , then  $G' \cap K_i = 1$  for  $i = 1, 2$ .

**Proof.** Write  $U = \bigcap_{\varphi \in \text{Irr}(G) - \text{Lin}(G)} \ker \varphi$  and  $h \in U$ . Then  $\varphi(h) = \varphi(1)$  for  $\varphi \in \text{Irr}(G) - \text{Lin}(G)$ . For any  $\lambda \in \text{Lin}(G)$ ,  $\lambda$  can act on  $\text{Irr}(G)$  by multiplication. Then  $\exists \theta \in \text{Irr}(G) - \text{Lin}(G)$  such that  $\varphi = \lambda\theta$ . Thus

$$\varphi(1) = \varphi(h) = \lambda(h)\theta(h) = \lambda(h)\theta(1) = \lambda(h)\varphi(1), \quad \lambda(h) = 1.$$

Since  $\lambda$  is arbitrary, it follows that  $h \in \bigcap_{\lambda \in \text{Lin}(G)} \ker \lambda = G'$ , where  $G'$  is the derived subgroup of  $G$ , and then

$$h \in U \cap G' = \bigcap_{\mu \in \text{Irr}(G)} \ker \mu = 1.$$

Therefore, we have that  $\bigcap_{\varphi \in \text{Irr}(G) - \text{Lin}(G)} \ker \varphi = 1$ . Immediately, if  $G$  has a unique nonlinear irreducible character  $\chi$ , then  $\ker \chi = 1$ .

If  $G$  satisfies Hypothesis (\*) and  $L > 1$ , then  $G$  has at least one nonlinear faithful irreducible character and so  $\mathbf{Z}(G)$  is cyclic.

Let  $G$  satisfy Hypothesis (\*) and let  $N$  be a normal subgroup of  $G$  not containing the derived subgroup  $G'$ . Then the number of nonlinear irreducible characters of  $G/N$  is no more than 2. Let  $\text{Irr}_1(G)$  denote the set of all the nonlinear irreducible characters of  $G$ . So we have the following two cases.

*Case (a):* When  $|\text{Irr}_1(G/N)| = 1$ , it follows that  $\text{Irr}_1(G/N) = \{\widehat{\chi}_1\}$  or  $\{\widehat{\chi}_2\}$ , where  $\widehat{\chi}_i(Ng) = \chi_i(g)$  for  $g \in G$ ,  $i = 1, 2$ . If  $|\text{Irr}_1(G/N)| = \{\widehat{\chi}_1\}$ , we deduce that  $\ker \widehat{\chi}_1 = \bar{1}$  by (i). Therefore  $N = \ker \chi_1$  since  $\ker \chi_1/N = \ker \widehat{\chi}_1$ . If  $|\text{Irr}_1(G/N)| = \{\widehat{\chi}_2\}$ , for the same reason as above, we have  $N = \ker \chi_2$ .

*Case (b):* When  $|\text{Irr}_1(G/N)| = 2$ , it follows that  $\text{Irr}_1(G/N) = \{\widehat{\chi}_1, \widehat{\chi}_2\}$ . By (i), we have that  $\ker \widehat{\chi}_1 \cap \ker \widehat{\chi}_2 = \bar{1}$ . Since  $(\ker \chi_1/N) \cap (\ker \chi_2/N) = \ker \widehat{\chi}_1 \cap \ker \widehat{\chi}_2$ , it follows that  $N = \ker \chi_1 \cap \ker \chi_2 = L$ .

By the above proof, we have that  $N$  is among  $K_1, K_2$  and  $L$ .

By (iii), it follows that  $G' \cap L = 1$  or  $G' \cap L = L$ . In both cases, we have that  $L$  is a minimal normal subgroup of  $G$ . If  $G' \subseteq \mathbf{Z}(G)$ , then  $G'$  is cyclic and so  $L \subseteq \mathbf{Z}(G)$  and then  $L$  is of a prime order. If  $\mathbf{Z}(G)$  is not containing  $G'$ , then it follows by (iii) that  $\mathbf{Z}(G)$  is among  $K_1, K_2$  and  $L$ . Thus  $L \subseteq \mathbf{Z}(G)$  and so  $L$  is of a prime order. Also, if  $G' \cap L = 1$ , then  $G' \cap K_i = 1$  for  $i = 1, 2$ . Otherwise,  $G' \cap K_i = L$  by (iii) and we have that

$$L = G' \cap K_i = G' \cap K_i \cap L = (G' \cap L) \cap K_i = 1,$$

a contradiction. □

Seitz proved in [7] the following lemma which will be used many times in the next section.

**Lemma 2.1.** *Let  $G$  be a finite group. Then  $G$  has exactly one nonlinear irreducible character if and only if one of the following conditions holds.*

- (i)  $G$  is an extraspecial 2-group.
- (ii)  $G$  is a Frobenius group with elementary abelian Frobenius kernel  $G'$  and a cyclic Frobenius complement  $H$ , where  $|G'| - 1 = |H|$ .

Zhang classified in [9] the groups with exactly two nonlinear irreducible characters.

**Lemma 2.2.** *Let  $G$  be a finite group with exactly two nonlinear irreducible characters. Then one of the following assertions holds.*

- (1)  $G$  is an extraspecial 3-group.
- (2)  $G$  is a Frobenius group with abelian Frobenius complement  $H$  and elementary abelian Frobenius kernel  $N$ , and  $2|H| = |N| - 1$ .
- (3)  $G = (C_3 \times C_3) \rtimes Q_8$  is a Frobenius group with Frobenius complement  $Q_8$ .
- (4)  $G$  is a 2-group with nilpotence class 3 with a normal series  $G \triangleright G' \triangleright \mathbf{Z}(G) \triangleright 1$ , and  $G/\mathbf{Z}(G)$  is an extraspecial 2-group,  $|G'| = 4$ ,  $|\mathbf{Z}(G)| = 2$ .
- (5)  $G$  is a 2-group with nilpotence class 2 with a normal series  $G \triangleright \mathbf{Z}(G) \triangleright G' \triangleright 1$ , and  $G/\mathbf{Z}(G)$  is elementary abelian,  $|\mathbf{Z}(G)| = 4$ ,  $|G'| = 2$ . Furthermore,  $|G| = 2^{2m}$ ,  $m \in \mathbb{Z}$ . (See [1], Theorem 6, page 281).

### 3. $p$ -GROUPS

Suppose  $G$  satisfies Hypothesis (\*), and  $L < K_1$ ,  $L < K_2$ . The following lemma indicates that we only need to consider 2-groups if we study  $p$ -groups satisfying those conditions.

**Lemma 3.1.** *Let a  $p$ -group  $G$  satisfy Hypothesis (\*). Let  $L = K_1 \cap K_2$  and  $L \leq K_1$ ,  $L \leq K_2$ . Then  $p = 2$  and  $|K_1| = |K_2| = 4$  if  $L > 1$ ,  $|K_1| = |K_2| = 2$  if  $L = 1$ .*

*Proof.* Notice that  $G/K_1$  has only one nonlinear irreducible character. Also, since  $G$  is a finite  $p$ -group, we have that  $p = 2$  by Lemma 2.1.

Assume that  $K_1/M$  is a chief factor of  $G$ . Then  $G' \not\leq M$ . Otherwise,  $G' \leq K_1$  and  $\chi_1 \in \text{Irr}(G/K_1)$ , a contradiction. Also, since  $L \leq K_2$ , we have  $M \neq K_2$ . Therefore,  $M = L$  by Proposition 2.1 (iii) and then  $K_1/L$  is a chief factor of  $G$ . Similarly,  $K_2/L$  is also a chief factor of  $G$ .

Since every chief factor of a  $p$ -group has order  $p$ , it follows that  $|K_1| = |K_2| = 2$  if  $L = 1$ . If  $L > 1$ , then it follows by Proposition 2.1 (iv) that  $|L| = 2$  and so  $|K_1| = |K_2| = 2^2$ . □

A  $p$ -group  $G$  is said to satisfy the *strong condition* on normal subgroups provided that for any  $N \trianglelefteq G$  either  $G' \leq N$  or  $N \leq \mathbf{Z}(G)$ . If for any  $N \trianglelefteq G$ , either  $G' \leq N$  or  $|N\mathbf{Z}(G) : \mathbf{Z}(G)| \leq p$ , then we say  $G$  satisfies the *weak condition* on normal subgroups. Fernández-Alcober and Moretó in [2] gave some results about finite groups satisfying the strong condition or the weak condition on normal subgroups.

**Lemma 3.2.** *Let  $G$  be a  $p$ -group.*

- (1) *If  $G$  satisfies the strong condition on normal subgroups, then it has nilpotence class  $c(G) \leq 3$ . Furthermore,*
  - (i) *if  $c(G) = 2$ , then  $\exp G/\mathbf{Z}(G) = \exp G' = p$ ;*
  - (ii) *if  $c(G) = 3$ , then  $|G : \mathbf{Z}(G)| = p^3$  and  $|G| \leq p^5$ . Moreover,  $|G| = 2^4$  for  $p = 2$ .*
- (2) *If  $G$  satisfies the weak condition on normal subgroups, then it has nilpotence class  $c(G) \leq 4$ . Furthermore,*
  - (i) *if  $c(G) = 2$ , then  $\exp G/\mathbf{Z}(G) = \exp G' = p$  or  $p^2$ . Moreover, in the latter case  $G/\mathbf{Z}(G) \cong C_{p^2} \times C_{p^2}$  and  $G' \cong C_{p^2}$ ;*
  - (ii) *if  $c(G) = 4$ , then  $|G : \mathbf{Z}(G)| = p^4$ , whereas for  $c(G) = 3$  we have  $|G : \mathbf{Z}(G)| = p^3, p^4$  or  $p^6$  for odd  $p$ , and  $|G : \mathbf{Z}(G)| = 2^3, 2^4$  when  $p = 2$ ;*
  - (iii) *if  $c(G) = 4$  and  $p = 2$ , then  $|G| = 2^5$ .*

*Proof.* See Theorem D, Theorem F and Theorem G of [2]. □

Let a  $p$ -group  $G$  satisfy Hypothesis (\*). Obviously, there are three cases for  $K_1 \cap K_2$ : case (i)  $L = 1$ ; case (ii)  $1 < L < K_i$ ,  $i = 1, 2$ ; and case (iii)  $K_1 \leq K_2$  (or  $K_2 \leq K_1$ , but without loss of generality, we may assume  $K_1 \leq K_2$ ). Next, we respectively discuss the structure of  $G$  according to the above cases. First, we have the following theorem.

**Theorem 3.1.** *A  $p$ -group  $G$  satisfies Hypothesis (\*) with  $L = 1$  if and only if  $G$  is a 2-group of nilpotence class 2,  $G/\mathbf{Z}(G)$  is elementary abelian,  $\mathbf{Z}(G) \cong C_2 \times C_2$  and  $|G'| = 2$ .*

*Proof.* When  $G$  is a 2-group of nilpotence class 2,  $G/\mathbf{Z}(G)$  is elementary abelian,  $\mathbf{Z}(G) \cong C_2 \times C_2$  and  $|G'| = 2$ , by Lemma 2.2 (5) and since  $\mathbf{Z}(G)$  is not cyclic, we know that  $G$  has exactly two nonlinear irreducible characters and both of them are non-faithful. It follows that  $L = 1$  from Proposition 2.1.

Now, we assume that a  $p$ -group  $G$  satisfies Hypothesis (\*) and  $L = 1$ . First we have  $p = 2$  and  $|K_1| = |K_2| = 2$  by Lemma 3.1. Hence both  $K_1, K_2$  are minimal normal subgroups of  $G$  and  $K_1, K_2 \leq \mathbf{Z}(G)$ . Using the fact that the nontrivial normal subgroups of  $G$  not containing  $G'$  are  $K_1$  and  $K_2$ , we have that  $G$  satisfies the strong condition and so  $c(G) \leq 3$ . If  $c(G) = 3$ , then  $G' \not\leq \mathbf{Z}(G)$  and hence  $\mathbf{Z}(G) = K_1$  or  $\mathbf{Z}(G) = K_2$ . When  $\mathbf{Z}(G) = K_1$ , we get that  $K_2 \leq K_1$ , contradicting that  $L = 1$ . Thus  $\mathbf{Z}(G) \neq K_1$ . For the same reason,  $\mathbf{Z}(G) \neq K_2$ . Therefore, we obtain that  $c(G) = 2$ . Also we have that  $G/\mathbf{Z}(G)$  is elementary abelian and  $\exp G' = 2$  by Lemma 3.2. Furthermore,  $G/K_i$  is extra-special, we can deduce that  $\mathbf{Z}(G)/K_i = \mathbf{Z}(G/K_i) \cong C_2$ , and hence  $|\mathbf{Z}(G)| = 4$ . We claim that  $|G'| = 2$ , otherwise, we must have that  $\mathbf{Z}(G) = G' \cong C_2 \times C_2$ . Thus we can obtain three normal

subgroups of  $G$  which are different from one another and do not contain  $G'$ . That is impossible as the nontrivial normal subgroups of  $G$  not containing  $G'$  are  $K_1$  and  $K_2$ . Then the claim follows. Therefore by Lemma 2.2 (5) we have that  $G$  has exactly two nonlinear irreducible characters. Thus  $G$  has no faithful irreducible characters, and so  $\mathbf{Z}(G)$  is not cyclic, which implies that  $\mathbf{Z}(G) \cong C_2 \times C_2$ .  $\square$

In the next Lemma, we give some properties of  $p$ -groups which satisfy Hypothesis (\*) and  $L > 1$ .

**Lemma 3.3.** *Let a finite  $p$ -group  $G$  satisfy Hypothesis (\*) and let  $L > 1$ . Then  $L < G'$  and  $G$  satisfies the weak condition on normal subgroups.*

*Proof.* By Proposition 2.1 (iv), we have that  $G' \cap L = 1$  or  $L < G'$ . If  $G' \cap L = 1$ , then  $G' \cap K_i = 1$ ,  $i = 1, 2$  by Proposition 2.1. Thus  $G'$  and  $L$  are all minimal normal subgroups of  $G$  and since  $G$  is a  $p$ -group, we have  $L, G' \subseteq \mathbf{Z}(G)$ . Since  $\mathbf{Z}(G)$  is cyclic, we have that  $L = G'$ , a contradiction. So  $L < G'$ .

Next, we prove that  $G$  satisfies the weak condition. First,  $L \leq \mathbf{Z}(G)$ . And by Proposition 2.1 (iii), we only need to prove that  $|K_i \mathbf{Z}(G) : \mathbf{Z}(G)| = |K_i / (K_i \cap \mathbf{Z}(G))| \leq p$ ,  $i = 1, 2$ . Since  $L \leq K_i \cap \mathbf{Z}(G) \leq K_i$ , and  $K_i/L$  are chief factors of  $G$  or  $K_i/L = 1$ ,  $i = 1, 2$ , the proof follows.  $\square$

In the rest of this paper, we consider the cases (ii) and (iii) stated above.

**Theorem 3.2.** *A  $p$ -group  $G$  satisfies Hypothesis (\*) with  $1 < L < K_1$  and  $1 < L < K_2$  if and only if  $G$  is a group of nilpotence class 3, order 32, and  $\mathbf{Z}(G) \cong C_2$  or  $C_4$ .*

*Proof.* A computation in GAP using the GAP libraries shows that the groups  $G$  of order 32 and nilpotence class 3 are  $((C_4 \times C_2) \rtimes C_2) \rtimes C_2$ ,  $(C_8 \times C_2) \rtimes C_2$ ,  $C_2((C_4 \times C_2) \rtimes C_2) = (C_2 \times C_2)(C_4 \times C_2)$ ,  $(C_2 \times D_8) \rtimes C_2$ ,  $(C_2 \times Q_8) \rtimes C_2$  with  $\mathbf{Z}(G) \cong C_2$  and  $(C_4 \times C_4) \rtimes C_2$ ,  $C_4 D_8 = C_4(C_4 \times C_2)$ ,  $(C_8 \times C_2) \rtimes C_2$  with  $\mathbf{Z}(G) \cong C_4$ . Using GAP's program to compute character tables we verified that in each case Hypothesis (\*) holds for  $G$  with  $1 < L < K_1$  and  $1 < L < K_2$ .

Now we assume that  $G$  satisfies Hypothesis (\*), and  $1 < L < K_1$ ,  $1 < L < K_2$ . First,  $p = 2$  by Lemma 3.1. Notice that  $L < G'$  by Lemma 3.3. Then  $L$  is the unique minimal normal subgroup of  $G$ . And we have that the nilpotence class satisfies  $c(G) \leq 4$  by Lemma 3.2.

If  $c(G) = 2$ , then  $G' \leq \mathbf{Z}(G)$  and so  $G'$  is cyclic. By Lemma 3.2, it follows that  $\exp G' = 2$  or  $2^2$ . We must have  $\exp G' = 2^2$ , otherwise,  $|L| = |G'| = 2$  and since  $L, G' \subseteq \mathbf{Z}(G)$ , we get that  $L = G'$ , a contradiction. Thus  $G' \cong C_4$  and  $G/\mathbf{Z}(G) \cong C_4 \times C_4$ . Note that  $G/L$  has exactly two nonlinear irreducible characters.

By Lemma 2.2 and  $c(G/L) = 2$ , it follows that  $|\mathbf{Z}(G/L)| = 4$  and then  $|\mathbf{Z}(G)/L| \leq 4$ ,  $|\mathbf{Z}(G)| \leq 8$ . Also, by [1], Theorem 6, page 281 we have that  $|G/L| = 2^{2m}$ ,  $m \in \mathbb{Z}$ . Since  $G' \leq \mathbf{Z}(G)$ , it follows that  $|\mathbf{Z}(G)| = 8$  or  $4$ . If  $|\mathbf{Z}(G)| = 4$ , then  $|G| = 2^6$  and  $|G/L| = 2^5$ , a contradiction. Thus  $\mathbf{Z}(G) \cong C_8$ . Also, note that  $G/K_1$  has only one nonlinear irreducible character. By Lemma 2.1 (Seitz's Theorem), it follows that

$$2 = |\mathbf{Z}(G/K_1)| = |\mathbf{Z}(\chi_1) : K_1|.$$

Since  $|K_1| = 4$ , it follows that  $|\mathbf{Z}(\chi_1)| = 8$ . Therefore  $\mathbf{Z}(G) = \mathbf{Z}(\chi_1) \supset K_1$ . Since  $K_1 \subseteq \mathbf{Z}(G)$  and  $G' \subseteq \mathbf{Z}(G)$ , we have that  $K_1 = G'$  as they have the same order, a contradiction.

If  $c(G) = 4$ , by Lemma 3.2 it follows that  $|G| = 2^5$ , that is,  $G$  has maximal class. And then  $G'$  has index 4 in  $G$  and so  $|G'| = 8$ . Since  $G/L$  has exactly two nonlinear irreducible characters  $\widehat{\chi}_1, \widehat{\chi}_2$  and  $\ker \widehat{\chi}_i = K_i/L \neq \bar{1}$  for  $i = 1, 2$ , it follows that  $\mathbf{Z}(G/L)$  is not cyclic and hence  $|(G/L)'| = |G'/L| = 2$  by Lemma 2.2. So  $|G'| = 4$ . We arrive at a contradiction.

The remaining case is  $c(G) = 3$ . By Lemma 3.2 it follows that  $|G : \mathbf{Z}(G)| = 2^3$  or  $2^4$ . Note that  $G' \not\leq \mathbf{Z}(G)$ . So  $\mathbf{Z}(G) \in \{L, K_1, K_2\}$  by Proposition 2.1 (iii). Thus  $|\mathbf{Z}(G)| = 2$  or  $2^2$ . Again by  $|G/L| = 2^{2m}$ , we have that  $|G| = 2^5$ .  $\square$

**Theorem 3.3.** *A  $p$ -group  $G$  satisfies Hypothesis (\*) and  $K_1 \leq K_2$  if and only if  $G$  is a group of nilpotence class 3 and order  $3^4$ .*

*Proof.* First assume that  $G$  satisfies Hypothesis (\*) and  $K_1 \leq K_2$ . Then  $L = K_1$  and so by Proposition 2.1 we have that  $\mathbf{Z}(G)$  cyclic and  $K_1$  is a minimal normal subgroup of  $G$  with  $K_1 \leq \mathbf{Z}(G)$ . Hence  $|K_1| = p$ . Moreover, Lemma 3.3 and Lemma 3.2 show that  $K_1 < G'$  and  $c(G) \leq 4$ .

Assume that  $K_1 \leq K_2$ . Then  $\text{Irr}_1(G/K_2) = \{\widehat{\chi}_2\}$ , where  $\text{Irr}_1(G/K_2)$  denotes the set of nonlinear irreducible characters of  $G/K_2$ . So  $p = 2$  by Lemma 2.1. Moreover, we have  $\text{Irr}_1(G/K_1) = \{\widehat{\chi}_1, \widehat{\chi}_2\}$ . It follows that  $c(G/K_1) = 2$  or  $3$  by Lemma 2.2.

Case (1), when  $c(G/K_1) = 3$ .

If  $c(G) = 4$ , then  $|G| = 2^5$  by Lemma 3.2, and so  $|G/K_1| = 2^4$ . But groups with order  $2^4$  and nilpotence class 3 have three nonlinear irreducible characters, which is a contradiction.

If  $c(G) = 3$ , then  $|G/\mathbf{Z}(G)| = 2^3$  or  $2^4$  by Lemma 3.2. Note that  $G' \not\leq \mathbf{Z}(G)$ , hence  $\mathbf{Z}(G) = K_1$  or  $K_2$ . If  $\mathbf{Z}(G) = K_1$ , then  $|\mathbf{Z}(G)| = 2$  and so  $|G| = 2^4$  or  $2^5$ . When  $|G| = 2^4$ , since  $c(G) = 3$ , it follows that  $G$  has just one nonlinear non-faithful irreducible character. This is a contradiction. When  $|G| = 2^5$ , we would obtain a contradiction in the same way as in the situation of  $c(G) = 4$ . If  $\mathbf{Z}(G) = K_2$ , then  $G$  satisfies the strong condition on normal subgroups. By Lemma 3.2, we have

$|G| = 2^4$  which is not possible as above. Since  $c(G/K_1) = 3$ , it is not possible that  $c(G) = 2$ .

Case (2), when  $c(G/K_1) = 2$ .

By Lemma 2.2, we get  $|G'/K_1| = 2$  and  $|\mathbf{Z}(G/K_1)| = 4$ . Since  $G/K_1$  has a faithful nonlinear irreducible character  $\widehat{\chi}_1$ , it follows that  $\mathbf{Z}(G/K_1) = C_4$ . Therefore, we deduce that  $G'/K_1$  is the unique minimal normal subgroup of  $G/K_1$ . Thus we get that all nonlinear irreducible characters of  $G/K_1$  are faithful, contradicting the fact that  $\widehat{\chi}_2$  is non-faithful for  $G/K_1$ .

Therefore we get that  $K_1 = K_2$ .

Write  $K_1 = K_2 = K$ . Then  $K$  is the only nontrivial normal subgroup of  $G$  which does not contain  $G'$ . Since  $K \leq \mathbf{Z}(G)$ , it follows that  $G$  satisfies the strong condition on normal subgroups and hence  $c(G) \leq 3$ . If  $c(G) = 2$ , then  $G'$  is cyclic as  $G' \leq \mathbf{Z}(G)$ . Therefore  $|G'| = p$  as  $\exp G' = p$ . It follows that  $K = G'$ , a contradiction. So  $c(G) = 3$ . If  $p = 2$ , then  $|G| = 2^4$  by Lemma 3.2. That is impossible as above. Hence  $p \neq 2$ . Note that  $G/K$  has exactly two nonlinear irreducible characters. Then  $G/K$  must be an extra-special 3-group by Lemma 2.2 and so  $p = 3$ . Moreover, by Lemma 3.2, we have  $|G : \mathbf{Z}(G)| = 3^3$ . Since  $G' \not\leq \mathbf{Z}(G)$  and by Proposition 2.1 (iii), we obtain that  $K = \mathbf{Z}(G)$ . It follows that  $|\mathbf{Z}(G)| = 3$ . Hence  $|G| = 3^4$  and  $G$  has maximal class.

Conversely, assume that  $|G| = 3^4$  and  $c(G) = 3$ . Then  $|\mathbf{Z}(G)| = 3$  and  $G/\mathbf{Z}(G)$  is an extra-special 3-group by the properties of groups of order  $3^4$  with maximal class. Hence  $G/\mathbf{Z}(G)$  has exactly two nonlinear irreducible characters by Lemma 2.2 and they are faithful. It indicates that there are  $\chi_1, \chi_2 \in \text{Irr}_1(G)$  and  $\ker \chi_1 = \ker \chi_2 = \mathbf{Z}(G)$ . Since  $\ker \chi_1 \cap \ker \chi_2 \neq 1$ , it follows that  $G$  must have  $\varphi \in \text{Irr}_1(G)$  and  $\varphi \neq \chi_1, \chi_2$ . Suppose  $\ker \varphi \neq 1$ . Since  $\mathbf{Z}(G) \leq \ker \varphi$ , it follows that  $\varphi \in \text{Irr}_1(G/\mathbf{Z}(G))$ . So we find three nonlinear irreducible characters in  $G/\mathbf{Z}(G)$ , which is impossible. Consequently,  $\varphi$  must be faithful and so  $G$  has exactly two nonlinear non-faithful irreducible characters.  $\square$

Finally, the proof of Theorem 1.1 in Introduction is immediately available by Theorems 3.1, 3.2 and 3.3.

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